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Existence Theorem and Structure of Solution Sets to Ordinary Differential Systems

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Abstract

We consider a general boundary value problem for nonlinear ordinary differential systems of the first order. By the transformation of this problem to the operator equation we observe the topological structure of solution and bifurcation sets and prove the existence theorem. There is used the theory of Fredholm operators.

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1. INTRODUCTION

V. Šeda in [3] studied the properties of the mapping $F : X \to Y$ which is the sum of a linear bounded Fredholm mapping of index zero and of a compact mapping. Here, X and Y are the Banach spaces over the same field $K = \mathbb{R}$ or $K = \mathbb{C}$. He studied especially generic properties, surjectivity and the bifurcation problem of such operators.

In this paper we shall study a general two points problem for the first order ordinary differential system.

First, we are interesting of the existence of given boundary value problem (BVP for short) by using the Schauder Fixed-Point theorem ([5]). Then, the generic properties of the solutions set shall study by using Nikolskiĭ theorem ([4]) on the structure of all linear bounded Fredholm mappings.

2. PRELIMINARIES

Let X and Y be two Banach spaces either both real or complex. If f is a mapping from X into Y (in general nonlinear), we denote the domain, the range and the kernel of f by $D(f)$, $R(f)$ and $N(f)$, respectivaly. In our considerations we will use the follow definitions:

Definition 2.1. 1. The mapping $f : X \to Y$ is *proper* (resp. *σ-proper*) if for each compact $K \subset Y$, $f^{-1}(K)$ is a compact (resp. is a countable union of compact sets) in X.

2. We call $f : X \to Y$ a coercive mapping in X(for short a coercive mapping) if for each bounded set $S \subset Y$, $f^{-1}(S)$ is bounded in X.

Remark 2.1. Clearly $f : X \to Y$ is coercive if and only if

$$\lim_{\|x\|_X \to \infty} \|f(x)\|_Y = \infty,$$

where $\|.\|_X$, resp. $\|.\|_Y$ is the norm in X, resp. in Y.

Definition 2.2. 1. If $M_1,\ M_2$ are two metric spaces and a mapping $f : M_1 \to M_2$, then f is said to be *locally injective* at a point $x_0 \in M_1$ if there is neighborhood $U(x_0)$ of x_0 such that f is injective in $U(x_0)$.

2. A mapping f is said to be *locally injective at the set* M_1 if it is locally injective at each point $x_0 \in M_1$.

Definition 2.3. If $M_1,\ M_2$ are two metric spaces and $f : M_1 \to M_2$ is a continuous mapping, then f is said to be *locally invertible* at a point $x_0 \in M_1$ if there is a neighborhood $U(x_0)$ of x_0 and a neighborhood $U_1(f(x_0))$ of $f(x_0)$ such that f is a homeomorphism of $U(x_0)$ onto $U_1(f(x_0))$.

2. A mapping f is said to be *locally invertible at the set* M_1 if it is locally invertible at each point $x \in M_1$.

Remark 2.2. 1. It is clear that if f is locally invertible at x_0, then f is locally injective at x_0.

2. We denote by Σ the set of all points $x \in X$ at which a continuous mapping $f : X \to Y$ $(f \in C(X, Y))$ is not locally invertinle. Since $X \setminus \Sigma$ is open then Σ is a closed subset of X.

Definition 2.4. Let $M \subset X$ be an open set and $A : X \to Y$, $u \in M$, $v \in X$.

1. The limit

$$\lim_{t \to 0} \frac{A(u + tv) - Au}{t} \in Y$$

is called a *G - differential* (Gateauxov differential) of the operator A at the point u and the direction v. We denote it as $dA(u; v)$.

2. If there exists G - differential $dA(u; v)$ for every $v \in X$,then we say that A is *G - differentiable* at the point u.

3. If the mapping $v \mapsto dA(u; v)$, $v \in X$ is linear and continuous, then we call it a G - derivation of A at the point u.

Definition 2.5. 1. The mapping $f : D(f) \subset X \to Y$ is called *Fréchet differentiable at the point* $x_0 \in D(f)$, iff there is a linear and continuous Fréchet derivative $f'(x_0) : X \to Y$ and the equality:

$$\lim_{\|h\|_X \to 0} \frac{\|f(x_0 + h) - f(x_0) - f'(x_0)(h)\|_Y}{\|h\|_X} = 0 \in R$$

holds.

2. The mapping f is called *Fréchet differentiable at an open set* $M \subset D(f)$, iff it is Fréchet differentiable at any point $x \in M$.

Remark 2.3. We say that mapping f is continuously Fréchet differentiable at M, i. e. $f \in C^1(M, Y)$ iff the mapping $f' : M \subset X \to L(X, Y)$ is continuous at M. (Here $L(X, Y)$ means the set of all linear and continuous mappings $X \to Y$.)

This is mutualy equivalent with the statement:

$\forall x \in M \quad \forall \varepsilon > 0 \quad \exists \delta(\varepsilon, x) > 0 \quad \forall y \in M$ such that $\|x - y\|_X < \delta$ implies

$$\|f'(x) - f'(y)\|_{L(X,Y)} < \varepsilon$$

$$(\|z\|_{L(X,Y)} = \sup_{\|x\|_X \le 1} \|z(x)\|_Y).$$

DEFINITION 2.6. 1. A linear operator $C : X \to Y$ is called a *Fredholm operator* if C is continuous mapping in X and $\dim N(C) < +\infty$, $\operatorname{codim} R(C) = \dim Y|_{R(C)} < +\infty$, where $N(C) = \{u \in X : Cu = 0\}$ and $R(C) = C(X)$. ($R(C)$ is closed set in Y.)

The index $\operatorname{ind} A$ of operator A is defined by

$$\operatorname{ind} C := \dim N(C) - \operatorname{codim} R(C).$$

2. A nonlinear operator $A : D(A) \subseteq X \to Y$ defined on the open set $D(A)$ is called *a Fredholm operator* iff $A \in C^1(D(A), Y)$ and Fréchet derivative (a linear operator) $A'(u) : X \to Y$ is Fredholm operator for every $u \in D(A)$.

The index of $A'(u)$ is called an index of A at the point u, i. e. for the index of A at u we have

$$\operatorname{ind} A := \operatorname{ind} A'(u).$$

DEFINITION 2.7. 1. Let $f : M \subset X \to Y$, where $M \neq \emptyset$ is a open set, be a differentiable mapping at M. Then $x_0 \in M$ is called *a regular point* of f if a linear mapping $f'(x_0)$ is a linear homeomorphism of X onto Y.

