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Nonlinear Vibration Problems

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Abstract

In this work we prove the existence, uniqueness and regularity of weak local solutions to the mixed problem associated with the nonlinear equation

$$u_{tt} = M(\|\nabla u\|^2 + \|u\|^2)(\Delta u - u) - \delta|u|^\alpha u + f.$$

We consider the three basic types of boundary conditions. Thus, our results extend [3] to the Neumann and Newton case with the operator - Δ + I.

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1. INTRODUCTION

This article is concerned with the study of the following nonlinear vibration equation with the generalized boundary condition

$$u_{tt} + M\left(\|u\|^2_{H^1(\Omega)}\right)(-\Delta u + u) + \delta|u|^\alpha u = f, \qquad x \in \Omega,\ 0 < t < T, \tag{1.1}$$

$$\gamma_1 \frac{\partial u}{\partial \nu} + \gamma_2 u = 0, \qquad x \in \partial\Omega,\ 0 \le t \le T, \tag{1.2}$$

$$u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \qquad x \in \Omega, \tag{1.3}$$

where $\|u\|^2_{H^1(\Omega)} = \int_\Omega \left(|\nabla u|^2 + |u|^2\right) dx$, $\Omega \subset R^n$ is of class $C^{0,1}$, ν is the outer normal vector to $\partial\Omega$ and $\delta > 0$, $\alpha \ge 0$, $T > 0$ are given constants, *M(r)* is a positive C^1-function on $[0,\infty)$ and $\gamma_1, \gamma_2 \in L^\infty(\partial\Omega)$ are non-negative real functions with $\gamma_1^2 + \gamma_2^2 \ne 0$ on $\partial\Omega$. In what follows, we will deal with the Dirichlet, Neumann and Newton boundary condition to equation (1.1) and take $\delta = 1$ without loss of generality.

Equation (1.1) arises in the study of the nonlinear vibration of particles with mass and with self-interaction assured by the restoring term $|u|^\alpha u$. Our model was inspired in the work of M. Hosoya and Y. Yamada [3], where the operator $-\Delta + I$ is replaced with

Laplacian and the Dirichlet boundary condition is considered. In such a case the equation appears in mathematical physics as a special case of the Kirchhoff-Carrier equation, when modeling planar vibrations. For a background and physical properties of the above mentioned model, we refer the reader to the work of the Japanese authors and their references. Other useful results for the nonlinear operators the reader can find in [5].

A weak solution of the problem (1.1)-(1.3) is a function $u(t) \in Y_k$ (k = 1, 2) satisfying (1.3) and

$$(u''(t), w) + M\left(\|u(t)\|^2_{H^1(\Omega)}\right)\left((\nabla u(t), \nabla w) + (u(t), w) + \beta(\gamma u(t), w)_{L^2(\partial\Omega)}\right) + \\ + \left(|u(t)|^\alpha u(t), w\right) = (f(t), w) \tag{1.4}$$

for all $w \in Y_k$, where $Y_1 = H^1(\Omega), Y_2 = H^1_0(\Omega)$ and $\beta \in \{0, 1\}$. Y_1 holds for the Neumann ($\beta = 0$) and Newton ($\beta = 1$) boundary condition and Y_2 for the Dirichlet ($\beta = 0$) one. Here $\gamma = \gamma_2/\gamma_1 \in L^\infty(\partial\Omega)$ for $\gamma_1 \neq 0$ on $\partial\Omega$.

The work of Hosoya and Yamada treats the first boundary condition, and in order to obtain the existence of weak solution, the authors employ Galerkin´s method and make use of a “special basis”.

In this work we also use Galerkin´s approximation and taking into account all the three types of the boundary conditions, we have to use basis $\{w_j\}_{j \in N}$ satisfying

$$-\Delta w_j = (\lambda_j - 1)w_j.$$

Note that the Laplacian in the Neumann case is non-coercive. Hence, the Lax-Milgram theorem assuring the existence of eigenfunctions cannot be applied but the operator $-\Delta + I$ avoids this difficulty.

Our paper is organized as follows. In section 2 we give the notations and state the main result. In Section 3 we introduce some lemmas and hint at the main steps of their proofs. In Section 4 we study the existence and uniqueness of weak solutions by making use of the lemmas given in Section 3.

2. NOTATIONS AND MAIN RESULT

In this section we present some notations that will be used throughout this paper and state the main result.

For any Banach space X, its norm is denoted by $\|.\|_x$. In particular, for $X = L^2(\Omega)$ its norm and inner product are simply denoted by $\|.\|$ and $(.,.)$, respectively. A bilinear form $a(.,.)$ on $H^1(\Omega)$ is defined by

$$a(u,v)=(\nabla u,\nabla v)+(u,v)\equiv\sum_{i=1}^{n}\int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx+\int_\Omega uvdx$$

and $a(u, u)$ is denoted by $a(u)$. We often write

$$a(u,v)+\beta(\gamma u,v)_{L^2(\partial\Omega)},$$

where $\beta\in\{0,\ 1\}$ and $\gamma\equiv\frac{\gamma_2}{\gamma_1}\in L^\infty(\partial\Omega)$ for $\gamma_1 \neq 0$ on $\partial\Omega$. Hereafter,

$\beta = 0$ will always hold for the Dirichlet and Neumann case and $\beta = 1$ for the Newton case.

