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On One-Sided Tolerance Intervals of Normal Distribution With Unknown Parameters[1]

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Abstract

In the paper there is derived an exact formula for computing one-sided tolerance factors of a normal distribution with both mean μ and variance σ^2 unknown. There are also included four approximations by means of which the one-sided tolerance factors can be computed roughly. Some of the possible applications of one-sided tolerance factors in the sampling inspection are presented on examples.

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1. INTRODUCTION

Let random sample $X_1, X_2, \dots, X_n$ be taken from distribution $N(\mu, \sigma^2)$. Parameters μ and σ^2 are unknown. Their unbiased estimators are $\overline{X}$ and S^2, where $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$, $S^2 = (n-1)^{-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$. Estimates of the parameters are computed by formulas

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{a} \quad s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2 \tag{1}$$

(x_i is a value of the variable X_i, thus the *i*th measured value).

We will find intervals, which with confidence $1-\alpha$ ($0 < \alpha < 1$) cover at least the fraction p ($0 < p < 1$) of values of the distribution $N(\mu, \sigma^2)$. Such intervals are called 100 p % tolerance intervals. The explanation in more detail can be found in [1], [2] and

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[3]. Tolerance intervals can be two-sided or one-sided. Widespread tables of two-sided tolerance factors can be found in monographs [4] and [5]. In the following text we are only dealing with one-sided tolerance intervals.

2. ONE-SIDED TOLERANCE INTERVALS

Right-hand tolerance interval is the interval $\left(-\infty, \overline{X}+kS\right)$, for which is valid

$$P\left[P(X < \overline{X}+kS) \geq p\right] = 1-\alpha \tag{2}$$

(X is a random variable normally distributed with mean μ and variability σ^2, its realization x is a value of any independent observation taken from the population, from which the random sample $X_1, X_2, \ldots, X_n$ was taken.). A realization of the right-hand tolerance interval (2) is $(-\infty, \overline{x}+ks)$.

Left-hand tolerance interval is the interval $(\overline{X}-kS, \infty)$, for which is valid

$$P\left[P\left(X > \overline{X}-kS\right) \geq p\right] = 1-\alpha \tag{3}$$

Its realization is $(\overline{x}-ks, \infty)$. The value of the factor *k* is determined so that the tolerance intervals with the confidence $1-\alpha$ cover at least fraction *p* of the values of the distribution $N(\mu, \sigma^2)$. Number $1-\alpha$ named the confidence level and constant *k* is the one-sided tolerance factor.

3. EXACT COMPUTATION OF TOLERANCE FACTORS

Let us consider random variables

$$Y = \frac{\overline{X} - \mu}{\sigma}\sqrt{n}\,, \qquad V = \frac{\nu S^2}{\sigma^2}, \qquad t' = \frac{Y+\delta}{\sqrt{V}}\sqrt{\nu}$$

The random variable Y has a standard normal distribution $N(0,1)$. It is independent from the random variable V, which has χ^2-distribution with $\nu = n-1$ degrees of freedom. The random variable t' has a non-central t distribution with a parameter of non-centrality δ $(-\infty < \delta < \infty)$. Its probability density function [6] is

$$f_\nu(t,\delta) = \frac{2^{-\frac{\nu-1}{2}}\, e^{-\frac{\nu\delta^2}{2(\nu+t^2)}}}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}\left(1+\frac{t^2}{\nu}\right)^{\frac{\nu+1}{2}}} \int_0^\infty y^\nu e^{-\frac{1}{2}\left(y-\frac{t\delta}{\sqrt{\nu+t^2}}\right)^2} dy \qquad -\infty < t < \infty \tag{5}$$

where $\Gamma(\nu) = \int_0^\infty x^{\nu-1}\, e^{-x}\, dx$ is the gamma function $(\nu > 0)$.

The non-central t distribution with the density function (5) will be denoted $t'(\nu,\delta)$. When $\delta = 0$ the distribution $t'(\nu,\delta)$ equals the central distribution $t(\nu)$.

