Journal of Applied Mathematics, Statistics and Informatics (JAMSI), **1** (2005), No. 2

Oscillation of a Class of Higher Order Forced Functional Differential Equations

MILAN GERA AND GHOLAMHOSS G. HAMEDANI

Abstract

It is shown that the oscillatory behavioral properties of the forced nonlinear functional differential equations and its corresponding unforced equations are maintained under the effect of certain forcing terms.

Mathematics Subject Classification 2000: 34K11
Additional Key Words and Phrases: forced and unforced nonlinear differential equations, oscillatory and nonoscillatory solution

1.V INTRODUCTIONV

The main purpose of this note is to study the oscillatory behavior of the n^{th} order forced functional differential equation

$$(x(t)-c(t)\,g\,(x(t-r)))^{(n)}+a(t)\,f(x(q(t)))=e(t),\quad t\geq t_0>0. \tag{1.1}$$

In what follows we shall restrict our attention to the solution of (1.1) that exists on some ray $[T,\infty)$, where $T\geq t_0$, and that are nontrivial in any neighborhood of infinity. Such a solution is called oscillatory if it has arbitrary large zeros. Otherwise, the solution is said to be nonoscillatory.

KARTSATOS [12], starts his paper by making the following statement: "One of the major, and generally unstudied, problems in the theory of oscillation of nonlinear equations, is the problem of maintaining oscillations under the effect of a forcing term." He pointed out that the oscillation of all solutions of the unforced equation

$$x^{(n)}(t)+a(t)\,f(x(t))=0,\quad t\geq t_0, \tag{1.2}$$

is not generally maintained if one considers the forced equation, by adding the term e to the righthand side of equation (1.2) . Thus, one must impose more conditions on the function e to ensure oscillation of the equation

$$x^{(n)}(t)+a(t)\,f(x(t))=e(t),\quad t\geq t_0. \tag{1.3}$$

KARTSATOS presented oscillation criteria for equation (1.3). We refer the reader for further results in this direction to the excellent survey paper of KARTSATOS [13] which includes a large number of related references. This important problem suggested above by KARTSATOS has been investigated in recent years; we shall mention some of the related works here. DAHIYA and AKINYELLE [1] established oscillation results for the linear equation

$$x^{(n)}(t)+a(t)\,x(q(t))=e(t), \quad t\ge t_0, \tag{1.4}$$

with $n\ge 2$.

Recently, GRACE and LALLI [8] and HAMEDANI [10] presented oscillation criteria which can be applied to a class of both forced and unforced differential equations of the form

$$x^{(n)}(t)+a(t)\,f(x(q(t)))=e(t), \quad t\ge t_0. \tag{1.5}$$

It is shown that the behavioral properties of the unforced equations are maintained under the effect of certain forcing terms. For further related results we refer the reader to [6], [7], [9], [14], [16], [17] and [19].

Recently, YANG and ZHANG [20] established oscillation criteria for the unforced equation

$$(x(t)-c(t)\,x(t-r))^{(n)}+a(t)\,x(t-\sigma)=0, \quad t\ge t_0, \tag{1.6}$$

where $r,\sigma>0$ are constants, $n\ge 1$ is an odd integer, $a,c:[t_0,\infty)\to[0,\infty)$ are continuous functions. Here are their results as stated in [20].

THEOREM YZ$_1$. If

$$\liminf_{t\to\infty} t^n\int_t^\infty a(s)\,ds>\frac{M_{n+1}}{n}r, \tag{1.7}$$

where M_{n+1} is the maximum of

$$P_{n+1}(x)=x(1-x)\ldots(n-1-x)(n-x) \text{ on } (0,1),$$

and

$$c(t-\sigma)\,a(t)\ge a(t-r) \text{ eventually}, \tag{1.8}$$

and

$$\text{there exists } t^* \text{ such that } c(t^*+ir)\le 1, \quad i=0,1,2,\ldots \tag{1.9}$$

hold, then equation (1.6) is oscillatory.

