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On the Solvability of Multi-point Boundary Value Problems for the Generalized Equation of the Pantograph with Constant and Proportional Delays

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Abstract

On the bounded interval $I = [0,T]$, we construct new efficient criteria for the solvability of the generalized equation of the pantograph

$$x'(t) = \sum_{i=1}^{m} P_i(t)x(\tau_i(t)) + f(t)$$

with multi-point condition

$$x(t) = u(t) \text{ for } t < 0, \quad \sum_{k=1}^{\nu} A_k x(t_k) = c_0,$$

where $T > 0$, $P_i \in L(I, \mathbb{R}^{n\times n})$ for $i = 1,\dots,m$, $f \in L(I,\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$, $t_k \in I$, $A_k \in \mathbb{R}^{n\times n}$ ($k = 1,\dots,\nu$), $\tau_i : I \to \mathbb{R}$ ($i = 1,\dots,m$) are absolutely continuous nondecreasing delays (i.e. $\tau_i(t) \le t$ for every $t \in I$), and $u : [-\infty, 0] \to \mathbb{R}^n$ is a continuous and bounded vector function.

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1. FORMULATION OF THE PROBLEM

On the bounded interval $I = [0,T]$, we consider the system of linear differential equations with deviating arguments

$$x'(t) = \sum_{i=1}^{m} P_i(t)x(\tau_i(t)) + f(t) \tag{1}$$

with the multi-point boundary condition

$$x(t) = u(t) \text{ for } t < 0, \quad \sum_{k=1}^{\nu} A_k x(t_k) = c_0, \tag{2}$$

where $T > 0$, $P_i \in L(I, \mathbb{R}^{n\times n})$ for $i = 1,\dots,m$, $f \in L(I,\mathbb{R}^n)$, $c_0 \in \mathbb{R}^n$, $t_k \in I$, $A_k \in \mathbb{R}^{n\times n}$ ($k = 1,\dots,\nu$), $\tau_i : I \to \mathbb{R}$ ($i = 1,\dots,m$) are absolutely continuous nondecreasing delays (i.e. $\tau_i(t) \le t$ for every $t \in I$), and $u :]-\infty, 0[\to \mathbb{R}^n$ is a continuous and bounded vector function.

By a solution of the boundary value problem (1), (2) we understand an absolutely continuous vector function $x : I \to \mathbb{R}^n$ which satisfies equation (1) almost everywhere on I and verifies condition (2).

Remark 1.1. If $m = 2$ and $\tau_1(t) \equiv t$ then system (1) represents equation of the pantograph that is frequently studied in the literature (usually for $n = 1$). The paper [Ockendon and Tayler 1971], about simulation of an electricity transmission between wiring and locomotive, increased interest in the equation of the pantograph. Authors studied an asymptotic behaviour of solutions for $t \to +\infty$ and their numerical approximations (see [Čermák 2000], [Čermák and Kundrát 2004], [Kundrát], [Lehninger and Liu 1998], [Lim 1976] and references therein). They used criteria of the existence and uniqueness of a solution of the Cauchy problem.

In this paper, we construct new efficient criteria for the solvability of the generalized equation of the pantograph with multi-point condition. It is clear that the boundary condition (2) also covers the initial condition

$$x(t) = u(t) \text{ for } t < 0, \quad x(t_0) = c_0, \tag{3}$$

where $t_0 \in I$, and the periodic condition

$$x(t) = u(t) \text{ for } t < 0, \quad x(T) - x(0) = c_0. \tag{4}$$

Note also that some solvability criteria for the linear system with one delayed argument and linear system of functional differential equations (in general meaning) are published in [Kuchyňková 2003].

2. BASIC NOTATION

$I = [0,T]$, $\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$;

χ_I is the characteristic function of the interval I, i.e.,

$$\chi_I(t) = \begin{cases} 1 & \text{for } t \in I \\ 0 & \text{for } t \notin I \end{cases};$$

$\mathbb{R}^n$ is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with the elements $x_i \in \mathbb{R}$ $(i = 1, \dots, n)$ and the norm

$$||x|| = \sum_{i=1}^n |x_i|;$$

$\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices $X = (x_{ik})_{i,k=1}^n$ with the elements $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$ and the norm

$$||X|| = \sum_{i,k=1}^n |x_{ik}|;$$

$$\mathbb{R}_+^n = \{(x_i)_{i=1}^n \in \mathbb{R}^n : x_i \geq 0 \ (i = 1, \dots, n)\};$$

$$\mathbb{R}_+^{n\times n} = \{(x_{ik})_{i,k=1}^n \in \mathbb{R}^{n\times n} : x_{ik} \geq 0 \ (i, k = 1, \dots, n)\};$$