For simplicity we will often denote some of the Hilbert spaces by Y_k, Z_k (k = 1, 2) as follows:

$$Y_1 = H^1(\Omega), \qquad Y_2 = H_0^1(\Omega)$$

and

$$Z_1 = H^2(\Omega), \qquad Z_2 = H_0^1 \cap H^2(\Omega)$$

Just as before, Y_k, Z_k with $k = 1$ will always hold for the Neumann and Newton case and $k = 2$ for the Dirichlet case.

Finally, as usual

$$u' = \frac{\partial u}{\partial t} \qquad \text{and} \qquad u'' = \frac{\partial^2 u}{\partial t^2}.$$

Now we are in condition to state our main result. Let M, α and f satisfy the following assumptions:

$$M\in C^1[0,\infty] \quad \text{and} \quad M(r)\geq m_0>0 \quad \text{for} \quad r\geq 0 \tag{A.1}$$

$$0\leq\alpha\leq\frac{2}{n-4} \quad \text{if } n\geq 5 \quad \text{and} \quad 0\leq\alpha<\infty \quad \text{if } n = 1, 2, 3, 4 \tag{A.2}$$

$$f\in L^2([0, T];Y_k) \quad (k = 1, 2) \tag{A.3}$$

Conditions for the existence , regularity and uniqueness are given in the following theorem:

THEOREM 2.1. Assume (A.1)-(A.3). Then, for any initial data

$u^0 \in Zk$ and $u^1 \in Y_k$ (k = 1, 2),

there exists a positive constant T_0 < T such that (1.1)-(1.3) possesses a unique weak solution u on $[0, T_0]$ satisfying

$$u \in C\left([0,T_0];Y_k \right) \cap C_w \left([0,T_0];H^2(\Omega) \right), \tag{B.1}$$

$$u_t \in C\left([0,T_0];L^2(\Omega) \right) \cap C_w\left([0,T_0];Y_k \right), \tag{B.2}$$

$$u_{tt} \in L^2([0, T_0]; L^2(\Omega)), \tag{B.3}$$

where the subscript "w" means the weak continuity with respect to t.

Moreover, if $f \in C_w([0, T_0]; L^2(\Omega))$, then u satisfies

$$u_{tt} \in C_w([0, T_0]; L^2(\Omega)). \tag{B.4}$$

3. LEMMAS

LEMMA 3.1. Let $\{\lambda_j\}$ be a sequence of eigenvalues of the weak problem to

$$-\Delta w + w = \lambda w \quad \text{in} \quad \Omega \in C^{0,1} \quad \text{and} \quad \gamma_1 \frac{\partial w}{\partial \nu} + \gamma_2 w = 0 \quad \text{on } \partial\Omega,$$

and $w_j \in Z_k$ (k = 1, 2) are the corresponding eigenfunctions to λ_j. Then $\{w_j\}$ is a complete orthonormal system in $L^2(\Omega)$.

PROOF. It is sufficient to show that the operator $(-\Delta + I)^{-1}:Y_k \to Y_k$ is non-negative, compact and self-adjoint and use the Hilbert-Schmidt theorem. □ .

LEMMA 3.2. Denote the approximate solutions u_m for a weak solution of (1.1)-(1.3) in the form

$$u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\, w_j, \qquad m = 1, 2, \ldots$$

where $\{w_j\}$ is a complete orthonormal system in $L^2(\Omega)$ from Lemma 3.1. and g_{jm} are determined by the following system of ordinary differential equations

$$(u''_m(t), w_j) + M\,(a\,(u_m(t)))\,(a(u_m(t), w_j) + \beta(u_m(t), w_j)_{L^2(\partial\Omega)}) + \\ + (|u_m(t)|^{\alpha}u_m\,(t), w_j) = (f(t), w_j) \qquad (3.1)$$

with $\beta \in \{0, 1\}$ and for $j = 1\ldots$m. The initial conditions are given by

$$u_m(0) = u_m^0 \equiv \sum_{j=1}^{m} b_{jm} w_j, \quad \to \mathrm{u}^0 \text{ in } Z_k,$$

$$u'_m(0) = u_m^1 \equiv \sum_{j=1}^{m} c_{jm} w_j, \quad \to \mathrm{u}^0 \text{ in } Y_k$$

(3.2)

for $m\to\infty$, where b_{jm} and c_{jm} are constants, which can be taken as Fourier coefficients of u^0 or u^1, respectively in $\mathrm{L}^2(\Omega)$ for the complete orthonormal system $\{\mathrm{w_j}\}$ from Lemma 3.1. Then the previous system determines a unique solution u_m on some interval $[0, t_m]$.

