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Linear Differential Equations with Property A and B[1]

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Abstract

On the basis of the results given by I. T. Kiguradze and T. A. Chanturija in [18] and U. Elias in [6], the definitions of the properties A, B, of the strict properties A, B and of the strong properties A,B for the linear differential equations are given. It is shown that there are some classes of linear differential equations having property A,strict property A and strong property A (property B, strict property B and strong property B).

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1. Introduction

In this paper are investigated certain classes of linear differential equations of order n with the properties A,B which were studied by I. T. Kiguradze and T. A. Chanturija. In investigating these properties the notion of the adjoint equation as it has been introduced by M. Gera in [7] plays an important role. Besides the oscillatory and asymptotic properties also the properties A, B of these equations belonging to these classes are studied. Since in the literature there are different definitions of properties A and B for different equations (see [6] and [18]), in this paper these diferent definitions are denoted as property A, B, or strict property A, B, or strong property A, B and they are applied to the same equation.

Using the results by I. T. Kiguradze and T. A. Chanturija in [18] it is shown that under some conditions the mentioned classes have the strong property A (the strong property B). Later on the results will be applied to solve problems in quasilinear differential equations (for similar results see e. g. [2]-[4], [10], [12]-[14] and [24]).

2. Monotonic properties of solutions

Consider the n-th order ($n \geqq 2$) linear differential equation

$$(1)\qquad (Lx \equiv) x^{(n)} + \sum_{k=1}^{n} p_k(t) x^{(n-k)} = 0,$$

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where p_k are continuous functions of the variable $t \in I = [a, \infty)$, $-\infty < a < \infty$, $k = 1, 2, \dots, n$. In the whole paper we assume that

$$p_n(t) \not\equiv 0 \quad \text{in each subinterval of } I.$$

Throughout the paper some of the assumptions will be used:

(H1) $$(-1)^k p_k(t) \leqq 0, \quad t \in I,\ k = 2, \dots, n;$$

(H1') $$(-1)^k p_k(t) \leqq 0, \quad t \in I,\ k = 1, 2, \dots, n;$$

(H2) $$p_k(t) \leqq 0, \quad t \in I,\ k = 2, \dots, n;$$

(H2') $$p_k(t) \leqq 0, \quad t \in I,\ k = 1, 2, \dots, n.$$

Remark 1. We see that the binomial equation

$$x^{(n)} + p_n(t)x = 0$$

in case n odd and $p_n(t) \geqq 0$, $t \in I$, ($p_n(t) \leqq 0$, $t \in I$) satisfies (H1') (satisfies (H2')) and in case n even, $p_n(t) \leqq 0$, $t \in I$, satisfies both (H1') and (H2'). Further, if equation (1) satisfies one of assumptions (H1)-(H2') so it has the following properties:

(i) It is disconjugate in I for $n = 2$.

(ii) It can be written as the two-term equation (for terminology see [6])

(2) $$L_n x + p_n(t)x = 0,$$

where $L_n x = 0$ is a disconjugate equation in I, for $n = 2, 3$.
Nevertheless,

(iii) the equation

$$x^{(n)} + cx' + p_n(t)x = 0$$

where $c \neq 0$ is a constant, cannot be written in the form (2) for $n \geqq 4$.

The asymptotic properties of nonoscillatory solutions of equation (1) and of its adjoint equation will be based on the following four lemmas.

Lemma 1. *Suppose that the functions $r_{ij} \in C(I)$, $i, j = 1, 2, \dots, n$, have the following properties:*

(3) $$r_{ij}(t) \geqq 0, \quad t \in I,\ i, j = 1, 2, \dots, n,\ r_{i,i+1}(t) > 0, \quad t \in I,\ i = 1, \dots, n-1,$$
$$r_{n,1}(t) \not\equiv 0 \quad \textit{in each subinterval of } I.$$

Let

(4) $$b_i \geqq 0, \quad i = 1, 2, \dots, n,\ \sum_{i=1}^{n} b_i > 0$$

and let $t_0 \in I$.

Then for the solution $(u_1, u_2, \dots, u_n)$ *of the initial-value problem*

$$u_i' = \sum_{j=1}^{n} r_{ij}(t) u_j, \quad i = 1, 2, \dots, n, \tag{5}$$

$$u_i(t_0) = b_i, \quad i = 1, 2, \dots, n, \tag{6}$$

the following statement holds:

The functions u_i, $i = 1, 2, \dots, n$, *are increasing in* $[t_0, \infty)$ *and thus*

$$u_i(t) > b_i, \quad t > t_0, \ i = 1, 2, \dots, n. \tag{7}$$

Proof. If all $b_i > 0$, $i = 1, 2, \dots, n$, then, with respect to (3), the functions u_i, $i = 1, 2, \dots, n$, are increasing in $[t_0, t_0 + \delta)$ for a $\delta > 0$ and hence in $[t_0, \infty)$. In the general case, under assumption (4), the theorem on continuous dependence of solutions on their initial values implies that the solution $(u_1, u_2, \dots, u_n)$ of problem (5), (6) satisfies the inequalities $u_i(t) \geqq b_i$, $t \geqq t_0$, $i = 1, 2, \dots, n$. If $b_k > 0$, then again u_{k-1}, $u_{k-2}, \dots$, u_1 as well as u_n, u_{n-1}, …, u_k are increasing in $[t_0, t_0 + \delta)$ which implies that all these functions are increasing in $[t_0, \infty)$.

Remark 2. With respect to (4) and (3), inequalities (7) imply that

$$u_i(t) > 0, \ t_0 < t < \infty, \ i = 1, 2, \dots, n, \text{ and } u_i'(t) > 0, \ t_0 < t < \infty,$$
$$i = 1, \dots, n-1, \ u_n'(t) \geqq 0, \quad t_0 < t < \infty.$$

Lemma 2. *Suppose that the functions* $r_{ij} \in C(I)$ *have the following properties:*

$$r_{ij}(t) \leqq 0, \ t \in I, \ i, j = 1, 2, \dots, n, r_{i,i+1}(t) < 0, \ t \in I,$$
$$i = 1, \dots, n-1, \ r_{n,1}(t) \not\equiv 0 \text{ in each subinterval } [b, c] \text{ of } I. \tag{8}$$

Let the numbers b_i, $i = 1, 2, \dots, n$, *satisfy (4) and let* $t_0 \in I$, $t_0 > a$. *Then for the solution* $(u_1, u_2, \dots, u_n)$ *of initial-value problem (5), (6) the following statement holds:*

The functions u_i, $i = 1, 2, \dots, n$, *are decreasing in* $[a, t_0]$ *and thus*

$$u_i(t) > b_i, \ a \leqq t < t_0, \ i = 1, 2, \dots, n. \tag{9}$$

Proof. The proof is similar to that of Lemma 1 and therefore it will be omitted.