if $x, y \in \mathbb{R}^n$ and $X, Y \in \mathbb{R}^{n\times n}$ then

$$x \leq y \Leftrightarrow y - x \in \mathbb{R}_+^n, \qquad X \leq Y \Leftrightarrow Y - X \in \mathbb{R}_+^{n\times n},$$

$$|x| = (|x_i|)_{i=1}^n, \qquad |X| = (|x_{ik}|)_{i,k=1}^n;$$

X^{-1} is the inverse matrix to the matrix $X \in \mathbb{R}^{n\times n}$;
$r(X)$ is the spectral radius of the matrix $X \in \mathbb{R}^{n\times n}$;
E is the unit matrix;
Θ is the zero matrix;
$C(I,\mathbb{R}^n)$ is the space of continuous vector functions $x : I \to \mathbb{R}^n$ with the norm

$$||x||_C = \max\{||x(t)|| : t \in I\};$$

if $x = (x_i)_{i=1}^n \in C(I,\mathbb{R}^n)$, then

$$|x|_C = (||x_i||_C)_{i=1}^n;$$

$\widetilde{C}(I,\mathbb{R}^n)$ is the space of absolutely continuous vector functions $x : I \to \mathbb{R}^n$;
$C(I,\mathbb{R}^{n\times n})$ is the set of continuous matrix functions $X : I \to \mathbb{R}^{n\times n}$; if $X = (x_{ik})_{i,k=1}^n \in C(I,\mathbb{R}^{n\times n})$ then

$$|X|_C = (||x_{ik}||_C)_{i,k=1}^n;$$

$L(I,\mathbb{R}^n)$ is the space of integrable vector functions $x : I \to \mathbb{R}^n$ with the norm

$$||x||_L = \int_0^T ||x(t)||\,\mathrm{d}t;$$

if $x = (x_i)_{i=1}^n \in L(I,\mathbb{R}^n)$ then

$$|x|_L = (||x_i||_L)_{i=1}^n;$$

$L(I,\mathbb{R}^{n\times n})$ is the space of Lebesque integrable matrix functions $X : I \to \mathbb{R}^{n\times n}$;
if $X = (x_{ik})_{i,k=1}^n \in L(I,\mathbb{R}^{n\times n})$ then

$$|X|_L = (||x_{ik}||_L)_{i,k=1}^n.$$

3. MAIN RESULTS

In this section, we establish some efficient criteria of the unique solvability of the problem (1), (2) using the results and methods from [Kiguradze and Půža 2003].

For any $i = 1,\dots,m$ and $t \in I$, we put

$$\tau_i^0(t) = \begin{cases} 0 & \text{if } \tau_i(t) < 0 \\ \tau_i(t) & \text{if } 0 \le \tau_i(t) \end{cases}.$$

PROPOSITION 3.1. *The problem* (1), (2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sum_{i=1}^{m} \chi_I(\tau_i(t))P_i(t)x(\tau_i^0(t)), \tag{1_0}$$

$$\sum_{k=1}^{\nu} A_k x(t_k) = 0 \tag{2_0}$$

has only the trivial solution.

THEOREM 3.2. *Let* $t_0 = \min\{t_k : k = 1,2,\dots,\nu\}$ *and let there exist matrices* $B_i \in \mathbb{R}_+^{n\times n}$ $(i = 1,\dots,m)$ *such that, for* $i = 1,\dots,m$,

$$\chi_I(\tau_i(t))|P_i(t)| \le B_i\tau_i'(t) \quad \textit{almost everywhere on } I. \tag{5}$$

Let, moreover, either the matrix

$$\Lambda_1 = \sum_{k=1}^{\nu} A_k \tag{6}$$

be nonsingular and $r(S_1) < 1$, *where*

$$S_1 = \sum_{i=1}^{m} B_i \tau_i^0(T) + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{m} |A_k| B_i (\tau_i^0(t_k) - \tau_i^0(t_0)), \tag{6_1}$$

or $\Lambda_1 = \Theta$, *the matrix*

$$\Lambda_2 = \sum_{k=1}^{\nu} \sum_{i=1}^{m} A_k \int_{t_0}^{t_k} \chi_I(\tau_i(t)) P_i(t)\, \mathrm{d}t \tag{7}$$

be nonsingular and $r(S_2) < 1$, *where*

$$S_2 = \sum_{i=1}^{m} B_i \tau_i^0(T) + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{m} \sum_{j=1}^{m} |A_k| B_i B_j (\tau_i^0(t_k) - \tau_i^0(t_0)) \tau_j^0(t_k). \tag{7_1}$$