PROOF. Denoting $g_j(t) =: g_{jm}(t)$, $b_j =: b_{jm}$, $c_j =: c_{jm}$ and $y_j(t) =: g'_j(t)$, (3.1)-(3.2) can be transformed to a system of ordinary differential equations of the first order with $2m$ variables

$$y'_j(t) = -\lambda_j \overline{M}(g_1 ... g_m)(t) g_j(t) - F(g_1 ... g_m)(t) + (f(t), w_j)$$

$$g'_j(t) = y_j(t)$$

with initial data

$$y_j(0) = c_j, \qquad g_j(0) = b_j$$

for $j = 1\ldots m$, where

$$\overline{M}(g_1 ... g_m)(t) = M\left(\left\|\sum_{i=1}^{m} g_i(t) w_i\right\|_{H^1(\Omega)}\right)$$

and

$$F(g_1 ... g_m)(t) = \left(\left|\sum_{i=1}^{m} g_i(t) w_i\right|^{\alpha} \sum_{i=1}^{m} g_i(t) w_i, w_j\right).$$

In what follows, the Carathéodory theorem (see [1]) of the existence and uniqueness of local solutions should be applied. □

LEMMA 3.3. Consider a linear operator A: $D(A) \to L^2[0, T]$ defined as $Au = u''$, where $D(A) = \{u \in C^1[0, T] : u, u'$ are absolutely continuous and $u'' \in L^2[0, T]\}$ is the domain of A. Then the operator A is closed in $L^2[0, T]$.

PROOF. It is sufficient to show that any $\{u_m\} \subset D(A)$, $u_m \to u$ in $C^1[0, T]$ and

$Au_m \to u_1$ in $L^2[0, T]$ implies $u \in D(A)$ and $Au = u_1$. □

LEMMA 3.4. Let $R^n \supset \Omega \in C^{0,1}$ be a domain with smooth boundary

and let $u \in H^2(\Omega)$. Then there exist positive constants c_1, c_2, c_3 independent of u such that

$$\|u\|^2_{H^2(\Omega)} \le c_1\|\Delta u\|^2 + c_2\|u\|^2 + c_3\|\varphi\|^2_{H^2(\Omega)},$$

where $\varphi(x)$ is an arbitrary function satisfying $u(x) - \varphi(x) \in H^2(\Omega) \cap H^1_0(\Omega)$.
(For the proof of the previous lemma see [4], pg. 217.)

4. PROOF OF THE EXISTENCE AND UNIQUENESS OF A WEAK LOCAL SOLUTIONS

In order to obtain a unique weak solution we employ Galerkin´s procedure.

Let be $\{\lambda_j\}_{j=1}^{\infty}$ a sequence of eigenvalues as defined in Lemma 3.1.

Denote by $w_j \in Z_k$ (k = 1, 2) the corresponding eigenfunctions to λ_j. Clearly, we may take $\{w_j\}$ as a complete orthonormal system in $L^2(\Omega)$. First we construct approximate solutions u_m in the form

where g_{jm} are determined by the $u_m(t) = \sum_{j=1}^{m} g_{jm}(t)\, w_j$, Cauchy problem

(3.1)-(3.2). The approximate system (3.1)-(3.2) is a system of m ordinary differential equations and on account of Lemma 3.2, it has a unique local solution in $[0, t_m]$. The restriction of the solution to some interval $[0, T_0]$ is a consequence of the second estimate we are going to obtain later.

A PRIORI ESTIMATES

THE FIRST ESTIMATE

The equation (3.1) is equivalent to

$$\left(u_m''(t), w\right) + M\left(a\left(u_m(t)\right)\right)\left(a\left(u_m(t), w\right) + \beta\left(u_m(t), w\right)_{L^2(\partial\Omega)}\right) + \\ + \left(\left|u_m(t)\right|^{\alpha} u_m(t), w\right) = \left(f(t), w\right) \tag{4.1}$$

for all $w \in V_m$, where V_m is an m-dimensional vector space spanned by $\{w_1, w_2, \ldots, w_m\}$. Putting $w = u'_m(t) \in V_m$ in (4.1) we get with $p = \alpha + 2$

$$\frac{1}{2}\frac{d}{dt}\left\{\| u_m'(t)\|^2 + \int_0^{(a(u_m(t)))} M(s)ds\right\} + \frac{\beta}{2} M\left(a\left(u_m(t)\right)\right)\frac{d}{dt}\left\|\gamma^{\frac{1}{2}} u_m(t)\right\|^2_{L^2(\partial\Omega)} +$$

$$+\frac{1}{p}\frac{d}{dt}\| u_m(t)\|^p_{L^p(\Omega)} = \left(f(t), u_m'(t)\right). \tag{4.2}$$

which can be verified directly by the derivation of (4.2).
Denote a functional by

$$E\left(u_m(t)\right) = \frac{1}{2}\left\{\| u_m'(t)\|^2 + \int_0^{(a(u_m(t)))} M(s)ds + \frac{1}{p}\| u_m(t)\|^p_{L^p(\Omega)}\right\}$$

Hence,

$$\frac{d}{dt}E\left(u_m(t)\right) + \frac{\beta}{2}M\left(a\left(um(t)\right)\right)\frac{d}{dt}\left\|\gamma^{\frac{1}{2}} u_m(t)\right\|^2_{L^2(\partial\Omega)} = \left(f(t), u_m'(t)\right).$$