Remark 3. With respect to (4) and (8), inequalities (9) imply that

$$u_i(t) > 0, \ a \leqq t < t_0, \ i = 1, 2, \dots, n, \text{ and } u_i'(t) < 0, \ a \leqq t < t_0,$$
$$i = 1, \dots, n-1, \ u_n'(t) \leqq 0, \ a \leqq t < t_0.$$

Let $t_0 \in I$, $S = \{(b_1, b_2, \dots, b_n) \in R^n : b_1^2 + b_2^2 + \dots + b_n^2 = 1\}$. Define two sets of initial values of the solutions of (5) by

$$S_{t_0}^+ = \{(b_1, b_2, \dots, b_n) \in S : \text{There exists a } t_1 \geqq t_0 \text{ (depending on } (b_1, b_2, \dots, b_n)) \text{ such that } u_i(t) > 0 \text{ for } t > t_1,\ i = 1, 2, \dots, n,\ u_i'(t) > 0,\ t > t_1,\ i = 1, \dots, n-1, u_n'(t) \geqq 0,\ t > t_1, \text{ for the solution } (u_1, u_2, \dots, u_n) \text{ of } (5), (6)\}.$$

$$S_{t_0}^- = \{(b_1, b_2, \dots, b_n) \in S : \text{There exists a } t_1 \geqq t_0 \text{ (depending on } (b_1, b_2, \dots, b_n)) \text{ such that } u_i(t) < 0 \text{ for } t > t_1,\ i = 1, 2, \dots, n,\ u_i'(t) < 0,\ t > t_1,\ i = 1, \dots, n-1, u_n'(t) \leqq 0,\ t > t_1, \text{ for the solution } (u_1, u_2, \dots, u_n) \text{ of } (5), (6)\}.$$

Lemma 3. *Suppose that the functions $r_{ij} \in C(I)$, $i, j = 1, 2, \dots, n$, satisfy conditions (3). Let $t_0 \in I$. Then:*

(i) $S_{t_0}^+$ *contains the set* $\{(b_1, b_2, \dots, b_n) \in S : b_i \geqq 0,\ i = 1, 2, \dots, n\}$.

(ii) $S_{t_0}^- = -S_{t_0}^+$.

(iii) $S_{t_0}^+$ *is open in* S.

(iv) $S \setminus (S_{t_0}^+ \cup S_{t_0}^-) \neq \emptyset$.

Proof. (i), (ii) The proof is clear.

(iii) Let $(b_1, b_2, \dots, b_n) \in S_{t_0}^+$, $(u_1, u_2, \dots, u_n)$ be the solution of (5), (6) and let $t_2 > t_0$ be such that $u_i(t_2) > 0$, $i = 1, 2, \dots, n$. Then by the theorem on continuous dependence of solutions on their initial values there exists a neighbourhood N of the point $(b_1, b_2, \dots, b_n)$ such that for each $(\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n) \in N \cap S$ the corresponding solution $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_n)$ of (5) satisfying $\bar{u}_i(t_0) = \bar{b}_i$, $i = 1, 2, \dots, n$, shows the property $\bar{u}_i(t_2) > 0$, $i = 1, 2, \dots, n$, and hence $(\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n) \in S_{t_0}^+$ by Lemma 1.

(iv) As $S_{t_0}^+$, $S_{t_0}^-$ are disjoint open nonempty subsets of the connected set S, $S \setminus (S_{t_0}^+ \cup S_{t_0}^-) \neq \emptyset$.

By Theorem 2.1, Chapter XIV, [15, p. 592], and by Lemma 2 and Remark 3, we obtain the following lemma.

Lemma 4. *Suppose that the functions $r_{ij} \in C(I)$, $i, j = 1, 2, \dots, n$, satisfy conditions (8). Then there exists a solution $(u_1, u_2, \dots, u_n)$ of (5) such that*

$$u_i(t) > 0,\ t \in I, i = 1, 2, \dots, n,\ u_i'(t) < 0,\ t \in I,\ i = 1, \dots, n-1,$$
$$u_n'(t) \leqq 0,\ t \in I, \text{ and } u_n \text{ is decreasing in } I.$$

Equation (1) can be transformed to differential system (5) in several ways. We shall consider two of them. Putting

$$x = x_1, \quad (-1)^i x^{(i)} = x_{i+1}, \quad i = 1, \dots, n-1, \tag{10}$$

equation (1) goes into the differential system

$$
\begin{aligned}
x_i' &= - x_{i+1}, \quad i = 1, \dots, n-1, \\
x_n' &= (-1)^n p_n(t) x_1 + (-1)^{n-1} p_{n-1}(t) x_2 + \dots \\
&\quad - p_3(t) x_{n-2} + p_2(t) x_{n-1} - p_1(t) x_n.
\end{aligned} \tag{11}
$$

To remove the term $-p_1(t)x_n$ in (11) we shall apply another transformation in which the function

$$
E(t,\tau) = \exp \int_\tau^t p_1(s)\, ds, \quad (t,\tau) \in I \times I, \tag{12}
$$

plays an important role.

Let us put

$$
x_i = y_i, \quad i = 1, \dots, n-1, \ x_n E(t,a) = y_n. \tag{13}
$$

Then (11) goes into the system

$$
\begin{aligned}
y_i' &= - y_{i+1}, \quad i = 1, \dots, n-2 \\
y_{n-1}' &= - E(a,t) y_n \\
y_n' &= (-1)^n p_n(t) E(t,a) y_1 + (-1)^{n-1} p_{n-1}(t) E(t,a) y_2 + \dots \\
&\quad - p_3(t) E(t,a) y_{n-2} + p_2(t) E(t,a) y_{n-1}.
\end{aligned} \tag{14}
$$

On the other hand, if we put

$$
x = x_1, \ x^{(i)} = x_{i+1}, \quad i = 1, \dots, n-1, \tag{15}
$$

equation (1) is transformed into the system

$$
\begin{aligned}
x_i' &= x_{i+1}, \quad i = 1, \dots, n-1 \\
x_n' &= - p_n(t) x_1 - p_{n-1}(t) x_2 - \dots - p_2(t) x_{n-1} - p_1(t) x_n
\end{aligned}
$$

and again, by using (13), we come to the differential system

$$
\begin{aligned}
y_i' &= y_{i+1}, \quad i = 1, \dots, n-2 \\
y_{n-1}' &= E(a,t) y_n \\
y_n' &= - p_n(t) E(t,a) y_1 - p_{n-1}(t) E(t,a) y_2 - \dots - p_2(t) E(t,a) y_{n-1}.
\end{aligned} \tag{16}
$$

Besides equation (1) we shall be interested in adjoint equation to equation (1). It can be written in the form ([7, p. 3])

$$
\begin{aligned}
(L^\star z \equiv) \quad & {}_{n-1}[{}_{n-2}[\dots {}_3[{}_2[{}_1[z' - p_1(t) z]_1' + p_2(t) z]_2' - p_3(t) z]_3' + \dots \\
& + (-1)^{n-2} p_{n-2}(t) z]_{n-2}' + (-1)^{n-1} p_{n-1}(t) z]_{n-1}' + (-1)^n p_n(t) z = 0.
\end{aligned} \tag{17}
$$

If we put

$$
\begin{aligned}
z &= z_1 \\
z_1' - p_1(t)z_1 &= z_2 \\
{}_{i-1}[\dots {}_2[{}_1[z_1' - p_1(t)z_1]_1' + p_2(t)z_1]_2' - \dots + (-1)^{i-1}p_{i-1}(t)z_1]_{i-1}' + \\
(-1)^i p_i(t)z_1 = z_{i+1}, i = 2, \dots, n-1,
\end{aligned} \tag{18}
$$

then (17) goes into the differential system

$$
\begin{aligned}
z_i' &= (-1)^{i-1}p_i(t)z_1 + z_{i+1}, \quad i = 1, \dots, n-1 \\
z_n' &= (-1)^{n-1}p_n(t)z_1.
\end{aligned} \tag{19}
$$

To remove the term containing p_1 we shall use the transformation

$$
\begin{aligned}
z_1 &= \tilde{z}_1 E(t,a) \\
z_i &= \tilde{z}_i, \quad i = 2, \dots, n,
\end{aligned} \tag{20}
$$

which transforms (19) into the form

$$
\begin{aligned}
\tilde{z}_1' &= E(a,t)\tilde{z}_2 \\
\tilde{z}_i' &= (-1)^{i-1}p_i(t)E(t,a)\tilde{z}_1 + \tilde{z}_{i+1}, \quad i = 2, \dots, n-1 \\
\tilde{z}_n' &= (-1)^{n-1}p_n(t)E(t,a)\tilde{z}_1.
\end{aligned} \tag{21}
$$