2. If $x_0 \in M$ is not a regular point of a mapping f, then it is called *a singular point* of f.

3. A point $x_0 \in M$ is called *a critical point* of mapping f, if $f'(x_0)(h) = 0$ has a nontrivial solution $h \in X$ $(h \neq 0)$.

4. *A singular value* of f is the image by f of a singular point of f. If S is the set of all singular points of f, then $f(S)$ is the set of all singular values of f.

5. The set $R_f = Y \setminus f(S)$ is called *the set of all regular values of* f.

REMARK 2.4. 1. If $f : M \subset X \to Y$ is a Fredholm mapping of index zero, where $M \neq \emptyset$ is a open set, then x_0 is a singular point of f if and only if x_0 is a critical point of f.

2. For each $q \in R_f$ either $f^{-1}(\{q\})$ is empty or it consists solely of regular points of f.

DEFINITION 2.8. Let a mapping $f : M \subset X \to Y$, where $M \neq \emptyset$ is a open set, $x_0 \in M$. Then we shall say that f is *a local C^1 - diffeomorphism* if and only if there exists a neighborhood $U(x_0)$ of x_0 and a neighborhood $U_1(f(x_0))$ of $f(x_0)$ such that f is a bijective mapping of $U(x_0)$ onto $U_1(f(x_0))$ and $f \in C^1(U(x_0), U_1(f(x_0)))$, $f^{-1} \in C^1(U_1(f(x_0)), U(x_0))$.

DEFINITION 2.9. Let $(X, \|.\|_X)$ be a real Banach space and let $F = I - f : X \to X$ be a field. We shall say that F is *strictly solvable* if it is a condensing field (see [5, p. 496]) and there is a sequence $\{r_k\}_{k \in N}$ such that

$$\lim_{k \to \infty} r_k = \infty$$

and

$$\deg(F, U(0, r_k), 0) \neq 0 \quad \forall k \in N,$$

where $U(0, r) = \{x \in X : ||x||_X < r\}$.

Now, we introduce quantitative and qualitative results of operators by different authors. We shall use it studying generic properties and existence questions of boundary value problems for ordinary differential systems.

PROPOSITION 2.1. (Schauder Fixed-Point theorem, [5, p. 56].) *Let M be a nonempty, closed, bounded, convex subset of Banach space X and suppose $T : M \to M$ be a compact operator. Then T has a fixed point from M.*

PROPOSITION 2.2. (Ascoli-Arzelá theorem, [2, p. 131].) *The set Φ of continuous functions at he interval $[a, b]$ is relatively compact in the space $C([a, b], \mathbb{R})$ if and only if the set Φ is uniformly bounded and equi-continuous in $C([a, b], \mathbb{R})$.*

As the simplified corollary of the previous Ascoli-Arzelá theorem is the following lemma about the relative compactness of a subset of the space $C([a, b], \mathbb{R}^n)$.

LEMMA 2.1. *The set M of continuous vector functions at the interval $[a, b]$ is relatively compact in the space $C([a, b], \mathbb{R}^n)$ if and only if the set M is uniformly bounded and equi-continuous in $C([a, b], \mathbb{R}^n)$.*

PROPOSITION 2.3. (Nikol̆skiĭ theorem, [4, p. 233].) *A linear bounded operator $A : X \to Y$ is Fredholm of index zero if and only if*

$$A = C + T,$$

where C is a linear homeorphism of X onto Y and $T : X \to Y$ is a linear compact operator.

PROPOSITION 2.4. (See Proposition 29.17, [6, p. 672].) *Let X and Y are Banach spaces over the real or complex numbers field, and let*

$$A : D(A) \subseteq X \to Y$$

be a C^k-Fredholm operator on the open set $D(A)$, where $1 \leq k \leq \infty$.

Then if $C : D(A) \subseteq X \to Y$ is compact and C^1, then $A + C$ is a Fredholm operator with the same index as A at each point of $D(A)$.

PROPOSITION 2.5. (See Theorem 4.F, [5, p. 172].) *Let $f : U(x_0) \subset X \to Y$ be a C^1-mapping, where both X and Y are Banach spaces over real or complex numbers field. Then f is a local C^1-diffeomorphism at x_0 if and only if x_0 is the regular point of mapping f (i. e. $f'(x_0) : X \to Y$ is bijective).*

In this paper we shall consider the mapping $F = A + N : X \to Y$ and the set Σ of all $x \in X$ in which F is not locally invertible. The following assumptions will be used:

(A.1) $A : X \to Y$ is a linear bounded Fredholm operator of the zero index;
(A.2) $N : X \to Y$ is a compact operator;
(A.3) $F = A + N : X \to Y$ is a coercive operator;
(A.4) $N : X \to Y$ is a continuously Fréchet differentiable mapping.

PROPOSITION 2.6. (See Lemma 3.1, [3, p. 23]). *Under the assumptions* (A.1), (A.2) *the following statements hold:*

(α) F *is a continuous and* σ *- proper mapping;*
(β) F *is locally invertible at* $x \in X$ *if and only if* F *is locally injective at* x.
Moreover, if also (A.3) *is assumed, then*
(γ) F *is a proper mapping.*

The most important properties of the mapping F satisfying (A.1), (A.2), (A.3) are given by the following theorem.

PROPOSITION 2.7. (See Theorem 3.1, [3, p. 23].) *If the assumptions* (A.1), (A.2), (A.3) *hold, then*

(a) *For each* $q \in Y$ *the set* $F^{-1}(q)$ *is compact (possibly empty) in* X*;*
(b) $R(F)$ *is closed and connected in* Y*;*
(c) Σ *and* $F(\Sigma)$ *are closed subsets of* X *and* Y*, respectively and* $F(X \setminus \Sigma)$ *is open in* Y*;*
(d) $\operatorname{card} F^{-1}(q)$ *is constant and finite (it may be even 0) on each connected component of the open set* $Y \setminus F(\Sigma)$*;*
(e) *if* $\Sigma = \emptyset$*, then* F *is a homeomorphism of* X *onto* Y*;*
(f) *if* $\Sigma \neq \emptyset$*, then for the boundary* $\partial F(X \setminus \Sigma)$ *of* $F(X \setminus \Sigma)$ *we have the inclusion*

$$\partial F(X \setminus \Sigma) \subset F(\Sigma).$$

Corollary of these properties of operator F are sufficient conditions for its surjectivity.