Integrating (4.2) over (0, t) and employing Young´s inequality yields

$$\frac{1}{2}\left\{\| u_m'(t)\|^2 + m_0 a(u_m(t)) + \beta m_0 \left\| \gamma^{\frac{1}{2}} u_m(t)\right\|^2_{L^2(\partial\Omega)}\right\} + \frac{1}{p}\| u_m(t)\|^p_{L^p(\Omega)} \leq$$

$$\leq E(u_m(0)) +$$

$$+ \int_0^t \frac{1}{2}\left\{\| f(s)\|^2 + \| u_m'(s)\|^2 + m_0 a(u_m(s)) + \beta m_0 \left\| \gamma^{\frac{1}{2}} u_m(t)\right\|^2_{L^2(\partial\Omega)}\right\} +$$

$$+ \frac{1}{p}\| u_m(s)\|^p_{L^p(\Omega)} ds. \tag{4.3}$$

Applying Gronwall´s lemma to (4.3) we obtain

$$\left\{\| u_m'(t)\|^2 + m_0 a(u_m(t)) + \beta m_0 \left\| \gamma^{\frac{1}{2}} u_m(t)\right\|^2_{L^2(\partial\Omega)}\right\} + \frac{2}{p}\| u_m(t)\|^p_{L^p(\Omega)} \leq$$

$$\leq \left\{2E(u_m^0) + \int_0^T \| f(s)\|^2 ds\right\} e^T. \tag{4.4}$$

Using $H^2(\Omega) \hookrightarrow L^p(\Omega)$ and (3.2) one can easily see the $E(u_m(0))$ is bounded by a positive constant independent of m which enables us to take $t_m = T_0$ for all m and so

$$\| u_m'(t)\| \leq C_1 \qquad \text{for} \qquad 0 \leq t \leq T_0, \tag{4.5}$$

$$\| \nabla u_m(t)\| \leq C_2 \qquad \text{for} \qquad 0 \leq t \leq T_0, \tag{4.6}$$

$$\| u_m(t)\| \leq C\| u_m(t)\|_{L^p(\Omega)} \leq C_3 \qquad \text{for} \qquad 0 \leq t \leq T_0, \tag{4.7}$$

$$\| u_m(t)\|_{L^2(\partial\Omega)} \leq C_4 \qquad \text{for} \qquad 0 \leq t \leq T_0, \tag{4.8}$$

with some positive constants C_i, $i = 1, \ldots, 4$ independent of m. From now on we denote by C or C_i various positive constants which depend on the given data but not on m.

THE SECOND ESTIMATE

In the following steps we will gradually put $w = \Delta u'_m(t) - u'_m(t)$ in (4.1). Then we have

$$\left(u''_m(t), \Delta u'_m(t) - u'_m(t)\right) = -\frac{1}{2}\frac{d}{dt}\left\{a\left(u'_m(t)\right) + \beta \left\| \gamma^{\frac{1}{2}} u'_m(t) \right\|_{L^2 \partial\Omega}\right\}. \tag{4.9}$$

Define a scalar product in $H^1(\Omega)$ x $H^1(\Omega)$ in the following way:

$$a_1(x, y) =: (\nabla x, \nabla y) + (x, y) + \beta(\gamma x, y)_{L^2(\Omega)}$$

with $\beta \in \{0,\ 1\}$. Putting $x = u_m(t), y = \Delta u'_m(t) - u'_m(t)$ into $a_1(x, y)$ one obtains

$$a_1\left(u_m(t), \Delta u'_m(t) - u'_m(t)\right) =$$

$$= -\frac{1}{2}\frac{d}{dt}\left\{\| \Delta u_m(t) \|^2 + 2 \| \nabla u_m(t) \|^2 + \| u_m(t) \|^2 + 2\beta \left\| \gamma^{\frac{1}{2}} u_m(t) \right\|^2_{L^2(\partial\Omega)}\right\}. \tag{4.10}$$

Define a functional

$$F(u) = \|\Delta u\|^2 + 2\|\nabla u\|^2 + \|u\|^2 + 2\beta \left\| \gamma^{\frac{1}{2}} u \right\|^2_{L^2(\partial\Omega)}. \qquad \text{for } u \in Z_k.$$

Also,

$$\left(f(t), \Delta u'_m(t) - u'_m(t) \right) = -\left\{ a\left(f(t), u'_m(t)\right) + \beta \left(f(t), \gamma u'_m(t) \right)_{L^2(\partial\Omega)} \right\}. \tag{4.11}$$

Finally,

$$\frac{d}{dt}\left\{ M\left(a\left(u_m(t)\right)\right) F\left(u_m(t)\right) \right\} =$$

$$= 2M'\left(a\left(u_m(t)\right)\right) a\left(u_m(t), u'_m(t)\right) F\left(u_m(t)\right) + M\left(a\left(u_m(t)\right) \frac{d}{dt} F\left(u_m(t)\right)\right). \tag{4.12}$$

Making use of (4.7)-(4.12) one gets

$$\frac{1}{2}\frac{d}{dt}\left\{a(u'_m(t))+\beta\left\|\gamma^{\frac{1}{2}}u'_m(t)\right\|_{L^2(\partial\Omega)}+M(a(u_m(t)))F(u_m(t))\right\}=$$

$$=\left(|u_m(t)|^{\alpha}u_m(t),\Delta u'_m(t)-u'_m(t)\right)+a(f(t),u'_m(t))+$$

$$+\beta(f(t),\gamma u'_m(t))_{L^2(\partial\Omega)}+M'(a(u_m(t)))a(u_m(t),u'_m(t))F(u_m(t)). \qquad (4.13)$$