Another way of transforming equation (17) consists of putting

$$
\begin{aligned}
z &= z_1 \\
z_1' - p_1(t)z_1 &= -z_2 \\
{}_{i-1}[\dots {}_2[{}_1[z_1' - p_1(t)z_1]_1' + p_2(t)z_1]_2' - \dots + (-1)^{i-1}p_{i-1}(t)z_1]_{i-1}' + \\
(-1)^i p_i(t)z_1 = (-1)^i z_{i+1}, \quad i = 2, \dots, n-1.
\end{aligned} \tag{22}
$$

This leads (17) to the system

$$
\begin{aligned}
z_i' &= p_i(t)z_1 - z_{i+1}, \quad i = 1, \dots, n-1 \\
z_n' &= p_n(t)z_1
\end{aligned} \tag{23}
$$

and by transformation

$$
\begin{aligned}
z_1 &= \tilde{z}_1 E(t,a) \\
z_i &= \tilde{z}_i, \quad i = 2, \dots, n
\end{aligned} \tag{24}
$$

to the system

$$\begin{aligned} \tilde{z}_1' &= - E(a,t)\tilde{z}_2 \\ \tilde{z}_i' &= p_i(t)E(t,a)\tilde{z}_1 - \tilde{z}_{i+1}, \quad i = 2, \dots, n-1 \\ \tilde{z}_n' &= p_n(t)E(t,a)\tilde{z}_1. \end{aligned} \tag{25}$$

Now let us denote the space $\{z \in C(I) : L^\star z \in C(I)\}$ by $C^n_\star(I)$. Then for any two functions $x \in C^n(I)$, $z \in C^n_\star(I)$ the following *Lagrange identity* holds ([7 , p. 4])

$$zLx + (-1)^{n-1} x L^\star z = \frac{d}{dt} F(x,z), \quad t \in I, \tag{26}$$

where

$$\begin{aligned} F(x,z) \equiv & z x^{(n-1)} - {}_1[z' - p_1(t)z]_1 x^{(n-2)} + {}_2[{}_1[z' - p_1(t)z]_1' + \\ & p_2(t)z]_2\, x^{(n-3)} - \dots + (-1)^{n-1}\, {}_{n-1}[\dots {}_2\, [{}_1[z' - p_1(t)z]_1' + \\ & p_2(t)z]_2' - \dots + (-1)^{n-2} p_{n-2}(t)z]_{n-2}' + (-1)^{n-1} p_{n-1}(t)z]_{n-1} x. \end{aligned} \tag{27}$$

With respect to (18), from the Lagrange identity we get the following lemma.

Lemma 5. *Let $(z_1^\star, z_2^\star, \dots, z_n^\star)$ be a solution of (19) such that the function $z_1^\star$ which is a solution of the adjoint equation (17) is positive in I. Then each solution x of (1) satisfies the differential equation of the $(n-1)-$st order*

$$(F_1(x, z_1^\star) \equiv) z_1^\star(t) x^{(n-1)} - z_2^\star(t) x^{(n-2)} + \dots + (-1)^{n-1} z_n^\star(t) x = c \tag{28}$$

where $c = [F_1(x(t), z_1^\star(t))]_{t=t_0}$ and t_0 is an arbitrary, but fixed point of I and conversely, for any constant c each solution x of (28) is a solution of (1).

If instead of (18) we consider transformation (22), then from (26), (27) we get the lemma.

Lemma 6. *Let $(z_1^\star, z_2^\star, \dots, z_n^\star)$ be a solution of (23) such that the function $z_1^\star$ which is a solution of the adjoint equation (17) is positive in I. Then each solution x of (1) satisfies the differential equation of the $(n-1)$-st order*

$$(F_2(x, z_1^\star) \equiv) z_1^\star(t) x^{(n-1)} + z_2^\star(t) x^{(n-2)} + \dots + z_n^\star(t) x = c, \tag{29}$$

where $c = [F_2(x(t), z_1^\star(t))]_{t=t_0}$ and t_0 is an arbitrary, but fixed point of I, and conversely, for any constant c each solution x of (29) is a solution of (1).

Now we introduce the notion of the band of solutions of (1). For similar results, see M. Greguš [11], V. Šeda [22], [23]. Let $t_0 \in I$. Denote by $x_0, \dots, x_{n-1}$ the solutions of (1) (defined on I) which are determined by the initial conditions

$$x_i^{(j)}(t_0) = \delta_{ij} \text{ (the Kronecker symbol)}, \quad i, j = 0, 1, \dots, n-1.$$

It is clear that for each $j \in \{0, 1, \dots, n-1\}$ each solution x of (1) such that $x^{(j)}(t_0) = 0$ is a linear combination $\sum_{\substack{k=0\\k\neq j}}^{n-1} c_k x_k$. The set of all such solutions will be called *the band of solutions of (1) of the j-th kind at the point t_0*. If the Wronskian $W(x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1})$ does not vanish on a subinterval $J \subset I$, then we say that this band is *regular on J*. Of course, $W(x_0, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1})(t_0) = 0$. We also see that each band is an $(n-1)$-dimensional subspace of solutions of (1).

The $(n-1)$-st order linear differential equation of the band of solutions of (1) of the j-th kind at t_0 which is regular on J is given by the formula

$$\frac{W(x, x_0(t), \dots, x_{j-1}(t), x_{j+1}(t), \dots, x_{n-1}(t))}{W(x_0(t), \dots, x_{j-1}(t), x_{j+1}(t), \dots, x_{n-1}(t))} = 0, \quad t \in J.$$

Criterion for the regularity of the band is given by the following lemma which can be proved in a similar way as Theorem 4 in [22, pp. 360–361] (see also Lemma 4 in [23]).

Lemma 7. *Let $t_0 \in I$ ($t_0 > a$), $j \in \{0, 1, \dots, n-1\}$. Then the band of solutions of (1) of the j-th kind at t_0 is regular in $[t_0, \infty)$ (in $[a, t_0)$) if and only if for each $t_1 > t_0$ (for each t_1, $a \leqq t_1 < t_0$) the solution x of the initial value problem (1),*

$$x^{(i)}(t_1) = 0, \quad i = 0, \dots, n-2, \quad x^{(n-1)}(t_1) = x_1^{n-1} \neq 0$$

is such that $x^{(j)}(t_0) \neq 0$.

If assumption (H1) is fulfilled, then fundamental properties of equation (1) and of system (19) corresponding to the adjoint equation (17) are given in

Theorem 1. *Suppose that assumption (H1) is satisfied. Then the following statements hold:*

(a) *Let $t_0 > a$ and let the numbers x_0^i, $i = 0, 1, \dots, n-1$, satisfy the inequalities*

$$(-1)^i x_0^i \geqq 0, \quad \sum_{i=0}^{n-1} |x_0^i| > 0.$$

Then the functions $(-1)^i x^{(i)}$, $i = 0, 1, \dots, n-2$, and $(-1)^{n-1} x^{(n-1)} E(., a)$ are decreasing in $[a, t_0]$ and thus,

$$(-1)^i x^{(i)}(t) > 0, \quad a \leqq t < t_0, \ i = 0, 1, \dots, n-1,$$

where x is the solution of the initial value problem (1),

$$x^{(i)}(t_0) = x_0^i, \quad i = 0, 1, \dots, n-1. \tag{30}$$

(b) *There exists a solution x_1 of (1) such that*

$$(-1)^i x_1^{(i)}(t) > 0, \quad t \in I, \ i = 0, 1, \dots, n-1. \tag{31}$$

(c) *Let $t_0 \in I$. Then the band of solutions of (1) of the j-th kind at the point t_0, $j = 0, 1, \ldots, n-1$, is regular in (t_0, ∞).*