THEOREM YZ$_2$. If

$$\liminf_{t\to\infty} t^n \int_t^\infty a(s)\,ds = \frac{\alpha}{n} r, \quad 0 < \alpha \le M_{n+1}, \tag{1.10}$$

and

$$\liminf_{t\to\infty} t^{n-\beta} \int_t^\infty s^\beta a(s)\,ds > \frac{M_{n+1}}{n-\beta} r, \quad 0 < \beta < \alpha_1, \tag{1.11}$$

where $0 < \alpha_1 < 1$ is a root of $P_n(x) = \alpha$, $0 < \alpha_1 \le M_n$, on $[0,1]$, and (1.8), (1.9) hold, then equation (1.6) is oscillatory.

REMARK 1.1. YANG and ZHANG [20] present oscillation criteria for (1.1) in the special case of $c(t) \ge 0, g(x) = x, f(x) = x, q(t) = t - \sigma, e(t)$ and *n* odd.

2.V MAIN\RESULTW

For the sake of completeness, we shall state a few lemmas which are basic in our discussion. They are taken from [17], [3], [15] and [11] respectively.

LEMMA 2.1. Let *x* be a positive, *n* times differentiable function on $[t_0, \infty)$. If $x^{(n)}$ is of constant sign and not identically zero in any interval $[t_1, \infty), t_1 > t_0$, then there exists $t_x > t_0$ and an integer *j*, $0 \le j \le n$ with $n + j$ even for $x^{(n)} \ge 0$ or $n + j$ odd for $x^{(n)} \le 0$; such that $j > 0$ implies $x^{(k)}(t) > 0$ for $t \ge t_x$ $(k = 0, 1, ..., j-1)$ and $j \le n-1$ implies that $(-1)^{j+k} x^{(k)}(t) > 0$ for $t \ge t_x, (k = j, j+1, ..., n-1)$.

LEMMA 2.2. Let $q \quad C[T, \infty)$, $q(t) \le t$, $\lim_{t\to\infty} q(t) = \infty$, and let $x(t) \quad C^2[T, \infty), x(t) > 0, x'(t) > 0,$ and $x''(t) \le 0$ for $t \ge T$. Then for each $k_1 \quad (0,1)$ there exists a $T_{k_1} \ge T$ such that

$$x(q(t)) \ge k_1 \frac{q(t)}{t} x(t), \quad t \ge T_{k_1}.$$

LEMMA 2.3. Let $x \quad C^2[T, \infty)$ with $x(t) > 0, x'(t) > 0,$ and $x''(t) \le 0$ for $t \ge T$. Then for each $k_2 \quad (0,1)$ there is a $T_{k_2} \ge T$ such that

$$x(t) \ge k_2 t x'(t), \quad \text{for} \quad t \ge T_{k_2}.$$

LEMMA 2.4. Let $x \quad C^3[T,\infty)$, and let

$$x(t)>0,\ x'(t)>0,\ x''(t)>0,\ x'''(t)<0, \text{ for } t\geq T.$$

Then for each $\varepsilon \quad \left(0,\frac{1}{2}\right)$ there is a $t_\varepsilon \geq T$ such that

$$x(t)\geq \varepsilon t x'(t), \quad \text{for} \quad t\geq t_\varepsilon .$$

That following lemma is similar to Lemma 5.1.4 of [4]; its proof follows that of Lemma 5.1.4 as well.

LEMMA 2.5. Assume that $n\geq 1$ is an odd integer, r is a positive number and

(i) $c,a,q,e \quad C([t_0,\infty))$, $a(t)\geq 0$ for $t\geq t_0$, $a\neq 0$ in each subinterval $[t_0,+\infty)$, $f,g \quad C(R)$, and $xf(x)>0$ for $x\neq 0$;

(ii) $q(t)\leq t$ for $t\geq t_0$ and $\lim_{t\to\infty} q(t)=\infty$;

(iii) there exists a continuous function $\eta:[t_0,\infty)\to R$ such that $\eta^{(n)}(t)=e(t)$, and $\lim_{t\to\infty}\eta(t)=0$;

(iv) $c\geq 0$ and there exists $t^*\geq t_0$ such that $c(t^*+ir)\leq L$ for a positive number L and $i=0,1,\ldots$;

(v) $\frac{g(u)}{u}\leq \frac{1}{L}$ for $u\neq 0$, where L as in (iv).