The mean $E[t'(\nu,\delta)]$ and variance $D[t'(\nu,\delta)]$ of the distribution $t'(\nu,\delta)$ are given by [7]

$$E[t'(\nu,\delta)] = \sqrt{\frac{\nu}{2}}\,\frac{\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\,\delta = \frac{\nu}{\nu-1}\, b_\nu\, \delta \qquad (\nu > 1) \tag{6}$$

$$D[t'(\nu,\delta)] = \frac{\nu}{\nu-2}(\delta^2+1) - \frac{\nu^2}{(\nu-1)^2}\, b_\nu^2\, \delta^2 \qquad (\nu > 2) \tag{7}$$

where $$b_\nu = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\sqrt{\frac{2}{\nu}} \qquad (\nu = n-1) \tag{8}$$

Let us denote the lower 100α percentage point $(0 < \alpha < 1)$ of the distribution $t'(\nu,\delta)$ by sign $t'_\alpha(\nu,\delta)$. It follows from (5) that for any real numbers t and δ holds

$$f_\nu(t,\delta) = f_\nu(-t,-\delta) \tag{9}$$

$$t'_\alpha(\nu,\delta) = -t'_{1-\alpha}(\nu,-\delta) \tag{10}$$

The most widespread tables of percentiles of non-central *t*-distributions can be found in the monograph [8], where they are computed with accuracy to five decimal places for $\alpha =$ 0,01; 0,025; 0,05; 0,10; 0,20; 0,30; 0,70; 0,80; 0,90; 0,95; 0,975; 0,99, $v = 1\,(1)\,60$ and $\delta = 0{,}1\,(0{,}1)\,8{,}0$.

For the right-sided tolerance interval $\left(-\infty, \overline{X} + kS\right)$ is

$$Z = P\left(X < \overline{X} + kS\right) = \Phi\left(\frac{\overline{X} + kS - \mu}{\sigma}\right) \tag{11}$$

where Φ is the distribution function of a standard normal distribution $N(0,1)$. For given p $(0 < p < 1)$ the inequality $Z \geq p$ is equivalent to

$$\frac{\overline{X} + kS - \mu}{\sigma} \geq u_p = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{p} \mathrm{e}^{-\frac{t^2}{2}} dt \tag{12}$$

where u_p is the lower $100p$ percentage point of $N(0,1)$. The tolerance factor k is determined so that for the given confidence coefficient $1-\alpha$ it is fulfiled the condition $P(Z \geq p) = 1-\alpha$. By using consecutive modifications and utilize (4) we obtain

$$P\left(\overline{X} + kS \geq \mu + \sigma u_p\right) = P\left(\frac{(\overline{X} - \mu)\sqrt{n}/\sigma - u_p\sqrt{n}}{S/\sigma} \geq -k\sqrt{n}\right) =$$

$$= P\left(t' \geq -k\sqrt{n}\right) = 1 - \alpha \tag{13}$$

The random value $t' = \dfrac{\overline{X} - \mu - \sigma u_p}{S}\sqrt{n}$ has a distribution $t'\left(n-1, -u_p\sqrt{n}\right)$.

From relations (10) and (13) it follows that

$$k = -\frac{t'_{\alpha}\left(n-1, -u_p\sqrt{n}\right)}{\sqrt{n}} = \frac{t'_{1-\alpha}\left(n-1, u_p\sqrt{n}\right)}{\sqrt{n}} \tag{14}$$

For the left-sided tolerance interval $(\overline{X} - kS, \infty)$ we will also get $P(t' \le k\sqrt{n}) = 1 - \alpha$, where t' has a distribution $t'(n-1, u_p\sqrt{n})$ and k is again given by (14).

The exact computation of the one-sided tolerance factors k is closely related with the exact computation of percentiles of non-central *t*-distribution. However tables of percentiles of non-central *t*-distribution [8] cannot be used to solve this problem because they do not contain a parameter of non-centrality $\delta = u_p\sqrt{n}$. Therefore the computation of the tolerance factors k was realised in the program system Mathematica [9], which contains the percentile function $t'_\alpha(\nu, \delta)$. But for large n $(n \ge 140)$ Mathematica is not able to compute the values of percentiles by this way. This problem was solved by using standard methods of numerical integration of the density (5). The computation is extremely time consuming when the parameter of non-centrality $\delta = u_p\sqrt{n} \ge 80$. The computed values were rounded up to four decimal places.