(d) *Let $t_0 \in I$ and let the numbers z_i^0, $i = 1, 2, \ldots, n$, satisfy the inequalities*

$$z_i^0 \geqq 0, \quad i = 1, 2, \ldots, n, \ \sum_{i=1}^{n} z_i^0 > 0. \tag{32}$$

Then the functions $z_1 E(a, .)$, z_i, $i = 2, \ldots, n$, are increasing in $[t_0, \infty)$ and thus, they satisfy the inequalities

$$z_i(t) > 0, \quad t > t_0, \ i = 1, 2, \ldots, n,$$

where $(z_1, z_2, \ldots, z_n)$ is the solution of the initial value problem (19),

$$z_i(t_0) = z_i^0, \quad i = 1, 2, \ldots, n. \tag{33}$$

(e) *There exists a solution $(z_1^\star, z_2^\star, \ldots, z_n^\star)$ of system (19) such that*

$$z_i^\star(t) > 0, \quad t \in I, \ i = 1, 2, \ldots, n, \tag{34}$$

and functions $z_1^\star E(a, .)$, $z_i^\star$, $i = 2, \ldots, n$, are increasing in I.

Proof. By (10), (13) where E is defined by (12), equation (1) is transformed into system (14). On the basis of (H1), system (14) satisfies conditions (8). Then by Lemma 2 and Remark 3 statement (a) follows. Lemma 4 implies statement (b). On the basis of Lemma 7, statement (c) is a consequence of statement (a).

By (20), system (19) is transformed into system (21) which in view of (H1) satisfies conditions (3). Then by Lemma 1 and Remark 2, statement (d) follows. Statement (e) follows from (d) for $t_0 = a$, $z_i^0 > 0$, $i = 1, 2 \ldots, n$.

Similarly, if (H2) is supposed, then by means of (15) and (13), equation (1) is transformed into system (16) and using (24), system (23) goes into system (25). In view of (H2), system (16) satisfies conditions (3) and system (25) conditions (8). Then in a similar way as we have proved Theorem 1, we obtain

Theorem 2. *Suppose that assumption (H2) is satisfied. Then the following statements hold:*

(a) *Let $t_0 \in I$ and let the numbers x_0^i, $i = 0, 1, \ldots, n-1$, satisfy the inequalities*

$$x_0^i \geqq 0, \quad \sum_{i=0}^{n-1} x_0^i > 0. \tag{35}$$

Then the functions $x^{(i)}$, $i = 0, \ldots, n-2$, and $x^{(n-1)} E(., a)$ are increasing in $[t_0, \infty)$ and thus

$$x^{(i)}(t) > 0, \quad t_0 < t < \infty, \ i = 0, 1, \ldots, n-1,$$

where x is the solution of the initial value problem (1), (30).

(b) *There exists a solution* x_1 *of (1) such that*

$$x_1^{(i)}(t) > 0, \quad t \in I,\ i = 0, 1, \dots, n-1. \tag{36}$$

(c) *Let* $t_0 > a$. *Then the band of solutions of (1) of the* j*-th kind at the point* t_0, $j = 0, 1, \dots, n-1$, *is regular in* $[a, t_0)$.

(d) *Let* $t_0 > a$ *and let the numbers* z_i^0, $i = 1, 2, \dots, n$, *satisfy inequalities (32). Then the functions* $z_1 E(a, .)$, z_i, $i = 2, \dots, n$, *are decreasing in* $[a, t_0]$ *and thus,*

$$z_i(t) > 0, \quad a \leqq t < t_0,\ i = 1, 2, \dots, n,$$

where $(z_1, z_2, \dots, z_n)$ *is the solution of the initial value problem (23), (33).*

(e) *There exists a solution* $(z_1^\star, z_2^\star, \dots, z_n^\star)$ *of system (23) such that*

$$z_i^\star(t) > 0, \quad t \in I,\ i = 1, 2, \dots, n \tag{37}$$

and the functions $z_1^\star E(a, .)$, $z_i^\star$, $i = 2, \dots, n$, *are decreasing in* I.

Remark 4. On the basis of Lemma 5, for any solution $(z_1^\star, z_2^\star, \dots, z_n^\star)$ of system (19) such that $z_1^\star(t) > 0$, $t \in I$, the space of all solutions of the equation

$$z_1^\star(t) x^{(n-1)} - z_2^\star(t) x^{(n-2)} + \dots + (-1)^{n-1} z_n^\star(t) x = 0 \tag{38}$$

is an $(n-1)$-dimensional subspace of solutions of (1) and we shall call it the *improper band of solutions of (1) corresponding to solution* $(z_1^\star, z_2^\star, \dots, z_n^\star)$ *of (19).* This band is *regular on* I.

Similarly, with respect to Lemma 6, for any solution $(z_1^\star, z_2^\star, \dots, z_n^\star)$ of (23) such that $z_1^\star > 0$, $t \in I$, the space of all solutions of the equation

$$z_1^\star(t) x^{(n-1)} + z_2^\star(t) x^{(n-2)} + \dots + z_n^\star(t) x = 0 \tag{39}$$

is an $(n-1)$-dimensional subspace of solutions of (1) and again we shall call it the *improper band of solutions of (1) corresponding to solution* $(z_1^\star, z_2^\star, \dots, z_n^\star)$ *of (23)* which is *regular on* I.

3. Properties A, B

With respect to Definitions 1.1, 1.2 in [18, p. 11] and Definitions 8.21, 8.22 in [6, p. 126], we introduce the following definitions.

Definition 1. Equation (1) has the property A if in case n even each nontrivial solution x of that equation is oscillatory and in case n is odd, each nontrivial solution x of that equation is either oscillatory or it satisfies the condition

$$x(t) x'(t) < 0 \quad \text{for } t \geqq t_1 \tag{40}$$

where $t_1 \in I$ depends on the solution x.

In case n is odd, we also have the following definition.

Definition 2. Let n be odd. Equation (1) has the strict property A (the strong property A) if each nontrivial solution x of that equation is either oscillatory or it satisfies the condition

$$(-1)^i x(t)x^{(i)}(t) > 0 \quad \text{for } t \geqq t_1, \quad i = 1, \dots, n-1 \tag{41}$$

(if each nontrivial solution x of that equation is either oscillatory or it satisfies the condition

$$\begin{aligned} &(-1)^i x(t)x^{(i)}(t) > 0 \quad \text{for } t \geqq t_1, \quad i = 1, \dots, n-1, \\ &\text{and } \lim_{t\to\infty} x^{(i)}(t) = 0, \quad i = 0, 1, \dots, n-1) \end{aligned} \tag{42}$$

where $t_1 \in I$ depends on solution x.

Definition 3. Equation (1) has the property B (the strict property B) [the strong property B] if in case n even each nontrivial solution x of that equation is either oscillatory or it satisfies (40) or it satisfies

$$x(t)x'(t) > 0 \quad \text{for } t \geqq t_1 \tag{43}$$

(each nontrivial solution x of that equation is either oscillatory or it satisfies (41) or it satisfies

$$x(t)x^{(i)}(t) > 0 \quad \text{for } t \geqq t_1,\ i = 1, \dots, n-1) \tag{44}$$

[each nontrivial solution x of that equation is either oscillatory or it satisfies (42) or it satisfies

$$\begin{aligned} &x(t)x^{(i)}(t) > 0 \quad \text{for } t \geqq t_1,\ i = 1, \dots, n-1, \\ &\text{and } \lim_{t\to\infty} x^{(i)}(t) = \infty.\text{sign}\,\{x(t) : t \geqq t_1\}, \quad i = 0, 1, \dots, n-1] \end{aligned} \tag{45}$$

where $t_1 \in I$ depends on x and in case n is odd x is either oscillatory or it satisfies (43) (x is either oscillatory or it satisfies (44)) [x is either oscillatory or it satisfies (45)].