PROPOSITION 2.8. (See Corollary 3.1, [3, p. 23]) *Under the assumptions* (A.1), (A.2), (A.3) *each of the following conditions is sufficient for the surjectivity of operator* F*:*

(g) $F(\Sigma) \subset F(X \setminus \Sigma)$*;*
(h) $Y \setminus F(\Sigma)$ *is connected set and*

$$F(X \setminus \Sigma) \setminus F(\Sigma) \neq \emptyset.$$

PROPOSITION 2.9. (See Corollary 3.3, [3, p. 25].) *Let* X *and* Y *be a real Banach spaces, and let* $F = A + N : X \to Y$*. Denote by* Σ *the set of all points* $x \in X$ *where* F *is not locally invertible. Suppose that the assumptions* (A.1), (A.2), (A.3) *and*

(A.5) *there exists a strictly solvable field* $G = I - g : X \to X$ *and an* $R > 0$ *such that*

$$F(x) = kC \circ G(x) \text{ implies } k \geq 0 \text{ for all } x \in X,\ ||x|| \geq R, \tag{2.1}$$

where $A = C + T : X \to Y$*,* C *is a linear homeomorphism of* X *onto* Y *and* $T : X \to Y$ *is a linear compact operator*

are satisfied.
Then the statement

(i) F *is surjective mapping.*

holds.

Remark 2.5. (See Remark 3.1, [3, p. 26].) The condition (2.1) can be expressed as an a priori estimate. Each possible solution x of the equation

$$F(x) = kC \circ G(x) \text{ pre } \forall k < 0$$

satisfies the estimate $||x|| < R$.

Proposition 2.10. (See Lemma 3.2, [3, p. 26].) *Suppose that the assumptions* (A.1), (A.2) *and* (A.4) *are satisfied. Then the folowing statement hold:*

(j) *if*

$$S = \{x \in X : F'(x)h = 0 \text{ has a solution } h \neq 0\},$$

then S is the set of all singular points of F and the set $R_F = Y \setminus F(S)$ of all regular values of F is dense in Y.

Remark 2.6. (See Remark 3.2, [3, p. 27].) If $F \in C^1(X, Y)$ and S, Σ have the meaning given above, then by Proposition 2.5, $\Sigma \subset S$.

3. THE FORMULATION PROBLEM AND EXISTENCE THEOREM

In this part we shall study the classic solution $u \in C^1([a, b], \mathbb{R}^n)$ of a boundary value problem for the differential system of the first order

$$u'(t) + f(t, u(t)) = g(t), \quad t \in [a, b] \tag{3.1}$$

with the general conditions

$$(l(u) \equiv) Mu(a) + Nu(b) = 0, \tag{3.2}$$

where $-\infty < a < b < \infty$, $n \geq 1$. In the whole text we assume $g =: (g_1, \ldots, g_n)^T \in C([a, b], \mathbb{R}^n)$ and $f =: (f_1, \ldots, f_n)^T : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ to be continuous and $l : C([a, b], \mathbb{R}^n) \to \mathbb{R}^n$. M and N are square constant matrices of the type $n \times n$. The topology in the space $C([a, b], \mathbb{R}^n)$ is given by the norm $||.||_n$, whereby $||v||_n = \max_{k=1,\ldots,n} \{ \sup_{a \leq t \leq b} |v_k(t)| \}$ for $v = (v_1, \ldots, v_n) \in C([a, b], \mathbb{R}^n)$. We shall use also the norm $||.||_{\mathbb{R}^n}$ in the space $\mathbb{R}^n$ defined by $||x||_{\mathbb{R}^n} = (\sum_{k=1}^n x_k^2)^{1/2}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. By the Schauder theorem (Proposition 2.1) we prove the existence theorem for the problem (3.1), (3.2).

Theorem 3.1. *Let $f \in C([a, b] \times \mathbb{R}^n, \mathbb{R}^n)$ be bounded function at the set $[a, b] \times \mathbb{R}^n$ (i. e. $\exists K_0 > 0$, such that $||f(t, z_1, \ldots, z_n)||_{\mathbb{R}^n} \leq K_0$ for $\forall\, t \in [a, b]$ and $(z_1, \ldots, z_n) \in C([a, b], \mathbb{R}^n)$,). If the rank of the matrix $M + N$ $h(M + N) = n$, then the boundary value problem* (3.1), (3.2) *has at least one solution.*

Proof. By the integration of the system (3.1),using the condition (3.2) we get that $u = (u_1, \ldots, u_n)^T$ is solution of (3.1), (3.2) if and only if it satysfies the integral equation

$$(M + N).u(t) = N. \int_t^b [f(\tau, u(\tau)) - g(\tau)]d\tau - M. \int_a^t [f(\tau, u(\tau)) - g(\tau)]d\tau \tag{3.3}$$

for $t \in [a,b]$. Since $h(M+N) = n$ we can write

$$\begin{aligned} u(t) &= (M+N)^{-1}.N.\int_t^b [f(\tau,u(\tau)) - g(\tau)]d\tau - \\ &- (M+N)^{-1}.M.\int_a^t [f(\tau,u(\tau)) - g(\tau)]d\tau \end{aligned}$$

for $t \in [a,b]$.

Define the operator $T : C([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ by the equation

$$\begin{aligned} (Tu)(t) &= (M+N)^{-1}.N.\int_t^b [f(\tau,u(\tau)) - g(\tau)]d\tau - \\ &- (M+N)^{-1}.M.\int_a^t [f(\tau,u(\tau)) - g(\tau)]d\tau \end{aligned}$$

for $u \in C([a,b],\mathbb{R}^n)$ and for $t \in [a,b]$.

Let

$$M = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ r_{21} & \cdots & r_{2n} \\ \vdots & & \\ r_{n1} & \cdots & r_{nn} \end{pmatrix}$$

and $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. Then

$$\begin{aligned} ||M.x||_{\mathbb{R}^n} &\leq \left[\sum_{i=1}^n \left(\sum_{k=1}^n |r_{ik}||x_k|\right)^2\right]^{1/2} \leq \\ &\leq c(n)\sqrt{\sum_{i=1}^n \left(\sum_{k=1}^n r_{ik}^2 x_k^2\right)} \leq c(n)\sqrt{n}\ r||x||_{\mathbb{R}^n}, \end{aligned}$$

where $c(n)$ is a real number depends of n and $r = \max_{i,k=1,\dots,n}\{|r_{ik}|\}$. Hence

$$||M.x||_{\mathbb{R}^n} \leq ||M||^*.||x||_{\mathbb{R}^n},$$

where $||M||^* = c(n)\sqrt{n}\ r$.

Since f is the bounded function on the set $[a,b]\times\mathbb{R}^n$, there is a positive constant K_1 such that

$$||f(\tau,u(\tau)) - g(\tau)||_{\mathbb{R}^n} \leq K_1$$

for all $u \in C([a,b],\mathbb{R}^n)$ and $t \in [a,b]$.