If x is an eventually positive (negative) solution of the inequality

$$\left(x(t)-c(t)g(x(t-r))\right)^{(n)}+a(t)f(x(q(t)))-e(t)\leq(\geq 0), \quad t\geq t_0 \tag{2.1}$$

and if

$$y(t)=x(t)-c(t)\,g(x(t-r))-\eta(t), \quad t\geq t_0 \tag{2.2}$$

then $y>0(<0)$ eventually.

PROOF. In view of (2.1), (2.2) and (i)-(iii), for sufficiently large t we have

$$y^{(n)}(t)\leq -a(t)\,f(x(q(t)))\leq 0 .$$

Since $a\neq 0$ in each subinterval $[t_0,+\infty)$, $y^{(n-1)}(t)$ is eventually positive or eventually negative. Hence by Lemma 2.1, $y^{(k)}(t)$ is strictly monotonic for $k=0,1,\ldots,n-2.$

Thus $y(t)$ is eventually positive or eventually negative. If $y(t)$ is eventually negative, then $y^{(n)}(t)y(t)\geq 0$ eventually and since n is odd, by Lemma 2.1, $y'(t)<0$ eventually for $n\geq 3$. In the case $n=1, y'\leq 0$ and not identically zero in any subinterval. This implies y is decreasing function. Therefore, there exists $l>0$ and $t_1\geq t_0$ such that $y(t)\leq -l,\ t\geq t_1$. Now, in view of (iii) there exists $t_2\geq t_1$ such that

$$x(t)<-\frac{l}{2}+c(t)\,g(x(t-r)),\ t\geq t_2.$$

From (v) we have

$$x(t)<-\frac{l}{2}+\frac{1}{L}c(t)\,x(t-r),\ t\geq t_2.$$

In view of (iv), we can choose the positive integer k so that $t^*+kr\geq t_2$ and hence

$$x\left(t^*+ir\right)<-\frac{l}{2}+x\left(t^*+(i-1)r\right),\quad i=k,\quad k+1,\ldots \tag{2.3}$$

From (2.3) we obtain

$$x\left(t^*+ir\right)<-(i-(k-1))\frac{l}{2}+x\left(t^*+(k-1)r\right),\quad i=k,\quad k+1,\ldots$$

Now as $i\to\infty,\ -(i-(k-1))\frac{l}{2}+x\left(t^*+(k-1)r\right)\to-\infty$, which is impossible. Thus, $y(t)$ must be eventually positive. In the case x is eventually negative solution of (1.1), the proof of lemma can be provided in the same way.

REMARK 2.6. If $c\leq 0$ for $t\geq t_0$, then conclusion of Lemma 2.5 holds also under (i)-(iii) and $ug(u)\geq 0$ for $u\neq 0$.

THEOREM 2.7. Let $n\geq 3$ be odd and in addition to (i)-(v) of Lemma 2.5 assume that

(vi) $ug(u)\geq 0$ for $u\neq 0$;

(vii) $\liminf\limits_{|u|\to\infty}\frac{f(u)}{u}>0$;

$$\text{(viii)} \quad \begin{cases} \displaystyle\int_{t_0}^{\infty} a(s)ds = \infty & \text{or} \\ \displaystyle\int_{t_0}^{\infty} a(s)\frac{q(s)}{s}ds < \infty \quad \text{and} & \displaystyle\liminf_{t\to\infty} t\int_{t}^{\infty} a(s)\frac{q(s)}{s}ds > 0 \end{cases}$$

Then every solution x of (1.1) is either oscillatory or nonoscillatory and $\liminf_{t\to\infty}|x(t)| = 0.$

PROOF. Let x be a positive nonoscillatory solution of equation (1.1) on an interval $[t_1, \infty), t_1 \geq t_0$. Let $t_2 \geq t_1$ be chosen so that $x(t-r) > 0$ and $x(q(t)) > 0$ for $t \geq t_2$, and let $y(t)$ be defined by (2.2).