Remark 5. By Theorem 1, statement (b), if n is even and (H1) is assumed, then equation (1) has no property A.

Theorem 3. *Suppose that n is odd and assumption (H1) is satisfied. Then equation (1) with the strict property A has the following properties:*

(i) *There exists a solution x_1 of (1) such that (31) is true.*

(ii) *Each nonoscillatory solution x of (1) satisfies the inequalities*

$$(-1)^i x(t)x^{(i)}(t) > 0, \quad i = 1, \dots, n-1 \tag{46}$$

in the whole interval I.

(iii) *If x is a solution of (1) such that*

$$(-1)^j x(t_0)x^{(j)}(t_0) \leqq 0 \tag{47}$$

is true for a $t_0 \in I$ and a $j \in \{0, 1, \dots, n-1\}$, then x is oscillating. Especially, the band of solutions of (1) of the j-th kind at the point t_0 for each $j \in \{0, 1, \dots, n-1\}$ and each $t_0 \in I$ consists of solely oscillatory solutions. The same is true for any improper band of solutions of (1) corresponding to solution $(z_1^\star, z_2^\star, \dots, z_n^\star)$ of (19) satisfying inequalities (34).

Proof. (i) The statement follows from statement (b) of Theorem 1.

(ii) If x is an eventually positive solution of (1), then (46) is true in an interval $[t_0, \infty)$, $a \leqq t_0$. By statement (a) of Theorem 1, these inequalities can be extended to the whole interval I. The same is true for eventually negative solutions of (1).

(iii) If a solution x of (1) satisfies (47), then by statement (ii), x is oscillatory. Further, if a solution $(z_1^\star, z_2^\star, \dots, z_n^\star)$ of (19) satisfies (34) and x is a nonoscillatory solution of (1), then by (46), the expression $z_1^\star(t)x^{(n-1)}(t) - z_2^\star(t)x^{(n-2)}(t) + \cdots + (-1)^{n-1}z_n^\star(t)x(t) \neq 0$ for each $t \in I$ and hence, each solution of (38) is oscillatory.

Theorem 4. *Suppose that n is odd and assumption (H2) is satisfied. Then equation (1) with the strict property B has the following properties:*

(i) *There exists a solution x_1 of (1) such that (36) is satisfied.*

(ii) *Each nonoscillatory solution x of (1) satisfies the inequalities*

$$x(t)x^{(i)}(t) > 0, \quad t \geqq t_1,\ i = 1, \dots, n-1, \tag{48}$$

where t_1 depends on x.

(iii) *The improper band of solutions of (1) corresponding to solution $(z_1^\star, z_2^\star, \dots, z_n^\star)$ of (23) satisfying (37) consists of solely oscillatory solutions.*

Proof. (i) Statement follows from statement (b) of Theorem 2.

(ii) Statement follows directly from Definition 3.

(iii) If $(z_1^\star, z_2^\star, \dots, z_n^\star)$ is a solution of (23) satisfying (37) and x is a nonoscillatory solution of (1), then by (48), $z_1^\star(t)x^{(n-1)}(t) + z_2^\star(t)x^{(n-2)}(t) + \cdots + z_n^\star(t)x(t) \neq 0$ for each $t \geqq t_1$ and hence, each solution of (39) is oscillatory.

Remark 6. (i) Under the assumtions in the Theorem 4 the existence of an oscillatory solution of (1) can be also proved by the use of Lemma 3 to system (16).

(ii) For $n = 3$ see similar results in [7], [8].

The next lemma will be based on Lemma 5 in [22, p. 354], which states that if a linear differential equation is eventually disconjugate, and the right-hand side does not change its sign, then the nonhomogeneous equation is nonoscillatory.

Lemma 8. *Suppose n is odd, equation (1) satisfies hypothesis (H1) and in case $n \geqq 5$ also hypothesis:*

(H3) The differential equations

$$x^{(j)} + \sum_{k=1}^{j} p_k(t)x^{(j-k)} = 0, \quad j = 3, \dots, n-2,$$

are eventually disconjugate.

Then the following implication holds:
If equation (1) has the property A, then it has the strict property A.

Proof. Let equation (1) have the property A and let x be a nonoscillatory solution of (1). Then there exists a $t_1 \in I$ such that (40) is true. Without loss of generality we will consider only the case that $x(t) > 0$, $x'(t) < 0$ in $[t_1, \infty)$. Then $x'' = u$ is a solution of the nonhomogeneous equation

$$u^{(n-2)} + \sum_{k=1}^{n-2} p_k(t) u^{(n-2-k)} = -p_{n-1}(t)x'(t) - p_n(t)x(t) \leqq 0, \quad t \geqq t_1,$$

the right-hand side of which is not identically zero on any subinterval of $[t_1, \infty)$ and the left-hand side is, in view of (H3), eventually disconjugate. Thus, by Lemma 5, [22], $x''(t) > 0$ or $x''(t) < 0$ in a subinterval $[t_2, \infty)$ of $[t_1, \infty)$. Because $x(t) > 0$, $x'(t) < 0$ in $[t_1, +\infty)$, it is obviously only possibility $x''(t) > 0$ can occur. Then repeating this process and taking into consideration (see Remark 1) that both equations $x'' + p_1(t)x' + p_2(t)x = 0$, $x' + p_1(t)x = 0$ are disconjugate in I, we get that $(-1)^i x^{(i)}(t) > 0$ holds in an interval $[t_3, \infty)$ for $i = 0, 1, \ldots, n-1$.

4. Sufficient conditions for properties A

Criteria for the property A of the differential equation (1) have been given by I. T. Kiguradze, T. A. Chanturija in Theorems 1.8-1.14 in [18]. Following this theory we suppose that

$$p_k(t) = p_{k,1}(t) + p_{k,2}(t), \quad t \in I, \ k = 1, 2, \ldots, n,$$

where $p_{k,1}$, $p_{k,2}$ are continuous on I, $k = 1, 2, \ldots, n$. Then choosing $\omega(t) \equiv 1$, $t \in I$, in Theorem 1.8, we get

Lemma 9. *If the functions $p_{k,1}$, $p_{k,2}$, $k = 1, 2, \ldots, n$, satisfy the conditions*

(a) $$\int_a^\infty |p_{k,2}(t)|\, dt < \infty, \quad k = 1, 2, \ldots, n,$$

(b) $$\limsup_{t\to\infty} |p_{k,1}(t)| < \infty, \quad k = 1, \ldots, n-1,$$

(c) $$\lim_{t\to\infty} p_{n,1}(t) = \infty,$$

then equation (1) has the property A.

The choice $\omega(t) \equiv t$, $t \in I$ (taken in Theorem 1.8, [18]) implies

Lemma 10. *If the functions $p_{k,1}$, $p_{k,2}$, $k = 1, 2, \ldots, n$, satisfy the conditions*

$$\text{(a)} \qquad \int_a^\infty t^{k-1}|p_{k,2}(t)|\,dt <\infty, \quad k = 1, 2, \ldots, n,$$

$$\text{(b)} \qquad \limsup_{t\to\infty} t^k|p_{k,1}(t)|\,dt <\infty, \quad k = 1, \ldots, n-1,$$

$$\text{(c)} \qquad \lim_{t\to\infty} t^n p_{n,1}(t) =\infty,$$

then equation (1) has the property A.