4. APPROXIMATE COMPUTATION OF ONE-SIDED TOLERANCE FACTORS

There exist a great number of approximations of the one-sided tolerance factors [1], [10] and [11]. The Wallis´s approximation [12] is the best known and up till now the most used. It was derived on the basis of the approximation of the statistic $\overline{X} + kS$ by the normal distribution $N\left(\mu + k\sigma, \frac{\sigma^2}{n} + \frac{\sigma^2 k^2}{2n-2}\right)$. The one-sided tolerance factor k is given by the relationship

$$k \approx \frac{u_p + \sqrt{u_p^2 - AB}}{A} \tag{15}$$

where $A = 1 - \frac{u_{1-\alpha}^2}{2(n-1)}$, $B = u_p^2 - \frac{u_{1-\alpha}^2}{n}$ and the percentiles u_p a $u_{1-\alpha}$ can be computed by the relationship (12). Slightly better results are given by Jennett´s and Welch´s approximation [13] by means of the percentiles of the non-central *t*-distribution

$$t'(\nu,\delta) \approx \frac{\delta b_\nu + u_\alpha \sqrt{b_\nu^2 + \left(1-b_\nu^2\right)\left(\delta^2 - u_\alpha^2\right)}}{b_\nu^2 - u_\alpha^2\left(1-b_\nu^2\right)} \qquad (16)$$

where b_ν is given by (8). For a small δ it is convenient the approximation by van Eeden [11], which was derived by means of Cornish-Fisher expansion [14] $(\nu = n-1)$

$$t'(\nu,\delta) \approx u_\alpha + \frac{u_\alpha^3 + u_\alpha}{4\nu} + \frac{5u_\alpha^5 + 16u_\alpha^3 + 3u_\alpha}{96\nu^2} +$$
$$+\delta\left(1 + \frac{2u_\alpha^2+1}{4\nu} + \frac{u_\alpha}{4\nu}\delta + \frac{4u_\alpha^4 + 12u_\alpha^2 + 1}{32\nu^2} + \frac{u_\alpha^3 + 4u_\alpha}{16\nu^2}\delta - \frac{u_\alpha^2 - 1}{24\nu^2}\delta^2 - \frac{u_\alpha}{32\nu^2}\delta^3\right) \qquad (17)$$

Neither of the above mentioned approximations can be considered to give good results in general. This non-availability does not contain Akahira´s approximation [15]. The percentiles of the non-central *t*-distribution $x = t'(\nu,\delta)$ can be found by solving the following equation

$$\frac{b_\nu x - \delta}{\sqrt{1 + x^2\left(1-b_\nu^2\right)}} = u_\alpha - \frac{x^3\left(u_\alpha^2 - 1\right)}{24\sqrt{\left[1 + x^2\left(1-b_\nu^2\right)\right]^3}}\left(\frac{1}{\nu^2} + \frac{1}{4\nu^3}\right) \qquad (\nu = n-1) \qquad (18)$$

where b_ν is given in (8) and u_α in (12). The approximation (18) yields good results even if ν is small. When $\nu \geq 200$ there are many cases where the error is less than 10^{-4}. There is a disadvantage that the equation (18) can be solved for unknown x only by using numerical methods.

In monograph [20] to be published, tables 1a, 1b, 1c till 16a, 16b and 16c contain the values of one-sided tolerance factors *k* rounded up to four decimal places for confidence levels $1-\alpha =$ 0,001; 0,005; 0,01; 0,025; 0,05; 0,10; 0,25; 0,50; 0,75; 0,90; 0,95; 0,975; 0,99; 0,995; 0,999; 0,9999, for the values of $p =$ 0,525; 0,55(0,05) 0,70; 0,725; 0,75(0,05) 0,90; 0,91(0,01) 0,97; 0,975; 0,98; 0,99; 0,991(0,001) 0,999; 0,9999 and for the samples of sizes $n =$ 2(1) 200; 205(5) 300; 310(10) 400; 425(25) 1000; 1500;

2000(1000) 5000; 10000. In the last line (∞) there are the values of quantiles u_p of the standard normal distribution. An abridged version of the tables 5a, 5b, 5c and 10a, 10b, 10c are tables numbered by 1 and 2 respectively. The tables of tolerance factors k published up to now are neither so widespread nor accurate as those given in the monograph [20]. In [16] they are computed for four decimal places and for confidence levels $1-\alpha=$ 0,90; 0,95; 0,99; 0,995; 0,999; for the values of $p=$0,75; 0,90; 0,95; 0,99; 0,995; 0,999 and for the samples of sizes $n=$2(1) 100; 102(2) 150; 155(5) 200; 220(20) 300, though in many cases the error is already on the third decimal place. In the most detailed tables [17] the factors are computed for three decimal places and for confidence levels $1-\alpha=$0,005; 0,01; 0,025; 0,05; 0,10; 0,25; 0,50; 0,75; 0,90; 0,95; 0,975; 0,99; 0,995; for the values of $p=$0,75; 0,90; 0,95; 0,975; 0,99; 0,999; 0,9999 and for the samples of sizes $n=$2(1) 100; 102(2) 180; 185(5) 300; 310(10) 400; 425(25) 650; 700(50) 1000; 1500; 2000; 3000; 5000; 10000 and ∞.