In view of (i) and (ii), from equation (1.1) we have

$$y^{(n)}(t) = -a(t)\, f(x(q(t))) \leq 0 \quad \text{for} \quad t \geq t_2. \tag{2.4}$$

By Lemma 2.1, there exists $t_3 \geq t_2$ such that $y^{(k)}(t),\ k = 0, 1, ..., n-1$ are monotonic and one-signed for $t \geq t_3$. Since $x > 0$, by Lemma 2.5, y is eventually positive, i.e., there exists $t_4 \geq t_3$ such that

$$y(t) > 0 \quad \text{for} \quad t \geq t_4.$$

From Lemma 2.1, we see that

$$y^{(n-1)}(t) > 0 \quad \text{for} \quad t \geq t_4.$$

Integrating (2.4) from t to ∞ $(t \geq t_4)$, we have

$$y^{(n-1)}(t) \geq \int_{t}^{\infty} a(s)\, f(x(q(s)))\, ds. \tag{2.5}$$

Now, let j (even) be as in Lemma 2.1 and let $\liminf_{t\to\infty} x(t) > 0$. Then exists a constant $M > 0$ and $t_5 \geq t_4$ such that

$$x(q(t)) \geq M \quad \text{for} \quad t \geq t_5.$$

If x is bounded, so $\min\limits_{s \ge t_5} f(x(q(s))) = \alpha > 0$. Let x is unbounded. In view of (vii), for any $\xi,\ 0 < \xi < \liminf\limits_{|u| \to \infty} \frac{f(u)}{u}$, there exists $u_1 > \mathrm{M}$ such that $\frac{f(u)}{u} > \xi$. Let

$$E_1 = \{t \quad [t_5, \infty); x(q(t)) > u_1\},\ E_0 = \{t \quad [t_5, \infty); M \le x(q(t)) \le u_1\}.$$

Then $f(x(q(t))) > \xi x(q(t)) \ge \xi M$ for $t \quad E_1$ and $\min\limits_{t \quad E_0} f(x(q(t))) = \alpha_0 > 0$. Therefore

$$f(x(q(t))) \ge d \quad \text{for} \quad t \ge t_5,$$

where $d = \alpha$ or $d = \min\{\alpha_0, \xi M\}$.

If $\int_{t_0}^{\infty} a(s)ds = \infty,$ then integrating (2.4) from t_5 to $t, t \ge t_5,$ we get

$$y^{(n-1)}(t) = y^{(n-1)}(t_5) - \int_{t_5}^{t} a(s) f(x(q(s))ds \le y^{(n-1)}(t_5) - d\int_{t_5}^{t} a(s)ds .$$

Hence it follows $\lim\limits_{t \to \infty} y^{(n-1)}(t) = -\infty$, which is a contradiction with $y^{(n-1)} > 0$. If $\int_{t_0}^{\infty} a(s)ds < \infty,$ from (2.5) we obtain

$$y^{(n-1)}(t) \ge d\int_{t}^{\infty} a(s)ds, \quad t \ge t_5$$

and hence

$$y^{(n-2)}(t) \ge y^{(n-2)}(t_5) + d\int_{t_5}^{t}\int_{\tau}^{\infty} a(s)ds d\tau,\ t \ge t_5. \qquad (2.6)$$