Then

$$\begin{aligned} ||(Tu)(t)||_{\mathbb{R}^n} &\leq ||(M+N)^{-1}.N||^*.\left|\int_t^b ||f(\tau,u(\tau)) - g(\tau)||_{\mathbb{R}^n} d\tau\right| + \\ &+ ||(M+N)^{-1}.M||^*.\left|\int_a^t ||f(\tau,u(\tau)) - g(\tau)||_{\mathbb{R}^n} d\tau\right| \leq \\ &\leq (K_2+K_3)(b-a)K_1, \end{aligned}$$

where $K_2 = ||(M+N)^{-1}.N||^*$ and $K_3 = ||(M+N)^{-1}.M||^*$, whence we obtain

$$||Tu||_n = \max_{k=1,\dots,n}\{\sup_{a\le t\le b}|[Tu]_k(t)|\} \le (K_2+K_3)(b-a)K_1.$$

Denote by $B(0;R)$ the ball $\{v \in C([a,b],\mathbb{R}^n),\ ||v||_n \le R,\ R = (K_2+K_3)(b-a)K_1\}$. Then $Tu \in B(0;R)$ and the operator T mapps $C([a,b],\mathbb{R}^n)$ into $B(0;R)$. To apply the Schauder theorem, we consider the operator $T : B(0;R) \to B(0;R)$. The ball $B(0;R) \subset (C([a,b],\mathbb{R}^n), ||.||_n)$ is a nonempty, closed, bounded and convex set.

Now, we prove that operator $T : B(0;R) \to B(0;R)$ is continuous. Let the sequence $\{u_l\}_{l=1}^{\infty} \subset B(0;R)$ tends to $u_0 \in B(0;R)$ in $C([a,b],\mathbb{R}^n)$ as $l \to \infty$. For an arbitrary $t \in [a,b]$ we have

$$\begin{aligned}
||(Tu_l)(t) \ &- \ (Tu_0)(t)||_{\mathbb{R}^n} = \\
&= ||(M+N)^{-1}.N.\int_t^b [f(\tau,u_l(\tau)) - f(\tau,u_0(\tau))]d\tau - \\
&- (M+N)^{-1}.M.\int_a^t [f(\tau,u_l(\tau)) - f(\tau,u_0(\tau))]d\tau||_{\mathbb{R}^n} \le \\
&\le K_2\left|\int_t^b ||f(\tau,u_l(\tau)) - f(\tau,u_0(\tau))||_{\mathbb{R}^n} d\tau\right| + \\
&+ K_3\left|\int_a^t ||f(\tau,u_l(\tau)) - f(\tau,u_0(\tau))||_{\mathbb{R}^n} d\tau\right|
\end{aligned}$$

and

$$\begin{aligned}
\lim_{l\to\infty} ||(Tu_l)(t) \ &- \ (Tu_0)(t)||_{\mathbb{R}^n} \le \\
&\le \lim_{l\to\infty}[K_2\left|\int_t^b ||f(\tau,u_l(\tau)) - f(\tau,u_0(\tau))||_{\mathbb{R}^n} d\tau\right| + \\
&+ K_3\left|\int_a^t ||f(\tau,u_l(\tau)) - f(\tau,u_0(\tau))||_{\mathbb{R}^n} d\tau\right|].
\end{aligned}$$

The assumption $u_l \to u_0$ in $C([a,b],\mathbb{R}^n)$ as $l \to \infty$ implies the convergence of the value sequence $\{u_l(t)\}_{l=1}^{\infty}$ to $u_0(t)$ in the space $\mathbb{R}^n$ (i.e. with respect to the elements of $u_l(t)$) as $l \to \infty$ for every $t \in [a,b]$. Hence and by the continuity of f on $[a,b]\times\mathbb{R}^n$ we obtain the inequality

$$0 \le \lim_{l\to\infty} ||(Tu_l)(t) - (Tu_0)(t)||_{\mathbb{R}^n} \le 0 \quad \text{for } t \in [a,b],$$

which means also the convergence of $\{(Tu_l)(t)\}_{l=1}^{\infty}$ in $\mathbb{R}^n$ and the convergence with respect to the elements of $(Tu_l)(t)$ as $l \to \infty$. And so

$$\lim_{l\to\infty} ||Tu_l - Tu_0||_n = \lim_{l\to\infty}\max_{k=1,\dots,n}\{\sup_{a\le t\le b}|[Tu_l]_k(t) - [Tu_0]_k(t)|\} = 0$$

which proves the continuity of the operator T in $C([a,b],\mathbb{R}^n)$.

Now, we have to prove the compat of T.

The set $S \subset B(0;R)$ bounded. For $u \in S$ it is $Tu \in B(0;R)$ and the norm $||Tu||_n \leq R$, where $R > 0$ is independent of u. This means the uniform boundedness of $T(S)$.

Further for $\varepsilon > 0$, $u \in S$ and $t_1, \ t_2 \in [a,b]$ we have

$$\begin{aligned}
||(Tu)(t_1) \ &- \ (Tu)(t_2)||_{\mathbb{R}^n} \leq \\
&\leq \ ||(M+N)^{-1}.N||^* \left| \int_{t_1}^{t_2} ||f(\tau,u(\tau)) - g(\tau)||_{\mathbb{R}^n} d\tau \right| + \\
&+ \ ||(M+N)^{-1}.M||^* \left| \int_{t_1}^{t_2} ||f(\tau,u(\tau)) - g(\tau)||_{\mathbb{R}^n} d\tau \right| \leq \\
&\leq \ (K_2+K_3)|t_2-t_1|K_1
\end{aligned}$$

Take $t_1, \ t_2$ such that $|t_2-t_1| < \delta(\varepsilon) = \frac{\varepsilon}{(K_2+K_3)K_1}$. Then $||(Tu)(t_1)-(Tu)(t_2)||_{\mathbb{R}^n} < \varepsilon$, which represents the equicontinuous of $T(S)$. According to Lemma 2.1 the set $T(S) \subset C([a,b],\mathbb{R}^n)$ is relative compact. Hence the operator $T: B(0;R) \to B(0;R)$ is compact. The Proposition 2.1 implies the existence of solution of the boundary value problem (3.1), (3.2). □

The following example shows that there exists a differential system with the continuum of solutions.

Example 3.1. Consider the differential systems

$$\begin{aligned}
u_1'(t) \ &= \ u_2(t) \\
u_2'(t) \ &= \ \lambda_k u_1(t) \quad \text{for } t \in [0,1] \quad \text{and } k = 1,2,\ldots,
\end{aligned}$$

where $\lambda_k = k^2\pi^2$.