The first term in the right-hand side of (4.13) is estimated as follows:

$$\left|\left(|u_m(t)|^{\alpha}u_m(t),\Delta u'_m(t)-u'_m(t)\right)\right|\le\left|\left(|u_m(t)|^{\alpha}u_m(t),\Delta u'_m(t)\right)\right|+$$

$$+\left|\left(|u_m(t)|^{\alpha}u_m(t),u'_m(t)\right)\right|. \qquad (4.14)$$

The first term in the right-hand side of (4.14) is further estimated as

$$\left|\left(|u_m(t)|^{\alpha}u_m(t),\Delta u'_m(t)\right)\right|\le\left|\left(\nabla\left(|u_m(t)|^{\alpha}u_m(t)\right)\right),\nabla u'_m(t)\right|+$$

$$+\beta\int_{\partial\Omega}\gamma|u_m(t)|^{\alpha}u_m(t)u'_m(t)ds\le(\alpha+1)\left|\left(|u_m(t)|^{\alpha}\nabla u_m(t),\nabla u'_m(t)\right)\right|+$$

$$+\beta C\int_{\partial\Omega}|um(t)|^{\alpha+1}u'_m(t)ds\le$$

$$\le(\alpha+1)\|u_m(t)\|^{\alpha}_{L^{q\alpha}(\Omega)}\|\nabla u_m(t)\|_{L^r(\Omega)}\|\nabla u'_m(t)\|+$$

$$+\beta C\|u_m(t)\|^{\alpha+1}_{L^{2(\alpha+1)}(\partial\Omega)}\|u'_m(t)\|_{L^2(\partial\Omega)}$$

We take q and r such that

$$\frac{1}{q}+\frac{1}{r}=\frac{1}{2},$$

$$\frac{1}{q\alpha}\geq\frac{1}{2}-\frac{2}{n} \qquad \text{and} \qquad \frac{1}{r}\geq\frac{1}{2}-\frac{1}{n}.$$

It is easy to check that $H^2(\Omega)\hookrightarrow L^{q\alpha}(\Omega)$ and $H^1(\Omega)\hookrightarrow L^r(\Omega)$ for the parameters q and r defined above. Also, $H^2(\Omega)\hookrightarrow L^{2(\alpha+1)}(\partial\Omega)$ and $H^1(\Omega)\hookrightarrow L^2(\partial\Omega)$. Hence,

$$\left|\left(|u_m(t)|^{\alpha}u_m(t),\Delta u_m'(t)\right)\right|\leq(\alpha+1)\|u_m(t)\|_{H^2(\Omega)}^{\alpha+1}+$$

$$+\beta C\|u_m(t)\|_{H^2(\Omega)}^{\alpha+1}\|u_m'(t)\|_{H^1(\Omega)}.$$

From Lemma 3.4 and (4.7) follows that

$$\|u_m(t)\|_{H^2(\Omega)}^2\leq C\left(\|\Delta u_m(t)\|^2+1\right). \tag{4.15}$$

Making use of (4.15) and (4.5) one obtains that

$$\left|\left(|u_m(t)|^{\alpha}u_m(t),\Delta u_m'(t)\right)\right|\leq$$

$$\leq C\left\{\left(\|\Delta u_m(t)\|^{\alpha+1}+1\right)\|\nabla u_m'(t)\|+\beta\left(\|\Delta u_m(t)\|^{\alpha+1}+1\right)\left(\|\nabla u_m'(t)\|+1\right)\right\}\leq$$

$$\leq C\left(\|\Delta u_m(t)\|^{\alpha+1}+1\right)\left(\|\nabla u_m'(t)\|+1\right). \tag{4.16}$$

The second term in the right-hand side of (4.14), using (4.5), is estimated as

$$\left|\left(|u_m(t)|^{\alpha}u_m(t),u_m'(t)\right)\right|\leq C\|u_m(t)\|_{L^{2(\alpha+1)}(\Omega)}^{\alpha+1}.$$

From (4.15) and $H^2(\Omega)\hookrightarrow L^{2(\alpha+1)}(\Omega)$ it follows that

$$\left|\left(|u_m(t)|^{\alpha}u_m(t),u_m'(t)\right)\right|\leq C\left(\|\Delta u_m(t)\|^{\alpha+1}+1\right). \tag{4.17}$$

Combining (4.16) and (4.17) we obtain

$$\left| \left(| u_m(t) |^{\alpha} u_m(t), \Delta u_m'(t) - u_m'(t) \right) \right| \leq$$

$$\leq C\left(\| \Delta u_m(t) \|^{\alpha+1} + 1 \right)\left(\| \nabla u_m'(t) \| + 1 \right). \tag{4.18}$$

By virtue of (4.5) the sum of the second and third term in the right-hand side of (4.14) is bounded by

$$| a(f(t), u_m'(t)) | + \beta | (f(t), \gamma u_m'(t)) | \leq$$

$$\leq C\left\{ \| \nabla f(t) \|^2 + \| \nabla u_m'(t) \|^2 + \| f(t) \|^2 + 1 \right\}. \tag{4.19}$$

Employing (4.5)-(4.8) we obtain the following estimate for the fourth term in the right-hand side for (4.14);

$$\left| M'(a(u_m(t)))\, a(u_m(t), u_m'(t))\, F(u_m(t)) \right| \leq$$

$$\leq M_1 C\left(\| \nabla u_m'(t) \| + 1 \right)\left(\| \Delta u_m(t) \|^2 + 1 \right), \tag{4.20}$$

where $M_1 = \max\left\{ M'(r) : 0 \leq r \leq C_2^2 + C_3^2 \right\}$ (use (4.6) and (4.9)).