Example 1 (variables acceptance sampling)
In variables acceptance sampling we are given generally a lower specification limit LSL or an upper specification limit USL. We assume that measurements of a quantitative quality characteristic $(x_1, x_2, \cdots, x_n)$ are realizations of a normal distribution $N(\mu,\sigma^2)$, where unknown parameters μ and σ^2 can be estimated by (1). In the case of an upper specification limit USL, a lot of products are accepted if the sampling inspection gives the result $\bar{x}+ks \le USL$. In the case of a lower specification limit LSL, a lot of products is accepted if the sampling inspection gives the result $\bar{x}-ks \ge LSL$. In both cases the one-sided tolerance factor k is tabulated. The choice of the confidence coefficient $1-\alpha$ depends upon whether the sampling plan is benefit for the producer or for the consumer. For example, the consumer chooses $1-\alpha=0{,}05$ and $p=0{,}90$. For $n=10$, the value of $k=0{,}7116$ can be found in table 1. In the case of the upper specification limit USL the lot of products is accepted if $\bar{x}+0{,}7116s \le USL$ and rejected if $\bar{x}+0{,}7116s > USL$. If this lot contains the fraction $100p\,\% = 90\,\%$ of measurements under the USL, the probability of acceptance is $1-(1-\alpha)=0{,}95.$ In the case of a lower specification limit LSL the lot

of products is accepted if $\underline{x} - 0{,}7116\,s \geq LSL$ and rejected if $\underline{x} - 0{,}7116\,s < LSL$. If this lot contains the fraction $100p\ \% = 90\ \%$ of measurements over the LSL, the probability of acceptance is $1-(1-\alpha) = 0{,}95$.

Example 2 (example 1 cotinued)

The quality characteristic of polyester 104 is viscosity which is measured in mPa·s (mili Pascal·sekunda) and the temperature is 25 °C [18]. The standard demand for the upper viscosity limit is $USL = 1\,000\ \text{mPa}\cdot\text{s}$. Polyester 104 is delivered in a constant number of tuns whose contents represent one production batch. The test results are not practically influenced by errors. A random sample yielded the $n = 10$ independent measurements:

939 945 947 945 948 941 943 944 946 940

By the W-criterion [19] was affirmed a good coincidence with the normal distribution (P value is 0,7385). We choose $1-\alpha = 0{,}05$ and $p = 0{,}90$ just like that given in example 1. Then for $n = 10$, the value of $k = 0{,}7116$ can be found in table 1. From the measurements by using (1) we can compute $\bar{x} = 943{,}8$ and $s = 3{,}0111$. Whereas

$$\bar{x} + ks = 943{,}8 + 0{,}7116 \cdot 3{,}0111 = 945{,}94 < 1\,000$$

there is not any reason to reject the lot of polyester 104. When this lot contains the fraction $100p\ \% = 90\ \%$ of measurements under the limit $USL = 1\,000\ \text{mPa}\cdot\text{s}$, then it will be accepted with probability $1-(1-\alpha) = 1-0{,}05 = 0{,}95$.

Example 3 (example 2 continued)

For data from example 2 it is required to compute the right-handed tolerance interval $(-\infty, \bar{x} + k\,s)$ with confidence $1-\alpha = 0{,}90$ which covers the fraction $p = 0{,}99$ if values from distribution $N(\mu, \sigma^2)$. Then for $n = 10$, the value of $k = 3{,}5317$ can be found in table 2. Then the upper limit is $\bar{x} + ks = 943{,}8 + 3{,}5317 \cdot 3{,}0111 = 954{,}43$ and the whole interval $(-\infty; 954{,}43)$ covers at least the fraction $100p\ \% = 99\ \%$ of the values of random sample, taken from the above mentioned production lot, with probability 0,90. On the basis of the 10 values from example 2 it can be expected with probability $0{,}90$, that at least the fraction of the $99\ \%$ measurements taken from the lot of polyester 104 will have a viscosity less then $954{,}43\ \mathrm{mPa}\cdot\mathrm{s}$.