We consider the boundary conditions

$$Mu(0) + Nu(1) = 0,$$

where $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $u = (u_1,u_2)^T \in C^1([0,1],\mathbb{R}^2)$.

This BVP has infinitely many solutions $u_k = (u_{k,1},u_{k,2})^T$, where

$$\begin{aligned}
u_{k,1}(t) \ &= \ c_1 \sin(\sqrt{\lambda_k}t) \\
u_{k,2}(t) \ &= \ c_2 \cos(\sqrt{\lambda_k}t)
\end{aligned}$$

for $t \in [0,1]$, $c_1 \in \mathbb{R}$, $c_2 = c_1\lambda_k$ and for all $k = 1,2,\ldots$

4. PRELIMINARY RESULTS

Putting $u =: (u_1,\ldots,u_n)^T$, $f =: (f_1,\ldots,f_n)^T$, $g =: (g_1,\ldots,g_n)^T$ we transform the problem (3.1), (3.2) to the following operator equation. Let

$$D(A) = \{u \in C^1([a,b],\mathbb{R}^n), \ lu = 0\}, \tag{4.1}$$

$||.||_n$ and $||.||_{\mathbb{R}^n}$ be the norms defined in the Chapter 3. The symbol $||.||_n^1$ means the norm in the space $C^1([a,b],\mathbb{R}^n)$ given by the equality

$$||v||_n^1 = \max_{k=1,\ldots,n} \{ \sup_{a\leq t\leq b} |v_k(t)|, \ \sup_{a\leq t\leq b} |v_k'(t)|\}$$

for $v \in C^1([a,b],\mathbb{R}^n)$. Then $X_0 := (D(A), ||.||_n^1)$ and $Y_0 := (C([a,b],\mathbb{R}^n), ||.||_n)$ are Banach spaces. Define a linear bounded operator $A : X_0 \to Y_0$ by

$$Au = u' \tag{4.2}$$

and a nonlinear operator $\mathcal{N} : X_0 \to Y_0$ by

$$(\mathcal{N}u)(t) = f(t, u(t)), \quad u \in X_0 \text{ and } t \in [a,b] \tag{4.3}$$

We shall deal with the operator equation

$$Fu := Au + \mathcal{N}u = g \quad \text{for } u \in D(A), \tag{4.4}$$

(F:$X_0 \to Y_0$).

LEMMA 4.1. *Let the operator A and its domain of definition $D(A)$ be given by* (4.2) *and* (4.1)*, respectively. Then*

1. *If* $\det(M+N) \neq 0$*, the linear bounded operator $A : X_0 \to Y_0$ is a Fredholm operator of zero index.*

2. *Suppose that* $\det(M+N) = 0$ *and the following condition*

(H.1) *There is the first order linear differential system*

$$((Cu)(t) \equiv)\ u' + D(t)u(t) = 0, \tag{4.5}$$

where $D(t)$ is a $n \times n$ type matrix of continuous functions on $[a,b]$ and the operator $C : X_0 \to Y_0$ is a linear homeomorphism.

Then the linear bounded operator $A : X_0 \to Y_0$ is a Fredholm operator of zero index.

Proof. 1. Denote $h =: (h_1, \ldots, h_n)^T \in C([a,b],\mathbb{R}^n)$ and $(Au)(t) = u'(t) =: h(t)$ for $u \in D(A)$ and $t \in [a,b]$. Since $\det(M+N) \neq 0$, homogeneous boundary value problem $Au = 0$, (3.2) has only a trivial solution. Hence there exists a linear inverse operator $A^{-1} : H(A) = C([a,b],\mathbb{R}^n)$ onto $D(A)$ defined by

$$(A^{-1}h)(t) = -(M+N)^{-1} \cdot N \cdot \int_a^b h(s)ds + \int_a^t h(s)ds \qquad \text{for } t \in [a,b].$$

Denote the vector

$$(\sigma_1, \ldots, \sigma_n)^T := \sigma := -(M+N)^{-1} \cdot N \cdot \int_a^b h(s)ds + \int_a^t h(s)ds \qquad t \in [a,b]$$

and matrix

$$\begin{pmatrix} d_{11} & d_{12} & \ldots & d_{1n} \\ d_{21} & d_{22} & \ldots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \ldots & d_{nn} \end{pmatrix} := -(M+N)^{-1} \cdot N, \quad d_{ij} \in \mathbb{R},\ i,j = 1, \ldots, n.$$

Then

$$\begin{aligned} ||A^{-1}h||_n^1 &= \max_{k=1,\ldots,n} \{ \sup_{a \le t \le b} |\sigma_k(t)|, \sup_{a \le t \le b} |h_k(t)| \} \le \\ &\le [(b-a) \cdot (d \cdot n + 1) + 1] \cdot \max_{k=1,\ldots,n} \sup_{a \le t \le b} |h_k(t)| = K \cdot ||h||_n, \end{aligned}$$

where $K = (b-a)\cdot(d\cdot n+1)+1 > 0$ and $d = \max_{i,j=1,\dots,n}|d_{ij}|$. Hence we have the continuity of the linear operator A^{-1}. The operator $A : D(A) \to H(A) = C([a,b],\mathbb{R}^n)$ is the linear homeomorphism. Since the zero mapping $O : D(A) \to C([a,b],\mathbb{R}^n)$ is compact, such the operator $A = A + O$ is the Fredholm operator of the zero index (see Proposition 2.3).

2. We can write

$$A = C + T$$

where $T = A - C : X_0 \to Y_0$. By Lemma 2.1 we prove the compactness of T.

Let $M_1 \subset X_0$ be a bounded set and $v = (v_1,\dots,v_n)^T \in M_1$ (i. e. there is a $K > 0$ such that $\|v\|_n^1 \le K$ for all $v \in M_1$). Denote the matrix

$$D(t) =: \begin{pmatrix} d_{11}(t) & d_{12}(t) & \dots & d_{1n}(t) \\ d_{21}(t) & d_{22}(t) & \dots & d_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1}(t) & d_{n2}(t) & \dots & d_{nn}(t) \end{pmatrix}$$

and

$$d := \max_{\substack{i,j=1,\dots,n \\ t\in[a,b]}} |d_{ij}(t)|.$$

Then the estimation

$$\begin{aligned} \|Tv\|_n &= \|-D(t)\cdot v\|_n \le d\cdot n\cdot \max_{k=1,\dots,n}\{\sup_{t\in[a,b]}|v_k(t)|\} = \\ &= d\cdot n\cdot\|v\|_n^1 \le d\cdot n\cdot K \end{aligned}$$

is true and the set $T(M_1)$ is uniformly bounded.