Then the problem (1), (2) *has a unique solution.*

Corollary 3.3. *Let there exist matrices* $B_i \in \mathbb{R}_+^{n \times n}$ $(i = 1, \dots, m)$ *such that the conditions* (5) *hold for* $i = 1, \dots, m$. *Let, moreover,* $r(S) < 1$, *where*

$$S = \sum_{i=1}^{m} B_i \tau_i^0(T).$$

Then the problem (1), (3) *has a unique solution.*

Corollary 3.4. *Let there exist matrices* $B_i \in \mathbb{R}_+^{n \times n}$ $(i = 1, \dots, m)$ *such that the conditions* (5) *hold for* $i = 1, \dots, m$. *Let, moreover,*

$$\Lambda = \sum_{i=1}^{m} \int_0^T \chi_I(\tau_i(t)) P_i(t)\, \mathrm{d}t$$

be a nonsingular matrix and $r(S) < 1$, *where*

$$S = \sum_{i=1}^{m} B_i \tau_i^0(T) + |\Lambda_2^{-1}| \sum_{i=1}^{m} \sum_{j=1}^{m} B_i B_j \tau_i^0(T) \tau_j^0(T).$$

Then the problem (1), (4) *has a unique solution.*

Let $\tau_i(t) = q_i t - \Delta_i$ for $t \in I$, where $q_i, \Delta_i \in \mathbb{R}_+$ are such that $(q_i - 1)T \le \Delta_i$ $(i = 1, \dots, m)$. Then we get the following criteria of the solvability of the problem (1), (2) with linear delays.

Corollary 3.5. *Let* $t_0 = \min\{t_k : k = 1, 2, \dots, \nu\}$, $\tau_i(t) = q_i t - \Delta_i$ *for* $t \in I$, *where* $q_i, \Delta_i \in \mathbb{R}_+$ *are such that* $(q_i - 1)T \le \Delta_i$ $(i = 1, \dots, m)$, *and let there exist matrices* $B_i \in \mathbb{R}_+^{n \times n}$ $(i = 1, \dots, m)$ *such that the inequality* (5) *holds for* $i = 1, \dots, m$. *Put* $\delta_i = \chi_{[0,T]}(\frac{\Delta_i}{q_i})(q_i T - \Delta_i)$ *and* $\delta_{ik} = \chi_{[0,t_k]}(\frac{\Delta_i}{q_i})(q_i t_k - \Delta_i)$ *for*

$i = 1, \dots, m, k = 0, 1, \dots, \nu$. *Let, moreover, either the matrix* Λ_1 *given by* (6) *be nonsingular and* $r(S_1) < 1$, *where*

$$S_1 = \sum_{i=1}^{m} B_i \delta_i + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{m} |A_k| B_i (\delta_{ik} - \delta_{i0}), \tag{8}$$

or $\Lambda_1 = \Theta$, *the matrix* Λ_2 *given by* (7) *be nonsingular and* $r(S_2) < 1$, *where*

$$S_2 = \sum_{i=1}^{m} B_i \delta_i + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{m} \sum_{j=1}^{m} |A_k| B_i B_j (\delta_{ik} - \delta_{i0}) \delta_{jk}. \tag{9}$$

Then the problem (1), (2) *has a unique solution.*

For the system with constant and proportional delays we get the following corollaries.

COROLLARY 3.6. *Let* $t_0 = \min\{t_k : k = 1, 2, \dots, \nu\}$, $\tau_i(t) = t - \Delta_i$ *for* $t \in I$, *where* $\Delta_i \in \mathbb{R}_+$ $(i = 1, \dots, m)$, *and let there exist matrices* $B_i \in \mathbb{R}_+^{n \times n}$ $(i = 1, \dots, m)$ *such that the inequality* (5) *holds for* $i = 1, \dots, m$. *Put* $\delta_i = \chi_{[0,T]}(\Delta_i)(T - \Delta_i)$ *and* $\delta_{ik} = \chi_{[0,t_k]}(\Delta_i)(t_k - \Delta_i)$ *for* $i = 1, \dots, m, k = 0, 1, \dots, \nu$. *Let, moreover, either the matrix* Λ_1 *given by* (6) *be nonsingular and* $r(S_1) < 1$, *where the matrix* S_1 *is defined by* (8), *or* $\Lambda_1 = \Theta$, *the matrix* Λ_2 *given by* (7) *be nonsingular and* $r(S_2) < 1$, *where the matrix* S_2 *is defined by* (9).
Then the problem (1), (2) *has a unique solution.*

EXAMPLE 3.7. *Let*

$$x'(t) = -10^{-1} x(t) + 10^{-1} x(t - 1/2) + f(t), \quad t \in [0, 1],$$

$$x(0) = x(1), \quad x(t) = u(t) \text{ for } t \in [-1/2, 0[,$$

where $f \in L(I, \mathbb{R}^n)$ *and* $u : [-1/2, 0[\to \mathbb{R}^n$ *is a continuous and bounded vector function.*