By Corollary 6.4, [18], a sufficient condition for eventual disconjugacy is given in

Lemma 11. *If the functions p_k, $k = 1, 2, \ldots, n$, satisfy the inequalities*

$$\int_a^\infty t^{k-1}|p_k(t)|\,dt < \infty, \quad k = 1, 2, \ldots, n,$$

then equation (1) is eventually disconjugate.

By Lemmas 8, 9 and 10 we get a criterion for the strict property A of equation (1) in case $n = 3$.

Theorem 5. *If $n = 3$, (H1) is true and*
either

$$\int_a^\infty |p_k(t)|\,dt < \infty \text{ or } \limsup_{t\to\infty} |p_k(t)| < \infty, \quad k = 1, 2 \tag{49}$$

and

$$\lim_{t\to\infty} p_3(t) = \infty$$

or

$$\int_a^\infty t^{k-1}|p_k(t)|\,dt < \infty \text{ or } \limsup_{t\to\infty} t^k|p_k(t)| < \infty, \quad k = 1, 2 \tag{50}$$

and

$$\lim_{t\to\infty} t^3 p_3(t) = \infty$$

then equation (1) has the strict property A.

If $n \geqq 5$ is odd, then again by Lemmas 8-11 we get

Theorem 6. *Suppose that $n \geqq 5$ is odd, (H1) is true,*

$$\int_a^\infty t^{k-1}|p_k(t)|\,dt < \infty, \quad k = 1, \ldots, n-2, \tag{51}$$

and either

$$\int_a^\infty |p_{n-1}(t)|\,dt < \infty \quad \text{and} \lim_{t\to\infty} p_n(t) = \infty, \tag{52}$$

or

$$\limsup_{t\to\infty} |p_{n-1}(t)| < \infty \text{ and } \lim_{t\to\infty} p_n(t) = \infty, \tag{53}$$

or

$$\int_a^\infty t^{n-2}|p_{n-1}(t)|\,dt < \infty \text{ and } \lim_{t\to\infty} t^n p_n(t) = \infty, \tag{54}$$

or

$$\limsup_{t\to\infty} t^{n-1}|p_{n-1}(t)| < \infty \text{ and } \lim_{t\to\infty} t^n p_n(t) = \infty, \tag{55}$$

then equation (1) has the strict property A.

Proof. Conditions (51) imply that (H3) is satisfied. If we put $p_{k,1}(t) = 0$, $k = 1,\dots,n-1$, $p_{n,2}(t) = 0$, $t \in I$, then (51) and (52) guarantee that all conditions of Lemma 9 are satisfied.

If $p_{k,1}(t) = 0$, $t \in I$, $k = 1,2,\dots,n-2$, then, by (53), again the conditions of Lemma 9 are satisfied. Similarly, from $p_{k,1}(t) = 0$, $k = 1,\dots,n-1$, $p_{n,2}(t) = 0$, $t \in I$, and by (51), (54), the conditions of Lemma 10 follow.

If we put $p_{k,1}(t) = 0$, $t \in I$, $k = 1,\dots,n-2$, $p_{n-1,2}(t) = 0$, $p_{n,2}(t) = 0$, $t \in I$, then (51) and (55) imply the conditions of Lemma 10.

By Lemma 8 the statement follows.

The result of Theorems 5 and 6 can be strengthened by the following lemma.

Lemma 12. *Suppose that n is odd, (H1') is true and*

$$\int_a^\infty (s-a)^{n-1} p_n(s)\,ds = \infty. \tag{56}$$

Then equation (1) with the strict property A has the strong property A.

Proof. Let x be a nonoscillatory solution of (1) such that $x(t) > 0$ in a neighbourhood of ∞. Then, by Theorem 3,

$$(-1)^i x^{(i)}(t) > 0 \quad \text{in } I,\ i = 0,1,\dots,n-1.$$

In view of (H1'), this implies that $x^{(n)}(t) \leqq 0$ in I and thus, there exists $\lim_{t\to\infty} x^{(i)}(t) = c_i$, $i = 0,1,\dots,n-1$, whereby $c_{n-1} = c_{n-2} = \dots = c_1 = 0$ and $c_0 \geqq 0$.

Consider the case $c_0 > 0$. Then integrating (1) we obtain for x with respect to (H1') the relations

$$\begin{aligned} x(t) =& c_0 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\left[\sum_{k=1}^n p_k(s) x^{(n-k)}(s)\right] ds \\ \geqq& c_0 + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} p_n(s) x(s)\,ds, \quad t \in I. \end{aligned}$$

Thus

$$x(a) \geqq c_0 + \int_a^\infty \frac{(s-a)^{n-1}}{(n-1)!} p_n(s)x(s)\,ds.$$

Hence according to inequality $x(t) \geqq c_0$, $t \in I$, we get

$$\int_a^\infty \frac{(s-a)^{n-1}}{(n-1)!} p_n(s)\,ds < \infty$$

which is a contradiction to (56).

As condition $\lim_{t\to\infty} p_n(t) = \infty$ or condition $\lim_{t\to\infty} t^n p_n(t) = \infty$ implies that (56) is fulfilled, by Lemma 12 we get

Corollary to Theorem 5. *If $n = 3$, (H1') is true and either (49), or (50) holds, then equation (1) has the strong property A.*

Corollary to Theorem 6. *Suppose that $n \geqq 5$ is odd, (H1') is true, (51) holds, and either (52), or (53), or (54), or (55) holds, then equation (1) has the strong property A.*

Using some decompositions of the functions p_k, $k = 1, 2, \dots, n$, as in the proof of the last theorem, by means of Lemmas 9 and 10 we can derive

Theorem 7. *Suppose that at least one from following conditions is fulfilled*

(a) $$\int_a^\infty |p_k(t)|\,dt < \infty \text{ or } \limsup_{t\to\infty} |p_k(t)| < \infty \quad k = 1, \dots, n-1,$$

and

$$\lim_{t\to\infty} p_n(t) = \infty.$$

(b) $$\int_a^\infty t^{k-1}|p_k(t)|\,dt < \infty \text{ or } \limsup_{t\to\infty} t^k |p_k(t)| < \infty, \quad k = 1, \dots, n-1,$$

and

$$\lim_{t\to\infty} t^n p_n(t) = \infty.$$

Then equation (1) has the property A.

Theorem 8. *Let n be odd, assumtion (H_1') is fulfilled and let*

$$\int_a^\infty t^{k-1}|p_k(t)|\,dt < \infty, \quad k = 1, \dots, n-1, \tag{57}$$

and

$$\int_a^\infty p_n(t)\,dt = \infty, \tag{58}$$

hold. Then equation (1) has strong property A.