Now, let us take $\varepsilon > 0$ and $t_1,\ t_2 \in [a,b]$. Then

$$\begin{aligned} \|(Tv)(t_1)-(Tv)(t_2)\|_{\mathbb{R}^n} &\le \|D(t_1)\cdot(v(t_2)-v(t_1))\|_{\mathbb{R}^n} + \\ &+ \|(D(t_2)-D(t_1))\cdot v(t_2)\|_{\mathbb{R}^n} \le \\ &\le \sqrt{\sum_{k=1}^n c_1\cdot\sum_{i=1}^n |d_{ki}(t)|^2\cdot|v_i(t_2)-v_i(t_1)|^2} + \\ &+ \sqrt{\sum_{k=1}^n c_2\cdot\sum_{i=1}^n |d_{ki}(t_2)-d_{ki}(t_1)|^2\cdot|v_i(t_2)|^2}\ , \end{aligned}$$

where $c_1,\ c_2 > 0$. The functions d_{ki} are uniformly continuous, whence there is $\delta_1 > 0$ such that $|d_{ki}(t_1)-d_{ki}(t_2)| < \varepsilon$ for all indices $k,\ i$ and $|t_1-t_2| < \delta_1$. Using the Lagrange mean value theorem for the continuously differentiable functions v_i and the boundedness of M_1 we have

$$\|(Tv)(t_1)-(Tv)(t_2)\|_{\mathbb{R}^n} \le \sqrt{c_1}\cdot d\cdot K\cdot n\cdot|t_1-t_2| + \sqrt{c_2}\cdot n\cdot\varepsilon\cdot K.$$

Take $t_1,\ t_2 \in [a,b]$ such that $|t_1-t_2| < \delta_2 = \varepsilon$ and $\delta = \min\{\delta_1,\varepsilon\}$. Hence

$$\|(Tv)(t_1)-(Tv)(t_2)\|_{\mathbb{R}^n} < K_1\cdot\varepsilon =: \varepsilon_1 \quad \text{for } t_1,\ t_2 \in [a,b], \text{ such that } |t_1-t_2| < \delta,$$

where $0 < K_1 = (\sqrt{c_1} \cdot d + \sqrt{c_2}) \cdot n \cdot K$. The set $T(M_1)$ is equi-continuous. This consideration proves that set $T(M_1)$ is relatively compact and the linear operator $T : X_0 \to Y_0$ is comapct. With respect to Proposition 2.3 $A = C + T$ is the Fredholm operator of zero index. □

REMARK 4.1. The operator given by (4.2), (4.1) is Gáteaux differentiable at the point $u \in X_0$ and at the direct $h \in X_0$ and the Gáteaux differential

$$dA(u;h)(t) = \lim_{s\to 0} \frac{A(u + s \cdot h)(t) - (Au)(t)}{s} = h'(t) \text{ in } Y_0 \text{ for } t \in [a,b].$$

Since

$$||dA(u;h)||_n \le 1 \cdot ||h||_n^1 \text{ for all } h \in X_0,$$

the linear operator $dA(u;\cdot)$ is continuos at X_0 and $dA(\cdot;h)$ is continuous at X_0, too. Then operator A belongs to the space $C^1(X_0, Y_0)$ (See [1, Theorem 2., p. 67]).

LEMMA 4.2. *Assume*
(H.2)

$$f = (f_1, \dots, f_n)^T \in C([a,b] \times \mathbb{R}^n, \mathbb{R}^n).$$

Then the associated nonlinear Nemitskij operator $\mathcal{N} : X_0 \to Y_0$ *defined by* (4.3) *is compact at* X_0.

Proof. By the hypothesis (H.2) we easly see the continuity of $\mathcal{N}$ at X_0.
Let, now, $M_1 \subset X_0$ be a bounded set and $v \in M_1$ ($\exists K > 0$ such that $||v||_n^1 \le K$ for all $v \in M_1$). From the continuity of functions f_k for $k = 1, \dots, n$ we obtain

$$||\mathcal{N}v||_n \le K_2,\ K_2 > 0,$$

which the uniform boundedness of $\mathcal{N}(M_1)$ proves. The equi-continuity of set $\mathcal{N}(M_1)$ can be proved again by the uniform continuity of the functions f_k at $[a,b] \times [-K,K]^n$ if $k = 1, \dots, n$. According to Lemma 2.1 the set $\mathcal{N}(M_1)$ is a relatively compact set and so the operator $\mathcal{N}$ is compact. □

LEMMA 4.3. *Let the hypothesis* (H.2) *and*
(H.3)

$$\frac{\partial f}{\partial u_i} \in C([a,b] \times \mathbb{R}^n, \mathbb{R}^n),\ i = 1, \dots, n$$

keep. Then the operator $\mathcal{N}$ *defined by* (4.3) *belongs to the space* $C^1(X_0, Y_0)$ *(it is Fréchet differentiable).*

Proof. First of all we find Gáteaux differential of operator $\mathcal{N}$ at the point $u \in X_0$ and at the direct $h = (h_1, \dots, h_n)^T \in X_0$:

$$d\mathcal{N}(u;h)(t) = \lim_{s\to 0} \frac{\mathcal{N}(u + sh)(t) - (\mathcal{N}u)(t)}{s} \quad \text{in } Y_0 \text{ for } t \in [a,b].$$

Again using the Lagrange mean value theorem we get from the last equality

$$d\mathcal{N}(u;h)(t) = \lim_{s\to 0} \left[\frac{\frac{\partial f}{\partial u_1}(t, \xi_1^1, \xi_2^1, \dots, \xi_n^1) \cdot s \cdot h_1(t)}{s} + \right.$$

$$+ \frac{\frac{\partial f}{\partial u_2}(t,\xi_1^2,\xi_2^2,\ldots,\xi_n^2)\cdot s\cdot h_2(t)+\cdots+\frac{\partial f}{\partial u_n}(t,\xi_1^n,\xi_2^n,\ldots,\xi_n^n)\cdot s\cdot h_n(t)}{s}\Bigg].$$

Here $\xi_k^i \in (u_k(t), u_k(t)+s\cdot h_k(t))$ for $i,k=1,\ldots,n$ and the Gáteaux differential

$$d\mathcal{N}(u;h)(t)=\sum_{i=1}^{n}\frac{\partial f}{\partial u_i}[t,u_1(t),u_2(t),\ldots,u_n(t)]\cdot h_i(t).$$