Now from (2.6), in view of (viii), we have for sufficiently large $t, y^{(n-2)}(t) > 0$. This implies that j must be $n-1$. Thus $y(t) > 0, y'(t) > 0, \ldots, y^{(n-1)}(t) > 0$ for $t \ge t_6 \ge t_5$ and so $y(t) \to \infty$ as $t \to \infty$. By Lemma 2.3, for $k_2 \quad (0,1)$, there exists $t_7 \ge t_6$ such that

$$y^{(n-2)}(t) \ge k_2 t y^{(n-1)}(t),\ t \ge t_7. \qquad (2.7)$$

Using (2.5) we obtain

$$y^{(n-2)}(t) \geq k_2 \left(t \int_t^\infty a(s)\, x(q(s))\, ds \right) \inf_{s \geq t} \frac{f(x(q(s)))}{x(q(s))}, \ t \geq t_7. \tag{2.8}$$

Because $c(t) \geq 0$ and in view (vi) $g(x(t-r)) \geq 0$ for $t \geq t_7$, we have that $x(t) \geq y(t) + \eta(t)$ for $t \geq t_7$. In regard of this and the fact that $y(t) \to \infty, \eta(t) \to 0$ as $t \to \infty$, we have $x(t) \to \infty$ as $t \to \infty$. Therefore by (vii), for each ξ, $0 < \xi < \liminf_{|u| \to \infty} \frac{f(u)}{u}$, there is a $t_8 \geq t_7$ such that

$$\inf_{s \geq t} \frac{f(x(q(s)))}{x(q(s))} \geq \xi, \ t \geq t_8.$$

Further, since $y'(t) > 0, t \geq t_8$ for each $\theta \quad (0,1)$, there exists $t_9 \geq t_8$ such that

$$x(q(t)) \geq \theta y(q(t)), \ t \geq t_9. \tag{2.10}$$

By Taylor´s formula, in the case $n \geq 5$, it can be shown that for each $\varepsilon \quad (0,1)$ there exists $t_{10} \geq t_9$ such that

$$y(q(t)) \geq \varepsilon q(t) y^{(n-2)}(q(t)) \quad \text{for} \quad t \geq t_9 \tag{2.11}$$

and in the case $n = 3$, by Lemma 2.4 for each $\varepsilon \quad \left(0, \frac{1}{2}\right)$ also (2.11) holds. By Lemma 2.2, for $k_1 \quad (0,1)$, there is $t_{10} \geq t_9$ such that

$$y^{(n-2)}(q(t)) \geq k_1 \frac{q(t)}{t} y^{(n-2)}(t) \quad \text{for} \quad t \geq t_{10} \tag{2.12}$$

Now from (2.8), (2.9), (2.10), (2.11) and (2.12), it follows that to $t \geq t_{10}$

$$y^{(n-2)}(t) \geq t \int_t^\infty a(s)\, \varepsilon \theta k_1 k_2\, \xi\, q(s) \frac{q(s)}{s} y^{(n-2)}(s)\, ds. \tag{2.13}$$

Since $q(t) \to \infty$ as $t \to \infty$, for any k positive, there exists $t_{11} \geq t_{10}$ such that $\varepsilon \theta k_1 k_2\, \xi\, q(t) \geq k$ for $t \geq t_{11}$. In view of this and the fact that $y^{(n-2)}$ is increasing, we have, for $t \geq t_{11}$,

$$1 \geq kt \int_t^\infty a(s) \frac{q(s)}{s} ds . \tag{2.14}$$

Hence

$$\liminf_{t\to\infty} t \int_t^\infty a(s) \frac{q(s)}{s} ds \leq \frac{1}{k}. \tag{2.15}$$

Because this holds for each $k > 0$, so $\liminf_{t\to\infty} t \int_t^\infty a(s) \frac{q(s)}{s} ds \leq 0$, which is a contradiction to (viii). If x is an eventually negative the proof theorem is possibly to provide analogous. The theorem is proved.

COROLLARY 2.8. Let $e = 0$, then the conclusion of Theorem 2.7 holds.

COROLLARY 2.9. Let $q(t) = t$ in (1.1), then the conclusion of Theorem 2.7 holds.

THEOREM 2.10. Let $n \geq 3$ be odd and in addition to (i)-(iii) of Lemma 2.5 and (vi) – (viii) of Theorem 2.7, assume that

(ix) g is bounded function;

(x) $c \leq 0$ and c is bounded function.