Proof. Let x be a nonoscillatory solution of (1). Without loss generality we can assume that $x(t) > 0$ in an interval $[t_0, \infty) \subset I$. On the basic (57), Lemma 11 implies that the equation

$$x^{(n-1)} + \sum_{k=1}^{n-1} p_k(t) x^{(n-1-k)} = 0, \tag{59}$$

is eventually disconjugate. We may suppose that (59) is disconjugate on the interval $[t_0, \infty)$. We observe that $y = x'$ is a solution of the equation

$$y^{(n-1)} + \sum_{k=1}^{n-1} p_k(t) u^{(n-1-k)} = -p_n(t)x(t) \leqq 0, \quad t \geqq t_0, \tag{60}$$

the right-hand side of which is not identically zero on any subinterval of $[t_0, \infty)$. Thus, by Lemma 5 [22], $x'(t) > 0$ or $x'(t) < 0$ in a subinterval $[t_1, \infty)$ of $[t_0, \infty)$. We assert that $x'(t) > 0$ is impossible. To prove this we assume the contrary. By (60)

$$x'(t) = u(t) - \int_{t_1}^{t} K(t,s) p_n(s) x(s)\, ds, \quad t \geqq t_1. \tag{61}$$

where u is the solution of (59) satisfying

$$u^{(i)}(t_1) = x^{(i+1)}(t_1), \quad i = 0, 1, \dots, n-2$$

and K is the Cauchy's function for (59), that is, $v = K(., s)$ is the solution of (59) which takes at $t = s$ the initial values

$$v^{(i)}(s) = 0,\ i = 0, \dots, n-3,\ v^{(n-2)}(s) = 1.$$

Moreover $K(t,s) > 0$ for $t > s \geqq t_1$, because (59) is disconjugate on $[t_1, \infty)$. If $x_1, \dots, x_{n-1}$ is a fundamental system of solutions of equation (59) then

$$K(t,s) = \frac{\begin{vmatrix} x_1(s) & x_2(s) & \dots & x_{n-1}(s) \\ \dots & \dots & \dots & \dots \\ x_1^{(n-3)}(s) & x_2^{(n-3)}(s) & \dots & x_{n-1}^{(n-3)}(s) \\ x_1(t) & x_2(t) & \dots & x_{n-1}(t) \end{vmatrix}}{W(x_1, \dots, x_{n-1})(s)} = \sum_{k=1}^{n-1} x_j(t) z_j(s), \quad a \leqq s \leqq t,$$

where $W(x_1, \dots, x_{n-1})(s)$ is the Wronskian of the functions $x_1, \dots, x_{n-1}$ at the point s and

$$z_j(s) = (-1)^{n-1+j} \frac{W(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n-1})(s)}{W(x_1, \dots, x_{n-1})(s)}, \quad \text{for } s \in I,$$

where $W(x_1,\dots,x_{j-1},x_{j+1},\dots,x_{n-1})$ is the Wronskian of $x_1,\dots,x_{j-1},x_{j+1},\dots,$ x_{n-1}, $j=1,2,\dots,n-1$. Under the assumtion (57), by Theorem 17.1 in [15], there exists the fundamental system of the solution $x_1,\dots,x_{n-1}$ of (59) such that

$$x_j^{(k)}(t)=\frac{t^j-1-k}{(j-1-k)!}(1+o(1)) \text{ as } t\to\infty,\quad j=1,\dots,n-1,\ k=0,1,\dots,j-1$$

$$x_j^{(k)}(t)=o(t^{j-1-k}) \text{ as } t\to\infty,\quad j=1,\dots,n-1,\ k=j,j+1,\dots,n-2. \tag{62}$$

From (61) and the fact that $x(t)\geqq x(t_1))$, for $t\geqq t_1$, we obtain

$$\begin{aligned} x'(t) &\leqq u(t)-x(t_1)\int_{t_1}^{T}K(t,s)p_n(s)\,ds-\int_{T}^{t}K(t,s)p_n(s)x(s)\,ds \\ &\leqq u(t)-x(t_1)\sum_{j=1}^{n-1}x_j(t)\int_{t_1}^{T}z_j(s)p_n(s)\,ds \quad \text{for } t\geqq T, \end{aligned} \tag{63}$$

where $T\geqq t_1$ is an arbitrary.

Since u is the solution of (59) there are constants $c_1,\dots,c_{n-1}$ such that $u=\sum_{j=1}^{n-1}c_ju_j$. Using (62), we get

$$\lim_{t\to\infty}\frac{x_j(t)}{t^{n-2}}=0,\ j=1,2,\dots,n-2;\ \lim_{t\to\infty}\frac{x_{n-1}(t)}{t^{n-2}}=\frac{1}{(n-2)!}$$

and so

$$\begin{aligned} &\lim_{t\to\infty}\frac{u(t)}{t^{n-2}}=c_{n-1},\ \lim_{t\to\infty}\sum_{j=1}^{n-1}\frac{x_j(t)}{t^{n-2}}\int_{t_1}^{T}z_j(s)p_n(s)\,ds \\ &=\frac{1}{(n-2)!}\int_{t_1}^{T}z_{n-1}(s)p_n(s)\,ds \quad (T\geqq t_1). \end{aligned}$$

Therefore by (63)

$$\limsup_{t\to\infty}\frac{x'(t)}{t^{n-2}}\leqq c_{n-1}-\frac{x(t_1)}{(n-2)!}\int_{t_1}^{T}z_{n-1}(s)p_n(s)\,ds \tag{64}$$

for each $T\geqq t_1$.

It is possible to verify that

$$z_{n-1}(s)=1+o(1) \text{ as } s\to\infty.$$

In regard of this and (58) we have

$$\lim_{t\to\infty}\int_{t_1}^{T}z_{n-1}(s)p_n(s)\,ds=\int_{t_1}^{\infty}p_n(s)\,ds=\infty.$$

Then from (64) we obtain

$$\limsup_{t\to\infty} \frac{x'(t)}{t^{n-2}} = -\infty.$$

Consequently $x' < 0$ near ∞, which is a contradiction. This contradiction shows that equation (1) has the property A. Therefore, in view of (57) (see Lemma 11), by Lemma 8, equation (1) has the strict property A. Now Lemma 12 completes the proof of the theorem.

Sufficient conditions for the strong property B

Now we shall give a sufficient condition for equation (1) to have a strong property B. First, we shall discuss the properties of the eventually disconjugate equation (1).

Hence, let equation (1) be eventually disconjugate. By A. Ju. Levin [19, p. 59] (see also [6, p. 5]) such an equation has a hierarchical fundamental system of solutions x_i, $i = 1, 2, \dots, n$, (principal system on an interval $[b, \infty) \in I$, by [6, p. 5]) which means that all functions x_i are positive in a neighbourhood of ∞ and

$$x_i(t) = o(x_{i+1}(t)) \quad \text{for } t \to \infty, \quad i = 1, \dots, n-1. \tag{65}$$

(See also [22, Lemma 6, pp. 355 and 361-362].)

Moreover, for any two nontrivial solutions x, y of (1) there exists a finite or infinite $\lim_{t\to\infty} \dfrac{x(t)}{y(t)}$. We shall call these two solutions *equivalent* if this limit is finite and different from zero. We denote this by $x \sim y$. The relation to be equivalent is reflexive, symmetric and transitive and hence, by this relation the set of all nontrivial solutions of (1) is decomposed into classes of equivalent solutions. In view of (65) and of the representation of the solution x of (1) in the form $x = \sum_{j=1}^{n} c_j x_j$, the following statements hold:

1. For each nontrivial solution x of (1) there exists one and only one $j \in \{1, 2, \dots, n\}$ such that x is equivalent to x_j and thus, there are exactly n classes U_j, $j = 1, 2, \dots, n$, of equivalent solutions of (1) possessing x_j as their representatives.
2. The class U_j consists of the solutions $\sum_{k=1}^{j} c_k x_k$ of (1), where $c_1, \dots, c_j$, $c_j \neq 0$ are arbitrary numbers.

Lemma 13. *Suppose that assumption (H2') is satisfied and that equation (1) is eventually disconjugate. Let x be the solution of initial value problem (1), (30), whereby $t_0 \in I$, x_0^i, $i = 0, 1, \dots, n-1$, satisfy inequalities (35). Then x belongs to the class U_n.*

Proof. By Theorem 2, the functions $x^{(i)}$ are increasing and $x^{(i)}(t) > 0$, $t_0 < t < \infty$, $i = 0, 1, \dots, n-1$. Choose $t_1 > t_0$. Comparing solution x with $x_n \in U_n$, we see that there exists a $k > 0$ such that $x_n^{(i)}(t_1) < kx^{(i)}(t_1)$, $i = 0, 1, \dots, n-1$, and thus, again by Theorem 2, $\lim_{t\to\infty} \dfrac{kx(t)}{x_n(t)} \geqq 1$, and since $x_n \in U_n$, $\lim_{t\to\infty} \dfrac{kx(t)}{x_n(t)}$ cannot be infinite. Hence $x \in U_n$.