Define a functional $G(u)$ by

$$G(u(t)) = a(u'(t)) + \beta \left\| \gamma^{\frac{1}{2}} u'(t) \right\|_{L^2(\partial\Omega)} + M(a(u(t)))\, F(u(t))$$

for $u(t) \in Y_k \cap H_2(\Omega)$ and $u'(t) \in Y_k$, $t \in [0, T]$.

Then, again, by virtue of (4.6) and (4.7) we have

$$\| \nabla u_m'(t) \|^2 + \| u_m'(t) \|^2 + \beta \left\| \gamma^{\frac{1}{2}} u_m'(t) \right\|^2_{L^2(\partial\Omega)} + m_0 F(u_m(t)) \leq G(u_m(t)) \leq$$

$$\leq \| \nabla u_m'(t) \|^2 + \| u_m'(t) \|^2 + \beta \left\| \gamma^{\frac{1}{2}} u_m'(t) \right\|_{L^2(\partial\Omega)} + M_2 F(u_m(t)), \tag{4.21}$$

where $M_2 = \max\left\{ M(r) : 0 \le r \le C_2^2 + C_3^2 \right\}$. Making use of (4.18), (4.19) and (4.20) we rearrange (4.13);

$$\frac{d}{dt} G_k(u_m(t)) \le C \{ \left(\| \Delta u_m(t) \|^{\alpha+1} + 1 \right) \left(\| \nabla u_m'(t) \| + 1 \right) + \| \nabla f(t) \|^2 +$$
$$+ \| \nabla u_m'(t) \|^2 + \| f(t) \|^2 + 1 + \left(\| \nabla u_m'(t) \| + 1 \right) \left(\| \Delta u_m(t) \|^2 + 1 \right) \} \le$$

$$\le C \{ G^{\frac{\alpha+2}{2}}(u_m(t)) + G^{\frac{\alpha+1}{2}}(u_m(t)) + G^{\frac{1}{2}}(u_m(t)) + G(u_m(t)) +$$
$$+ G^{\frac{3}{2}}(u_m(t)) + \| \nabla f(t) \|^2 + \| f(t) \|^2 + 1 \},$$

where (4.21) is used in the last inequality.

Taking $\omega = \max\left\{ \frac{\alpha+2}{2}, \frac{3}{2} \right\}$ if $G(u_m(t)) \ge 1$ and $\omega = \frac{1}{2}$ if $G(u_m(t)) < 1$ we get

$$\frac{d}{dt} G(u_m(t)) \le C \left\{ G^{\omega}(u_m(t)) + 1 \right\}.$$

One can easily show that there exists a positive number T_0 such that

$$G(u_m(t)) \le C_5 \qquad \text{for} \quad 0 \le t \le T_0,$$

which yields by (4.21)

$$\| \nabla u_m'(t) \| \le C_6 \qquad \text{for} \quad 0 \le t \le T_0, \tag{4.22}$$

$$\| \Delta u_m(t) \| \le C_7 \qquad \text{for} \quad 0 \le t \le T_0, \tag{4.23}$$

$$\| u_m'(t) \|_{L^2(\partial\Omega)} \le C_8 \qquad \text{for} \quad 0 \le t \le T_0, \tag{4.24}$$

From (4.15) and (4.23) also follows

$$\| u_m(t) \|_{H^1(\Omega)} \le C \|u_m(t)\|_{H^2(\Omega)} \le C_9 \qquad \text{for} \qquad 0 \le t \le T_0. \qquad (4.25)$$

THE THIRD ESTIMATE

Put $w = u''_m(t)$ in (4.1). We estimate

$$\left| a(u_m(t), u''_m(t)) + (\gamma u_m(t), u''_m(t))_{L^2(\partial\Omega)} \right| \le$$

$$\le C \left(\|\Delta u_m(t)\| + \| u_m(t)\| \right) \| u''_m(t) \|.$$

Thus, it is easy to obtain form (4.1)

$$\| u''_m(t) \| \le M_2 C\left(\| \Delta u_m(t) \| + \| u_m(t) \| \right) + \| u_m(t) \|^{\alpha+1}_{L^{2(\alpha+1)}(\Omega)} + \| f(t) \|.$$

Therefore, using $H^2(\Omega) \hookrightarrow L^{2(\alpha+1)}(\Omega)$ and (4.7), (4.23), (4.25), it follows from the previous inequality that

$$\| u''_m(t) \| \le C_{10} \qquad \text{for} \quad 0 \le t \le T_0 \qquad (4.26)$$

PASSAGE TO THE LIMIT

1. From (4.25) we have that $\{u_m\}$ is uniformly bounded in $C([0, T_0], Z_k))$. Since the embedding from $H^2(\Omega)$ into $H^1(\Omega)$ and from $H^1_0(\Omega) \cap H^2(\Omega)$ into $H^1_0(\Omega)$ is compact, $\{u_m\}$ is uniformly bounded in $C([0, T_0]; Y_k)$.

Moreover, the sequence $\{u_m(t)\}$ is relatively compact in Y_k for every $t \in [0, T_0]$. From (4.5) and (4.22) we have the equicontinuity of $\{u_m(t)\}$ in Y_k, that is

$$\lim_{\delta \to 0+} \sup_{n \in N} \left\{ \| u_m(t_1) - u_m(t_2) \|_{Y_k} ; | t_1 - t_2 | < \delta \right\} = 0 .$$

Generalized Ascoli-Arzela theorem enables us to choose a subsequence, again denoted by $\{u_m\}$, such that

$$\lim_{m\to\infty} u_m = u \qquad \text{in } C([0, T_0];Y_k) \tag{4.27}$$

with some $u \in C([0, T_0]; Y_k)$.