Then every solution x of (1.1) is either oscillatory or nonoscillatory and $\liminf_{t\to\infty} |x(t)| = 0.$

PROOF. Let x be a positive nonoscillatory solution of (1.1) on an interval $[t_1, \infty), t_1 \geq t_0$. Follow the proof of Theorem 2.7 to arrive at (2.8). Because c, q, η are bounded functions and $y(t) \to \infty$ as $t \to \infty$ so it is also $x(t) \to \infty$ as $t \to \infty$. Further, the proof of this theorem we can proceed in the same way as that of Theorem 2.6. We get (2.9)-(2.15) and finally we arrive at a contradiction.

COROLLARY 2.11. Let $e = 0$, then the conclusion of Theorem 2.10 holds.

THEOREM 2.12. Let the conditions of Theorem 2.7 (Theorem 2.10) be satisfied and

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} = \infty.$$

Moreover, let $q^2(t) \le t$ for $t \ge t_0$. Then conclusion of Theorem 2.7 (Theorem 2.10) holds also under the assumption that (viii) is replaced by

$$\text{(xi)} \qquad \int_t^\infty a(s)\frac{q^2(s)}{s}ds < \infty \quad \text{and} \quad \liminf_{t\to\infty} t\int_t^\infty a(s)\frac{q^2(s)}{s}ds > 0.$$

PROOF. If x is a nonoscillatory positive solution of (1.1), then similarly as in the proof of Theorem 2.7 (or Theorem 2.10) we come to the inequality (2.13). Now for any k positive, we can take ξ sufficiently large so as to $\varepsilon\theta k_1 k_2 \xi \ge k$. Further proceeding as in the proof of Theorem 2.7 we obtain the inequalities

$$1 \ge kt\int_t^\infty a(s)\frac{q^2(s)}{s}ds, \quad t \ge t_{11},$$

$$\liminf_{t\to\infty} t\int_t^\infty a(s)\frac{q^2(s)}{s}ds \le \frac{1}{k},$$

which leads finally to contradiction with (xi).

COROLLARY 2.13. Let $e = 0$, then the conclusion of Theorem 2.12 holds.

REMARK 2.14. (a) It is clear, from Theorem 2.7 and Corollary 2.8 (as well as from Theorem 2.10 and Corollary 2.11 or Theorem 2.12 and Corollary 2.13) that the oscillation of the unforced equation

$$\left(x(t) - c(t)g(x(t-r))\right)^{(n)} + a(t)f(x(q(t))) = 0$$

is maintained under the effect of certain forcing terms.

(b) Our results can directly be extended to the forced functional equation

$$\left(x(t) - c(t)\,g\,(x\,(t-r))\right)^{(n)} + a(t)f(x\,(q_1(t)),\ldots,x\,(q_m(t))) = e(t),\ t \ge t_0,$$

under appropriate conditions imposed on the functions involved.

3. EXAMPLES

EXAMPLE 3.1. Consider the forced and associated unforced linear differential equations

$$\left(x(t) - c(t)\,x(t-1)\right)''' + \frac{k}{t^\varepsilon}x(t) = \frac{-10}{t^4}\left[\cos\ln t + \sin\ln t\right],\ t \ge 2, \tag{3.1}$$

and

$$\left(x(t)-c(t)\,x(t-1)\right)''' + \frac{k}{t^{\varepsilon}}\,x(t)=0\,,\ t\geq 2\,,$$

(3.2)

where $1<\varepsilon\leq 2,\ k>0$ and $c(t)\geq 0$ satisfies condition (iv) of Theorem 2.7 (e.g. $c(t)=1+\frac{1}{2}\sin 2\pi t$ or $c(t)=1+\frac{1}{2}\cos 2\pi t$).