We prove that linear operator $d\mathcal{N}(u;\cdot): X_0 \to Y_0$ is continuous in X_0 for all $u\in X_0$. By the hypothesis (H.3) for all $u\in X_0$ we have

$$\left\|\frac{\partial f}{\partial u_i}[t,u_1(t),\ldots,u_n(t)]\right\|_{\mathbb{R}^n}\le K_i(u)\quad i=1,\ldots,n,$$

when K_i is a positive constant depended of u. Put $K(u)=\max_{i=1,\ldots,n}\{K_i(u)\}$, then

$$||d\mathcal{N}(u;.)||_{Y_0}\le n\cdot K(u)\cdot||h||_{X_0},\ u\in X_0 \text{ and } h\in X_0.$$

Consider the mapping $d\mathcal{N}(\cdot;h): X_0\to Y_0$ and a sequence $\{v^p\}_{p\in N}\subset X_0$ for which $v^p\to v^0\in X_0$ as $p\to\infty$. Then the sequence $\{v_k^p\}_{p\in N}$ for $k=1,\ldots,n$ uniformly converges to the function v_k^0 on the interval $[a,b]$. Hence

$$\lim_{p\to\infty}\left\|d\mathcal{N}(v^p;h)-d\mathcal{N}(v^0;h)\right\|_n=$$

$$=\lim_{p\to\infty}\max_{k=1,\ldots,n}\left\{\sup_{t\in[a,b]}\left|\left[\sum_{i=1}^{n}\left(\frac{\partial f}{\partial u_i}[t,v_1^p(t),\ldots,v_n^p(t)]-\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-\ \frac{\partial f}{\partial u_i}\left[t,v_1^0(t),\ldots,v_n^0(t)\right]\right)\cdot h_i(t)\right]_k\right|\right\}=0$$

which proves that operator $d\mathcal{N}(\cdot;h)$ is continuous at X_0. The Gáteaux differential is equal to Fréchet differential (see [1, Theorem 2., p. 67]). This ends the proof of Lemma 4.3 . □

LEMMA 4.4. *Let $F=A+\mathcal{N}: X_0\to Y_0$, where A is defined by* (4.2) *and* (4.1) *and $\mathcal{N}$ is given by* (4.3). *Then $u\in X_0$ is a solution of the boundary value problem* (3.1), (3.2) *if and only if $Fu=g$.*

Proof. The solution u of (3.1), (3.2) is an element of X_0 and F is the operator defined by the left hand side of (3.1). So the equality $Fu=g$ means that u is a solution of problem (3.1), (3.2) and contrarily. □

LEMMA 4.5. *Let the hypothesis* (H.2) *and*

(H.4) *For each bounded set $S\subset Y_0$ there is $R>0$ such that the inequality*

$$||u||_n\le R$$

hold for every solution u of the problem (3.1), (3.2) *wiht the right hand side $g\in S$.*

hold. Then the operator $F=A+\mathcal{N}: X_0\to Y_0$ is coercive, if A is defined by (4.2) *and* (4.1) *and $\mathcal{N}$ is defined by* (4.3).

Proof. Let $S \subset Y_0$ be a bounded set. The assumtion (H.4) and Lemma 4.4 implies the boundedness of $F^{-1}(S)$ in the norm $\|\cdot\|_n$. Take $u \in F^{-1}(S)$. Then $u_k(t) \in [-R, R]$ for k=1,...,n. From the hypothesis (H.2) we have

$$\|\mathcal{N}u\|_n = \max_{k=1,\dots,n}\{\sup_{t\in[a,b]} \|f_k(t,u(t))\|\} \le R_1,$$

where $R_1 > 0$. Since $Fu \in S$ such $\|Fu\|_n \le R_2$ for a positive constant R_2. Hence

$$\|u'\|_n = \|Au\|_n = \|(F - \mathcal{N})\|_n \le \|Fu\|_n + \|\mathcal{N}u\|_n \le R_2 + R_1 = R_3.$$

If we put $R_4 = \max\{R, R_3\}$ such $u \in F^{-1}(S)$ satisfies the inequality

$$\|u\|_n^1 \le R_4,\ R_4 > 0,$$

which means the boundedness of $F^{-1}(S)$ in the norm $\|\cdot\|_n^1$. This completes the proof of Lemma 4.5. □

LEMMA 4.6. *Let the hypotheses* (H.1), (H.2), (H.3) *satisfy. Then the operator* $F = A + \mathcal{N} \in C^1(X_0, Y_0)$ *is the Fredholm operator of zero index at* X_0*. Here* A *is given by* (4.2) *and* (4.1) *and* $\mathcal{N}$ *by* (4.3).

Proof. From Lemma 4.1 and Remark 4.1 we see that operator A is a C^1 - Fredholm operator of zero index at X_0. Lemma 4.2 and Lemma 4.3 imply the compactness and the Fréchet differentiability of the operator $\mathcal{N}$ at X_0. The Proposition 2.4 gives the assertion of this Lemma. □

DEFINITION 4.1. 1. A couple $(u, g) \in X_0 \times Y_0$ will be called *the bifurcation point of the boundary value problem* (3.1), (3.2) if u is a solution of this problem and there exists a sequence $\{g_k\} \subset Y_0$ such that $g_k \to g$ in Y_0 as $k \to \infty$ and the problem (3.1), (3.2) for $g = g_k$ has at least two different solutions u_k, v_k for each $k \in N$ and $u_k \to u, v_k \to u$ in X_0 as $k \to \infty$.

2. The set of all solution $u \in X_0$ of (3.1), (3.2) (or the set of all functions $g \in Y_0$) such that (u, g) is a bifurcation point of the boundary value problem (3.1), (3.2) will be called *the domain of bifurcation (the bifurcation range)* of this problem.

The domain of bifurcation we denote by D_b and the bifurcation range by R_b.

LEMMA 4.7. *Let the hypotheses* (H.1), (H.2) *hold and let the operator* $F = A + \mathcal{N} : X_0 \to Y_0$*, where* A *is defined by* (4.2), (4.1) *and* $\mathcal{N}$ *by* (4.3)*. Then* (u, g) *is a bifurcation point of boundary value problem* (3.1), (3.2) *if and only if* $Fu = g$ *and* $u \in \Sigma$*, where* Σ *means the set of Remark* 2.2.

Proof. Let (u, g) be a bifurcation point of the problem (3.1), (3.2). Lemma 4.4 gives $Fu = g$. Denote by $U(u)$ a neighbourhood of point u. There exists at least two different point $u_1,\ v_1 \in U(u)$ such that $F(u_1) = g_1 = F(v_1)$. Then F is not locally injective. Proposition 2.6 (β) gives $u \in \Sigma$.