5. CONCLUSION

Tolerance intervals are considerable in the statistical quality control. The methods of the statistical quality control are given in [21], [22].

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TABLE 1

$1 - \alpha = 0{,}05$								
p \ n	0,75	0,90	0,95	0,975	0,99	0,995	0,999	0,9999

2	-0,9346	0,1381	0,4748	0,7110	0,9539	1,1079	1,4093	1,7609
3	-0,3493	0,3345	0,6392	0,8748	1,1297	1,2958	1,6265	2,0175
4	-0,1679	0,4439	0,7434	0,9826	1,2462	1,4198	1,7681	2,1827
5	-0,0659	0,5188	0,8178	1,0607	1,3310	1,5100	1,8707	2,3018
6	0,0032	0,5749	0,8748	1,1209	1,3965	1,5797	1,9500	2,3937
7	0,0545	0,6191	0,9204	1,1693	1,4493	1,6359	2,0139	2,4676
8	0,0949	0,6553	0,9581	1,2095	1,4931	1,6826	2,0669	2,5291
9	0,1277	0,6856	0,9899	1,2435	1,5304	1,7223	2,1121	2,5813
10	0,1552	0,7116	1,0173	1,2729	1,5626	1,7566	2,1511	2,6264
11	0,1787	0,7342	1,0413	1,2986	1,5908	1,7867	2,1853	2,6661
12	0,1991	0,7542	1,0625	1,3214	1,6158	1,8134	2,2157	2,7013
13	0,2171	0,7719	1,0815	1,3418	1,6383	1,8373	2,2429	2,7328
14	0,2330	0,7879	1,0985	1,3602	1,6585	1,8589	2,2675	2,7613
15	0,2474	0,8023	1,1140	1,3770	1,6769	1,8786	2,2899	2,7872
16	0,2603	0,8155	1,1282	1,3923	1,6938	1,8966	2,3104	2,8110
17	0,2721	0,8275	1,1412	1,4063	1,7093	1,9131	2,3293	2,8329
18	0,2829	0,8387	1,1532	1,4193	1,7236	1,9284	2,3468	2,8531
19	0,2928	0,8490	1,1643	1,4314	1,7369	1,9426	2,3630	2,8719
20	0,3020	0,8585	1,1746	1,4426	1,7493	1,9559	2,3781	2,8894
25	0,3394	0,8979	1,2174	1,4891	1,8006	2,0108	2,4408	2,9622
30	0,3671	0,9276	1,2499	1,5244	1,8397	2,0527	2,4886	3,0176
35	0,3887	0,9511	1,2756	1,5525	1,8708	2,0859	2,5267	3,0618
40	0,4062	0,9703	1,2966	1,5755	1,8963	2,1133	2,5580	3,0982
45	0,4208	0,9864	1,3143	1,5948	1,9178	2,1363	2,5843	3,1288
50	0,4332	1,0001	1,3294	1,6114	1,9362	2,1561	2,6070	3,1551
60	0,4532	1,0225	1,3542	1,6385	1,9664	2,1884	2,6440	3,1983
70	0,4689	1,0401	1,3737	1,6599	1,9902	2,2140	2,6734	3,2324
80	0,4816	1,0545	1,3896	1,6774	2,0097	2,2350	2,6975	3,2604
90	0,4922	1,0665	1,4030	1,6921	2,0261	2,2525	2,7176	3,2839
100	0,5011	1,0767	1,4144	1,7047	2,0401	2,2676	2,7349	3,3040
110	0,5089	1,0856	1,4243	1,7155	2,0522	2,2806	2,7499	3,3215
120	0,5157	1,0934	1,4329	1,7251	2,0629	2,2921	2,7631	3,3369
130	0,5217	1,1003	1,4406	1,7336	2,0724	2,3023	2,7748	3,3505
140	0,5270	1,1065	1,4475	1,7412	2,0809	2,3114	2,7853	3,3628
150	0,5318	1,1120	1,4538	1,7481	2,0886	2,3197	2,7948	3,3738
160	0,5362	1,1171	1,4594	1,7543	2,0956	2,3272	2,8034	3,3839
170	0,5402	1,1217	1,4646	1,7600	2,1020	2,3341	2,8113	3,3931
180	0,5439	1,1260	1,4694	1,7653	2,1078	2,3404	2,8186	3,4016