Here

$$\eta(t)=\frac{\cos\ln t-\sin\ln t}{t} \tag{3.3}$$

and condition of Corollaries 2.8 and 2.9 are satisfied for equations (3.2) and (3.1) respectively. Thus, every solution *x* of (3.1) or (3.2) is either oscillatory or nonoscillatory and $\liminf_{t\to\infty}|x(t)|=0$.

EXAMPLE 3.2. Consider the equation

$$\left(x(t)-c(t)\,x(t-1)\right)''' + \frac{k}{t^{2}}\,x(t)=0\,,\ t\geq 2\,, \tag{3.4}$$

where $k>0$ and c satisfies (iv).

It is easy to see that all the conditions of Corollary 2.8 are satisfied and every solution *x* of (3.4) is oscillatory or nonoscillatory and $\liminf_{t\to\infty}|x(t)|=0$.

EXAMPLE 3.3. The nonlinear equation

$$\left(x(t)-c(t)x(t-1)\right)^{(n)} + \frac{k}{t^{2}}\,x(t)e^{|\sin x(t)|}=(-1)^{n}\,\frac{n!}{t^{n+1}}\,,\ t\geq 1,$$

where $n\geq 3$ is an odd integer satisfies the conditions of Theorem 2.7 (or Theorem 2.10) for $k>0$ and appropriate c. Here $\eta(t)=\frac{1}{t}$.

EXAMPLE 3.4. The equation

$$\left(x(t)-c(t)\,x\,(t-1)\right)''' + \frac{2}{t}\,x\left(t^{\beta}\right)=e(t),\ \ t\geq 1,$$

where $0<\beta<1$ and c and e are appropriate functions, and its corresponding unforced equation satisfy the conditions of Theorem 2.7 (Theorem 2.10).

EXAMPE 3.5. Consider the unforced nonlinear differential equation

$$\left(x(t)-\alpha\, arctg\, x(t-1)\right)' + \left(\frac{e^{-\frac{1}{t}}}{t^2} - \alpha \frac{e^{\frac{1}{t-1}-\frac{2}{t}}}{1+e^{-\frac{2}{t}}} \frac{1}{(t-1)^2} \right) x\,(\mathrm{t}/2) = 0, \quad t \geq 2,$$

where $\alpha < 1$.

This equation has a solution

$$x(t) = e^{\frac{1}{t}},\ t \geq 2.$$

We observe that the conclusion of Theorem 2.7 $(0 \leq \alpha < 1)$ and Theorem 2.10 $(\alpha \leq 0)$ proves false for $n = 1$.

EXAMPLE 3.6. The linear differential equation

$$x''(t) + \frac{1}{2t^2} x\,(\mathrm{t}/4) = 0, \quad t \geq 1,$$

has a solution

$$x(t) = k\sqrt{t}, \quad t \geq 1,$$

where k is an arbitrary constant. We see, if $n = 2$, and all remaining assumptions of Theorem 2.7 (or Theorem 2.10) are satisfied, so its assertion need not be true.

EXAMPLE 3.7. Consider the linear differential equation

$$x''(t) + \frac{2}{9t^2} x\,(t) = 0, \quad t \geq 1.$$

Every solution of this equation has the form

$$x(t) = k_1 t^{1/3} + k_2 t^{2/3}, \quad t \geq 1,$$

where k_1, k_2 are constants.

Again, we notice that in the case $n = 2$, the conclusion of Theorem 2.7 and Theorem 2.10 may fail to hold true although all remaining hypotheses of these theorems are fulfilled.

EXAMPLE 3.8. The unforced nonlinear functional differential equation

$$x'''(t) + \frac{21}{64t^2} x^3\,(\mathrm{t}^{1/7}) = 0, \quad t \geq 1$$

has a nonoscillatory solution

$$x(t) = t^{7/4},\ t \geq 1.$$

Condition (viii) of Theorem 2.7 (Theorem 2.10) and also (xi) of Theorem 2.12 is not satisfied.

REFERENCES

[1] DAHIYA, R.S., AND AKINYELE, O. Oscillation Theorems of n_{th} Order Functional Differential Equations with Forcing Terms. *J. Math. Anal. Appl. 109*, *1985*, 325-332.