Contrarily, if $Fu = g$ and $u \in \Sigma$, then u is a solution of (3.1), (3.2). Further, Proposition 2.6(β) ensures that in any nieghbourhood $U_k(u)$ of the point u there exists two different points $v_k,\ z_k$ for which $F(v_k) = F(z_k) = g_k$, k=1,...,n. Consider a sequence of neighbourhood $\{U_k(u)\}_{k\in N}$ such that $U_{k+1} \subset U_k$ for k=1,...,n and the radii sequence of $U_k(u)$ converges to zero as $k \to \infty$. Then $u_k \to u,\ z_k \to u$ in X_0 as $k \to \infty$. Since F is continuous at X_0, $g_k = F(v_k) \to F(u) = g$ in Y_0 as $k \to \infty$. Hence it follows that the pair (u, g) is a bifurcation point of problem (3.1), (3.2). □

5. PROPERTIES OF SET SOLUTIONS OF THE DIFFERENTIAL SYSTEM

Using the previous lemmas we obtain the following set results for the solutions set of the boundary value problem (3.1), (3.2). Let the operator A be defined by the relations (4.1), (4.2), the operator $\mathcal{N}$ by (4.3) and $F = A + \mathcal{N} : X_0 \to Y_0$.

THEOREM 5.1. *Suppose that the assumption* (H.1), (H.2), (H.4) *are satisfied. Then the following statements are true:*

1. *For each $g \in Y_0$ the set S_g of all solutions of the BVP* (3.1), (3.2) *is compact (possibly empty).*
2. *The set $R(F)$ of all $g \in Y_0$ for which there exists at least one solution of the BVP* (3.1), (3.2) *is closed and connected in Y_0.*
3. *The domain of bifurcation of the BVP* (3.1), (3.2) *D_b is closed in X_0 and the bifurcation range of that BVP, R_b, is closed in Y_0.*
4. *If $Y_0 \setminus R_b \neq \emptyset$, then each component of that set is nonempty, open and hence a region. The number n_g of solutions of the problem* (3.1), (3.2) *is finite, constant (it may be even zero) on each component of the set $Y_0 \setminus R_b$.*
5. *If $R_b = \emptyset$, then the problem* (3.1), (3.2) *has a unique solution u for each $g \in Y_0$ and this solution continuously depends on g as a mapping from Y_0 onto X_0.*

Proof. With the respect to Lemma 4.4, the equality $F = g$ holds if and only if u is a solution of the boundary value problem (3.1), (3.2). Then the properties of mapping F imply the corresponding properties of the problem (3.1), (3.2). By the Lemma 4.1, 4.2, 4.5 we get the assumptions (A.1), (A.2), (A.3) for F.

The statement 1 and 2 follow directly, by the Proposition 2.7 (a) and (b), respectivaly.

The equation $D_b = \Sigma$ and $R_b = F(\Sigma)$ follow by the Lemma 4.7, where Σ is defined as in the second chapter.Then the Propositon 2.7 (c) implies the statement 3.

Since every component of an open set in a Banach space is open and $Y_0 \setminus R_b$ is open in Y_0, then every component of this set is open. The last assertion of the point 4 follows from the Proposition 2.7 (d).

The Proposition 2.7 (e) implies the assertion 5. □

THEOREM 5.2. *Under the assumptions* (H.1), (H.2), (H.4) *each of the following conditions is sufficient for the surjectivity of F:*

6. *For each $g \in R_b$ there is a solutions u of the BVP* (3.1), (3.2) *such that $u \in X_0 \setminus D_b$;*
7. *$Y_0 \setminus R_b$ is connected and there exists a $g \in R(F) \setminus R_b$;*
8.

(H.5) *There exists an $R_1 > 0$ such that all possible solutions u of the BVP for the equation*

$$u' + D(t)u + \mu[-D(t)u + f(t,u)] = 0 \qquad t \in [a,b], \tag{5.1}$$

where $0 < \mu < 1$ and the condition (3.2)*, satisfy the inequality*

$$||u||_n \leq R_1.$$

Proof. The statement for 6 and 7 follows by the Proposition 2.8 (g) and (h), respectivaly.

Let $C : X_0 \to Y_0$ be a linear homeomorphism defined by the relation (4.5). We show that (H.5) implies (A.5) from the Proposition 2.9 for $G(u) = u$, $u \in X_0$. With respect to Remark 2.5, it is sufficient to show that every solution u of the equation $F(u) = \lambda Cu$ satisfies the estimate

$$||u||_n^1 < R, \; R > 0. \tag{5.2}$$

for all $\lambda < 0$.

By $F(u) = \lambda Cu$ we can write

$$(1-\lambda).[u' + D(t)u] - D(t)u + f(t,u) = 0.$$

The inequality $\lambda < 0$ holds if and only if $(1-\lambda)^{-1} = \mu \in (0,1)$. Hence, every solutionof the equation $F(u) = \lambda Cu$ for $\lambda < 0$ satisfies (5.1), (3.2). According to (H.5) the estimate $||u||_n < R_1$ holds. Since we can rewrite the equation $F(u) = \lambda Cu$ to

$$u' = \frac{\lambda}{1-\lambda} D(t)u - \frac{1}{1-\lambda} \mathcal{N}u,$$

then we obtain the estimate $||u'||_n < R_2, \; R_2 > 0$ similarly as in Lemma 4.5. If we put $R = \max\{R_1, R_2\}$ such we have (5.2). With respect to the Proposition 2.9 (i), the mapping F is surjective. □

THEOREM 5.3. *If R_b has the same meaning as in Theorem* 5.1 *and the assumptions* (H.1), (H.2), (H.3), (H.4) *are satisfied, then the following statement holds:*

9. *The open set $Y_0 \setminus R_b$ is dense in Y_0 and hence R_b is nowhere dense in Y_0.*

Proof. Let S and Σ denote the sets from the chapter 2. Using Lemma 4.6 and Remark 2.6 we see that $\Sigma \subset S$. The Lemmas 4.1, 4.2 and 4.3 yield the assumptions (A.1), (A.2) and (A.4) of the Proposition 2.10 (j), respectivaly.

Hence, the set $Y_0 \setminus F(S)$ is dence in Y_0. Since $Y_0 \setminus F(S) \subset Y_0 \setminus F(\Sigma) = Y_0 \setminus R_b$ then the set $Y_0 \setminus R_b$ is dence in Y_0. The set R_b is nowhere dence in Y_0. □

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