A similar result as given by Lemma 12 is stated by the following lemma.

Lemma 14. *Suppose that n is odd, assumption (H2') is satisfied and*

$$\int_a^\infty s^{n-1}|p_n(s)|\,ds = \infty. \tag{66}$$

Then equation (1) with the strict property B has the strong property B.

Proof. Let x be a nonoscillatory solution of (1). By Theorem 4, we can suppose that $x^{(i)}(t) > 0$ in an interval $[t_1, \infty) \subset I$, $i = 0, 1, \dots, n-1$. From (1) it follows $x^{(n)}(t) \geqq 0$ for $t \geqq t_1$. Thus there exists $\lim_{t\to\infty} x^{(i)}(t) = c_i$, $i = 0, 1, \dots, n-1$, whereby $0 < c_{n-1} \leqq \infty$ and $c_i = \infty$ for $i = 0, 1, \dots, n-2$.

Consider the case $c_{n-1} < \infty$. Let $0 < \varepsilon < c_{n-1}$ be an arbitrary number. Then there exists $t_2 \geqq t_1$ such that

$$c_{n-1} - \varepsilon \leqq x^{(n-1)}(t) \leqq c_{n-1} \quad \text{for } t \geqq t_2.$$

Hence

$$\begin{aligned} x(s) =& \sum_{j=0}^{n-2} \frac{x^{(j)}(t_2)}{j!}(s-t_2)^j + \frac{1}{(n-2)!}\int_{t_2}^s (s-\tau)^{n-2} x^{(n-1)}(\tau)\,d\tau \\ \geqq& (c_{n-1} - \varepsilon)\frac{(s-t_2)^{n-1}}{(n-1)!}, \quad t_2 \leqq s, \end{aligned}$$

and thus,

$$\begin{aligned} x^{(n-1)}(t) =& x^{(n-1)}(t_2) - \int_{t_2}^t \Big(\sum_{k=1}^n p_k(s) x^{(n-k)}(s)\Big)\,ds \\ \geqq& x^{(n-1)}(t_2) - \int_{t_2}^t p_n(s)x(s)\,ds \\ \geqq& x^{(n-1)}(t_2) - \frac{c_{n-1}-\varepsilon}{(n-1)!}\int_{t_2}^t (s-t_2)^{n-1} p_n(s)\,ds, \quad t \geqq t_2. \end{aligned}$$

Hence we get

$$\int_{t_2}^\infty (s-t_2)^{n-1}|p_n(s)|\,ds < \infty.$$

This contradicts (66).

Theorem 9. *Let n be odd, assumption (H2') is fulfilled and*

$$\int_a^\infty |p_n(t)|\,dt = \infty, \tag{67}$$

and let

$$\int_a^\infty t^{k-1}|p_k(t)|\,dt < \infty, \quad k = 1, 2, \dots, n-1, \tag{68}$$

hold. Then equation (1) has the strong property B.

Proof. Let x be a nonoscillatory solution of (1). Without loss of generality we may assume that $x(t) > 0$ in $[t_0, \infty) \subset I$. Then x satisfies the equation

$$x^{(n)} + \sum_{k=1}^{n-1} p_k(t) x^{(n-k)} = -p_n(t)x(t). \tag{69}$$

On the basis of (68), Lemma 11 implies that the equation

$$x^{(n)} + \sum_{k=1}^{n-1} p_k(t) x^{(n-k)} = 0, \quad t \in I, \tag{70}$$

is eventually disconjugate and by Lemma 2.1, [18], there exists a $t_1 \geqq t_0$ such that the quasiderivatives $D^0(x; a_0) > 0$, $D^1(x; a_0, a_1) > 0$ for $t \geqq t_1$ where $D^n(x; a_0, a_1, \dots, a_n)$ is the canonical form of (70) at ∞ ([26, p. 322]). The functions $a_0, a_1, \dots, a_n$ are uniquely determined up to positive multiplicative constants with product 1 (Theorem 1, [26, p. 322]). By Theorem 17.1 in [15], there exists a hierarchical fundamental system of solutions of (70) in the form $x_1(t) \equiv 1$, $x_j(t) \sim t^{j-1}$, $j = 2, \dots, n$. In view of 2.1 in [26, p. 321], $a_0(t) \equiv 1$ and thus,

$$D^0(x; a_0)(t) = x(t), \quad D^1(x; a_0, a_1)(t) = \frac{1}{a_1(t)} x'(t), \quad t \geqq t_1.$$

Since $a_1(t) > 0$,

$$x'(t) > 0 \quad \text{for } t \geqq t_1. \tag{71}$$

Then instead of (69) we consider the equation

$$x^{(n)} + \sum_{k=1}^{n-2} p_k(t) x^{(n-k)} = -p_{n-1}(t)x'(t) - p_n(t)x(t).$$

By this equation,

$$x''(t) = u(t) + \int_{t_1}^{t} K(t,s)[-p_{n-1}(s)x'(s) - p_n(s)x(s)]\, ds,$$

where u is the solution of the equation

$$u^{(n-2)} + \sum_{k=1}^{n-2} p_k(t) u^{(n-2-k)} = 0 \tag{72}$$

satisfying

$$u^{(i)}(t_1) = x^{(i+2)}(t_1), \quad i = 0, 1, \dots, n-3,$$

and K is the Cauchy's function for (72), that is, $K(.,s)$ is the solution v of (72) satisfying

$$v^{(i)}(s) = 0, \quad i = 0, \dots, n-4, \; v^{(n-3)}(s) = 1.$$

By Theorem 2, statement (a), for the solution $v = K(.,s)$ of (72) we get $v(t) > 0$, $v'(t) > 0, \dots, v^{(n-2)}(t) > 0$, $v^{(n-3)}(t) > 1$ for $t > s$. Therefore $v(t) = K(t,s) > \frac{(t-s)^{n-3}}{(n-3)!}$ for $t > s$, and hence in regard of (71),

$$\begin{aligned} x''(t) \geqq & u(t) + \int_{t_1}^{t} K(t,s)(-p_n(s))x(s)\,ds \\ \geqq & u(t) + x(t_1) \int_{t_1}^{t} \frac{(t-s)^{n-3}}{(n-3)!} |p_n(s)|\,ds = f(t), t \geqq t_1. \end{aligned}$$

Again in view of (68), by Theorem 17.1 in [15], there exists a $j \in \{0, 1, \dots, n-3\}$ and a constant $\tilde{c}_j \neq 0$ such that $u(t) \sim \tilde{c}_j t^j$.

Thus $\lim_{t\to\infty} \frac{u(t)}{t^{n-3}} \in R$. Furthermore, on the basic of (67), we have

$$\lim_{t\to\infty} \frac{1}{t^{n-3}} x(t_1) \int_{t_1}^{t} \frac{(t-s)^{n-3}}{(n-3)!} |p_n(s)|\,ds = \infty.$$

From these facts we obtain that $f(t)$ is ultimately positive and thus, there exists a $t_2 > t_1$ such that

$$x''(t) > 0 \quad \text{for } t \geqq t_2.$$

Proceeding in this way, we get that there exists a $t_3 > t_2$ such that

$$x^{(j)}(t) > 0 \quad \text{for } t \geqq t_3, \; j = 0, 1, \dots, n-1.$$

Thus we have proved that equation (1) has the strict property B. Lemma 14 then completes the proof of the theorem.

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