[2] DAS, P. On Oscillations of Third Order Forced Equations. *J. Math. Anal. Appl. 196*, 1995, 502-513.

[3] ERBE, L.H. Oscillation Criteria for Second Order Nonlinear Delay Equations. *Canad. Math. Bull. 16*, 1973, 49-56.

[4] ERBE, L. H., KONG, Q., AND ZHANG, B. G. *Oscillation Theory for Functional Differential Equations*. Marcell Dekker, 1995

[5] GRACE, S. R. Oscillation Theorems of Comparison Type for Neutral Nonlinear Functional Differential Equations. *Czech. Math. J. 45*, 1995, 609-626.

[6] GRACE, S. R. On the Oscillation of Certain Forced Functional Differential Equations. *J. Math. Anal. Appl. 202*, 1996, 555-577.

[7] GRACE, S. R., AND LALLI, B. S. On Oscillation of Higher Order Forced Functional Differential Inequalities. *J. Math. Anal. Appl. 91*, 1983, 525-535.

[8] GRACE, S. R., AND LALLI, B. S. Asymptotic and Oscillatory Behavior of n^{th} Order Forced Functional Differential Equations. *J. Math. Anal. Appl. 140*, 1989, 10-25.

[9] GRACE, S. R., AND LALLI, B. S. Oscillatory Criteria for Forced Neutral Differential Equations. *Czech. Math. J. 44*, 1994, 713-724.

[10] HAMEDANI, G. G. Oscillatory Behavior of n^{th} Order Forced Functional Differential Equations. *J. Math. Anal. Appl. 195*, 1995, 123-134.

[11] HAMEDANI, G. G., AND KRENZ, G. S. Oscillatory Properties for n^{th} Order Functional Differential Equations. *Indian J. Pure Appl. Math. 21*, 1990, 309-319.

[12] KARTSATOS, A. G. On the Maintenance of Oscillations under the Effect of a Small Forcing Term. *J. Differential Equations 10*, 1971, 355-363.

[13] KARTSATOS, A. G. Recent Results on Oscillations of Solutions of Forced and Perturbed Nonlinear Differential Equations of Even Order. *Stability of Dynamical Systems: Theory and Application, Lecture Notes in Pure and Applied Mathematics 28,* Dekker, New York, 1977, 17-72

[14] KARTSATOS, A. G. The Oscillation of a Forced Equation Implies the Oscillation of the Unforced Equation-Small Forcing. *J. Math. Anal. Appl. 76*, 1980, 98-106.

[15] OHRISKA, J. Oscillation of Second Order Delay and Ordinary Differential Equations. *Czech. Math. J. 34*, 1984, 107-112.

[16] PARHI, N., AND MOHANTY, P. K. Oscillation of Solutions of Forced Neutral Differential Equation of n^{th} Order. *Czech. Math. J. 45*, 1995, 413-433.

[17] STAIKOS, V. A. Basic Results on Oscillations for Differential Equations. *Hiroshima Math. J. 10*, 1980, 495-516.

[18] ŠEVELO, V. N. *Oscillation of Solutions of Differential Equations with Deviating Argument*. Kijev, Naukova dumka. (Russian), 1978

[19] WONG, P. J. Y., AND AGARWAL, R. P. Oscillatory Behavior of Solutions of Certain Second Order Nonlinear Differential Equations. *J. Math. Anal. Appl.* *198*, 1996, 337-354.

[20] YANG, B., AND ZHANG, B. G. Oscillation of a Class of Higher Order Neutral Differential Equations. *Math. Nachr.* *193*, 1998, 243-253.

Milan Gera
Department of Applied Mathematics,
Uni ersity of SS. Cyril and Metodius in Trna a
Slo ak Republic
e-mail: geramv ucm.sk

G. G. Hamedani
Department of Mathematics and Computer Science,
Marquette Uni ersity
Milwaukee, WI 53201-1881
USA

Received February 2005