2. We can also study some weak convergence properties of the second derivatives of u_m.

Consider the $L^2(\Omega)$- inner product of $D^\mu u_m \left(|\mu| = 2\right)$ with the fixed function $\varphi \in C_0^\infty(\Omega)$.

From (4.6), (4.8), (4.22) and (4.24) follows that $\left\{\left(D^\mu u_m, \varphi\right)\right\}$ is a family of uniformly bounded equicontinuous functions on $[0, T_0]$ for every μ ($\mu = 2$). Thus, we may choose a subsequence denoted by $\left\{\left(D^\mu u_m, \varphi\right)\right\}$, such that

$$\lim_{m\to\infty}\left(D^\mu u_m, \varphi\right) = \left(D^\mu u, \varphi\right) \qquad \text{in } C([0, T_0]; R) \tag{4.28}$$

by (4.27). Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, (4.28) remains valid for every $\varphi \in L^2(\Omega)$; that is

$$\lim_{m\to\infty} D^\mu u_m = D^\mu u \qquad \text{in } C_w([0, T_0]; L^2(\Omega)).$$

3. Also,

$$\lim_{m\to\infty}\left(u_m, \varphi\right) = \left(u, \varphi\right) \quad \text{and} \quad \lim_{m\to\infty}\left(\frac{\partial u_m}{\partial x_i}, \varphi\right) = \left(\frac{\partial u}{\partial x_i}, \varphi\right) \quad \text{in } C([0, T_0]; R),\ i = 1, \ldots, n$$

so that

$$\lim_{m\to\infty} u_m = u \qquad \text{in } C_w([0, T_0]; H^2(\Omega)). \tag{4.29}$$

4. Now we can study the convergency of $\{u'_m\}$. By (4.5) and (4.22) the sequence $\{u'_m\}$ is uniformly bounded in $C([0,T_0]; Y_k)$. The compact embedding from Y_k into $L^2(\Omega)$ and (4.26) assure that $\{u'_m\}$ is a family of uniformly bounded equicontinuos functions in $C([0,T_0]; L^2(\Omega))$ and that $\{u'_m(t)\}$ is relatively compact in $L^2(\Omega)$ for every $t \in [0,T_0]$. Hence,

$$\lim_{m\to\infty} u'_m = u' \quad \text{in} \quad C([0, T_0]; L^2(\Omega)). \tag{4.30}$$

5. Just as before, consider the $L^2(\Omega)$-inner product of $\nabla u'_m$ with any fixed function $\varphi \in L^2(\Omega)$. From (4.22) and (4.26) follows that $\{(\nabla u'_m, \varphi)\}$ is a family of uniformly bounded equicontinuous functions on $[0,T_0]$. Again,

$$\lim_{m\to\infty} (\nabla u'_m, \varphi) = (\nabla u', \varphi) \quad \text{in} \quad C([0, T_0]; R). \tag{4.31}$$

(4.30) and (4.31) assure that

$$\lim_{m\to\infty} u'_m = u' \quad \text{in} \quad C_w([0, T_0]; Y_k). \tag{4.32}$$

6. Furthermore, (4.26) implies that $\{u''_m\}$ is uniformly bounded in $L^2([0,T_0]; L^2(\Omega))$. The reflexivity enables us to choose a subsequence denoted by $\{u''_m\}$, which converges weakly to some $u_1 \in L^2([0,T_0]; L^2(\Omega))$. Lemma 3.3 implies that $u_1 = u''$. Thus, we get

$$\lim_{m\to\infty} u''_m = u'' \quad \text{weakly in} \quad L^2([0, T_0]; L^2(\Omega)). \tag{4.33}$$

7. Similarly, we can show that

$$\lim_{m\to\infty} u_m = u \quad \text{weakly in} \quad L^2([0, T_0]; L^2(\partial\Omega)). \tag{4.34}$$

The estimate (4.8) implies that $\{u_m\}$ is uniformly bounded in $L^2([0,T_0]; L^2(\partial\Omega))$ and on account of the Eberlein-Schmuljan theorem there exists a subsequence denoted by $\{u_m\}$ converging weakly to some $u_2 \in L^2([0, T_0]; L^2(\partial\Omega))$. The continuous embedding from $H^1(\Omega)$ into $L^2(\partial\Omega)$ assures that $u_m \to u$ in $L^2([0, T_0]; L^2(\partial\Omega))$ and $u_2 = u$ by (4.27). Hence, we get (4.34).