190	0,5472	1,1299	1,4738	1,7701	2,1132	2,3462	2,8253	3,4094
200	0,5503	1,1335	1,4778	1,7746	2,1183	2,3516	2,8315	3,4167
250	0,5631	1,1484	1,4945	1,7930	2,1389	2,3738	2,8571	3,4465
300	0,5725	1,1595	1,5069	1,8068	2,1543	2,3904	2,8761	3,4687
350	0,5799	1,1682	1,5167	1,8176	2,1664	2,4034	2,8911	3,4862
400	0,5858	1,1752	1,5246	1,8263	2,1762	2,4139	2,9033	3,5005
450	0,5908	1,1810	1,5312	1,8336	2,1844	2,4228	2,9135	3,5123
500	0,5950	1,1860	1,5368	1,8398	2,1913	2,4302	2,9221	3,5224
600	0,6018	1,1941	1,5458	1,8499	2,2026	2,4424	2,9361	3,5388
700	0,6071	1,2004	1,5529	1,8578	2,2114	2,4519	2,9471	3,5516
800	0,6114	1,2055	1,5587	1,8641	2,2186	2,4596	2,9560	3,5620
900	0,6149	1,2097	1,5635	1,8694	2,2245	2,4660	2,9634	3,5706
1000	0,6179	1,2133	1,5675	1,8739	2,2296	2,4715	2,9697	3,5780
1500	0,6282	1,2256	1,5814	1,8893	2,2469	2,4901	2,9912	3,6031
2000	0,6343	1,2329	1,5897	1,8986	2,2573	2,5013	3,0041	3,6182
3000	0,6416	1,2417	1,5997	1,9097	2,2697	2,5147	3,0196	3,6364
4000	0,6460	1,2470	1,6056	1,9163	2,2772	2,5228	3,0289	3,6472
5000	0,6490	1,2506	1,6097	1,9208	2,2823	2,5283	3,0353	3,6547
10000	0,6564	1,2596	1,6199	1,9322	2,2951	2,5421	3,0512	3,6733
∞	0,6745	1,2816	1,6449	1,9600	2,3264	2,5759	3,0903	3,7191