8. Finally, we will prove

$$\lim_{m\to\infty}|u_m|^{\alpha}u_m = |u|^{\alpha}u \qquad \text{in } C([0, T_0]; L^2(\Omega)). \qquad (4.35)$$

Rearranging the following inequality we obtain

$$\int_\Omega \left\{|u_m|^{\alpha}u_m - |u|^{\alpha}u\right\}^2 dx \le C\int_\Omega \left(|u|^{\alpha} + |u_m|^{\alpha}\right)^2 |u_m - u|^2 dx \le$$

$$\le C\left\{\|u\|^{\alpha}_{L^{\alpha q}(\Omega)} + \|u_m\|^{\alpha}_{L^{\alpha q}(\Omega)}\right\}^2 \| \Delta u_m - u \|^2_{L^r(\Omega)},$$

where $\frac{1}{q}+\frac{1}{r}=\frac{1}{2}$ and the parametres q, r are taken such that $H^2(\Omega) \hookrightarrow L^{\alpha q}(\Omega)$ and

$H^1(\Omega) \hookrightarrow L^r(\Omega)$, which yields to

$$\left\| \, |u(t)|^{\alpha}u(t) - |u_m(t)|^{\alpha}u_m(t) \right\| \le C \| u_m - u \|_{C([0,T_0],Y_k)} \qquad (4.36)$$

by (4.25). Therefore, (4.35) follows from (4.36) with use of (4.27).

Now we are ready to prove that u is a solution of (1.1)-(1.3). From (4.27), (4.29), (4.30) and (4.32)-(4.35) one can pass to the limit in (4.1) to get

$$(u''(t), w) + M\left(\| u(t) \|^2_{H^1(\Omega)}\right)(-\Delta u(t) + u(t), w) + \qquad (4.37)$$

$$+\left(|u|^{\alpha}u(t), w\right) = (f(t), w) \qquad \text{on } [0, T_0] \text{ for all } w \in L^2(\Omega).$$

Regularity properties are easy consequences of (4.27), (4.29), (4.30) and (4.32).

Moreover, since $|u|^{\alpha}u \in C\left([0,T_0]; L^2(\Omega)\right)$

by (4.35), (B.4) follows from (4.37).

UNIQUENESS

Let u and v be two solutions of (1.1)-(1.3) on $[0,T_0]$. Substracting the two distributive solutions (1.1) with u and v we get that $w = u - v$ satisfies

$$w_{tt} + M(a(u))(-\Delta w + w) = \{ M(a(u)) - M(a(v)) \}(\Delta v - v) + \tag{4.38}$$

$$+ \left(|v|^{\alpha} v - |u|^{\alpha} u \right) \quad \text{in} \quad \Omega \times [0, T_0],$$

with $\gamma_1 \dfrac{\partial w}{\partial \nu} + \gamma_2 w = 0$ on $\partial\Omega \times [0, T_0]$ and $w(x, 0) = 0$, $w_t(x, 0) = 0$ in Ω.

Taking the $L^2(\Omega)$ – inner product of (4.38) with w_t we get

$$\frac{d}{dt}\left\{ \|wt\|^2 + M(a(u))\left(a(w) + \beta \left\| \gamma^{\frac{1}{2}} w \right\|^2_{L^2(\partial\Omega)} \right) \right\} =$$

$$= 2M'(a(u))\, a(u, u_t)\left(a(w) + \beta \left\| \gamma^{\frac{1}{2}} w \right\|^2_{L^2(\partial\Omega)} \right) +$$

$$+ 2\{ M(a(u)) - M(a(v)) \}(\Delta v - v, w_t) + 2\left(|v|^{\alpha} v - |u|^{\alpha} u, w_t \right). \tag{4.39}$$

The first and second terms in the right-hand side of (4.39) are bounded by

$$C\| w(t) \|^2_{Y_k} \qquad \text{and} \qquad C\| w(t) \|_{Y_k} \| w_t(t) \|.$$

Making use of (3.36) (with u_m replaced by v) one can see that the last term in (4.39) is bounded by

$$C\| w(t) \|_{Y_k} \| w_t(t) \|.$$

Thus, integrating (4.39) over (0, t) yields

$$\| w_t(t) \|^2 + m_0 \| w(t) \|^2_{Y_k} \leq C \int_0^t \| w(s) \|^2_{H^1(\Omega)} + \| w_t(s) \|^2 ds.$$

We take

$$\| w_t(t) \|^2 + m_0 \| w(t) \|^2_{Y_k} \leq C \int_0^t \left\{ \| w_t(s) \|^2 + m_0 \| w(s) \|^2_{Y_k} \right\} ds, \quad \text{if} \quad m_0 \geq 1$$

and

$$m_0 \left\{ \| w_t(t) \|^2 + \| w(t) \|^2_{Y_k} \right\} \leq C \int_0^t \left\{ \| w_t(s) \|^2 + \| w(s) \|^2_{Y_k} \right\} ds, \quad \text{if} \quad m_0 < 1.$$

Gronwall´s lemma with regard to the boundary and initial conditions implies $w \equiv 0$. The proof of Theorem 2.1. is complete. □

REFERENCES

[1] Ďurikovič, V., Funkcionálna analýza Nelineárne metódy, Univerzita Komenského, Bratislava, 1989.

[2] Greguš, G., Švec, M. and Šeda, V., Obyčajné diferenciálne rovnice, Alfa, Bratislava, SNTL, Praha, 1985.

[3] Hosoya, M. and Yamada, Y., On some nonlinear wawe equations I: local existence and regularity of solutions, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. 38, 1991.

[4] Ladyženskaja, O. A., Uraľceva, N. N., Linejnyje i kvazilinejnyje uravnenija eliptičeskovo tipa, Nauka, Moskava, 1973.

[5] Martin, R. H., Nonlinear Operators and Differential Equations in Banach Spaces, John Villey and Sons, New York, London, Sydney, Toronto, 1976.

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