TABLE 2

$1 - \alpha = 0,90$								
p / *n*	0,75	0,90	0,95	0,975	0,99	0,995	0,999	0,9999

2	5,8425	10,2528	13,0898	15,5860	18,5001	20,4862	24,5816	29,5872
3	2,6029	4,2582	5,3115	6,2438	7,3405	8,0924	9,6512	11,5663
4	1,9723	3,1879	3,9566	4,6370	5,4383	5,9882	7,1294	8,5330
5	1,6978	2,7424	3,3999	3,9814	4,6660	5,1360	6,1113	7,3114
6	1,5399	2,4937	3,0919	3,6205	4,2426	4,6695	5,5556	6,6457
7	1,4353	2,3327	2,8938	3,3892	3,9721	4,3719	5,2018	6,2226
8	1,3599	2,2186	2,7543	3,2269	3,7826	4,1638	4,9547	5,9275
9	1,3024	2,1329	2,6500	3,1057	3,6415	4,0089	4,7711	5,7084
10	1,2568	2,0657	2,5684	3,0113	3,5317	3,8885	4,6286	5,5385
11	1,2195	2,0113	2,5027	2,9353	3,4435	3,7918	4,5142	5,4023
12	1,1883	1,9662	2,4483	2,8725	3,3707	3,7121	4,4201	5,2903
13	1,1617	1,9281	2,4025	2,8197	3,3095	3,6452	4,3410	5,1963
14	1,1387	1,8954	2,3632	2,7745	3,2572	3,5879	4,2735	5,1161
15	1,1186	1,8669	2,3290	2,7352	3,2119	3,5384	4,2151	5,0466
16	1,1008	1,8418	2,2990	2,7008	3,1721	3,4949	4,1639	4,9858
17	1,0850	1,8195	2,2725	2,6703	3,1369	3,4564	4,1186	4,9321
18	1,0707	1,7996	2,2487	2,6430	3,1055	3,4221	4,0782	4,8841
19	1,0577	1,7816	2,2273	2,6185	3,0772	3,3912	4,0419	4,8411
20	1,0459	1,7653	2,2078	2,5963	3,0516	3,3633	4,0090	4,8021
25	0,9996	1,7016	2,1323	2,5100	2,9524	3,2551	3,8820	4,6516
30	0,9668	1,6571	2,0799	2,4502	2,8838	3,1804	3,7943	4,5478
35	0,9421	1,6239	2,0408	2,4058	2,8329	3,1249	3,7294	4,4710
40	0,9227	1,5979	2,0103	2,3711	2,7932	3,0818	3,6789	4,4114
45	0,9068	1,5769	1,9857	2,3432	2,7613	3,0471	3,6383	4,3635
50	0,8937	1,5595	1,9653	2,3201	2,7349	3,0184	3,6048	4,3239
60	0,8728	1,5321	1,9333	2,2839	2,6936	2,9735	3,5523	4,2620
70	0,8568	1,5113	1,9091	2,2565	2,6623	2,9396	3,5127	4,2154
80	0,8441	1,4948	1,8899	2,2349	2,6377	2,9128	3,4816	4,1786
90	0,8337	1,4813	1,8743	2,2173	2,6177	2,8911	3,4562	4,1488
100	0,8250	1,4701	1,8613	2,2026	2,6010	2,8730	3,4351	4,1239
110	0,8176	1,4605	1,8502	2,1901	2,5868	2,8576	3,4172	4,1028
120	0,8111	1,4523	1,8406	2,1793	2,5745	2,8443	3,4017	4,0846
130	0,8055	1,4450	1,8323	2,1699	2,5638	2,8327	3,3882	4,0687
140	0,8004	1,4386	1,8249	2,1615	2,5543	2,8224	3,3762	4,0547
150	0,7959	1,4329	1,8182	2,1541	2,5459	2,8132	3,3656	4,0421
160	0,7919	1,4277	1,8123	2,1474	2,5383	2,8050	3,3560	4,0309
170	0,7882	1,4231	1,8069	2,1413	2,5314	2,7976	3,3474	4,0207
180	0,7849	1,4188	1,8020	2,1358	2,5251	2,7908	3,3395	4,0114
190	0,7818	1,4149	1,7975	2,1308	2,5194	2,7846	3,3323	4,0030

200	0,7790	1,4113	1,7934	2,1261	2,5141	2,7789	3,3256	3,9952
250	0,7675	1,3969	1,7767	2,1074	2,4930	2,7559	3,2990	3,9638
300	0,7591	1,3863	1,7646	2,0938	2,4775	2,7392	3,2796	3,9411
350	0,7527	1,3782	1,7552	2,0833	2,4657	2,7264	3,2647	3,9236
400	0,7475	1,3717	1,7478	2,0750	2,4562	2,7161	3,2528	3,9096
450	0,7432	1,3663	1,7416	2,0681	2,4484	2,7077	3,2430	3,8981
500	0,7396	1,3618	1,7365	2,0623	2,4418	2,7006	3,2348	3,8884
600	0,7338	1,3546	1,7282	2,0530	2,4314	2,6893	3,2216	3,8730
700	0,7293	1,3490	1,7218	2,0459	2,4233	2,6805	3,2115	3,8611
800	0,7257	1,3445	1,7167	2,0401	2,4168	2,6735	3,2033	3,8516
900	0,7227	1,3408	1,7124	2,0354	2,4114	2,6677	3,1966	3,8437
1000	0,7202	1,3377	1,7089	2,0314	2,4069	2,6629	3,1910	3,8371
1500	0,7117	1,3272	1,6968	2,0180	2,3918	2,6465	3,1720	3,8148
2000	0,7067	1,3210	1,6897	2,0100	2,3828	2,6368	3,1608	3,8017
3000	0,7007	1,3136	1,6814	2,0007	2,3723	2,6254	3,1476	3,7862
4000	0,6972	1,3093	1,6764	1,9952	2,3660	2,6187	3,1398	3,7770
5000	0,6948	1,3063	1,6731	1,9914	2,3618	2,6141	3,1345	3,7708
10000	0,6888	1,2990	1,6647	1,9821	2,3513	2,6028	3,1214	3,7555
∞	0,6745	1,2816	1,6449	1,9600	2,3264	2,5759	3,0903	3,7191