Then $\nu = 2, m = 2$, $t_0 = t_1 = 0$, $t_2 = 1$, $\tau_1(t) \equiv t$, $\tau_2(t) = t - 1/2$, $P_1 = -10^{-1} E$, $P_2 = 10^{-1} E$, $A_1 = E$, $A_2 = -E$ *and we get* $\Lambda_1 = \Theta$, *nonsingular matrix* $\Lambda_2 = \frac{1}{20} E$ *and* $S_2 = \frac{12}{20} E$ *with* $r(S_2) < 1$. *According to the theory the solution of the problem is an absolutely continuous vector function on the interval* $[0, 1]$. *It needn't generally be (and here it isn't) a continuous extension of the solution* $x(t) = u(t)$ *defined outside the interval* $[0, 1]$.
From Corollary 3.6 follows that the problem has a unique solution.

COROLLARY 3.8. *Let* $t_0 = \min\{t_k : k = 1, 2, \dots, \nu\}$, $\tau_i(t) = q_i t$, *for* $t \in I$, *where* $q_i \in]0, 1]$ $(i = 1, \dots, m)$, *and let there exist matrices* $B_i \in \mathbb{R}_+^{n \times n}$ $(i = 1, \dots, m)$ *such that* (5) *holds for* $i = 1, \dots, m$. *Let, moreover, either the matrix* Λ_1 *given by* (6) *be nonsingular and* $r(S_1) < 1$, *where*

$$S_1 = \sum_{i=1}^{m} B_i q_i T + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{m} |A_k| B_i q_i (t_k - t_0),$$

or $\Lambda_1 = \Theta$, *the matrix* Λ_2 *given by* (7) *be nonsingular and* $r(S_2) < 1$, *where*

$$S_2 = \sum_{i=1}^{m} B_i q_i T + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{m} \sum_{j=1}^{m} |A_k| B_i B_j q_i q_j (t_k - t_0) t_k.$$

Then the problem (1), (2) *has a unique solution.*

Remark 3.9. From Corollaries 3.5 - 3.8 we can easily derive analogous criteria for the solvability of the problems (1), (3) and (1), (4).

4. PROOFS OF THE MAIN RESULTS

For any $x \in C(I, \mathbb{R}^n)$ and almost all $t \in I$, we set

$$p(x)(t) = \sum_{i=1}^{m} \chi_I(\tau_i(t)) P_i(t) x(\tau_i^0(t)), \tag{10}$$

$$q(t) = \sum_{i=1}^{m} (1 - \chi_I(\tau_i(t))) P_i(t) u(\tau_i(t)) + f(t).$$

It is obvious that $p : C(I, \mathbb{R}^n) \to L(I, \mathbb{R}^n)$ is a linear operator and $q \in L(I, \mathbb{R}^n)$. Moreover, $||p(x)(t)|| \leq \alpha(t)||x||_C$ for any $x \in C(I, \mathbb{R}^n)$ and almost all $t \in I$, where $\alpha(t) = \sum_{i=1}^{m} ||\chi_I(\tau_i(t)) P_i(t)||$. It is also clear that $\alpha \in L(I, \mathbb{R}_+)$. Therefore p is a strongly bounded operator. Concurrently, $l : C(I, \mathbb{R}^n) \to \mathbb{R}^n$ defined by $l(x) = \sum_{k=1}^{\nu} A_k x(t_k)$ is a linear bounded functional.

Consequently, the problem (1), (2) can be rewritten to the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = p(x)(t) + q(t),$$

$$l(x) = c_0$$

and thus Proposition 3.1 is valid (see [Kiguradze and Půža 2003], Theorem 1.1.1).

Proof of Theorem 3.2. According to Proposition 3.1, it is sufficient to show that if x is a solution of the problem (1_0), (2_0) then $x(t) \equiv 0$. Let x be such solution and let $t_0 \in I$ be an arbitrary point. The integration of (1_0) from t_0 to t, in view of (10), results in

$$x(t) = c + \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s \tag{11_1}$$

and, by iteration in (11_1), we get

$$x(t) = \left[E + \int_{t_0}^{t} p(E)(s)\,\mathrm{d}s\right] c + \int_{t_0}^{t} p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s)\,\mathrm{d}s, \tag{11_2}$$

where $c = x(t_0)$ and

$$p(E)(s) = \sum_{i=1}^{m} \chi_I(\tau_i(s)) P_i(s),$$

$$p\left(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\right)(s) = \sum_{i=1}^{m}\sum_{j=1}^{m} \chi_I(\tau_i(s)) P_i(s) \int_{t_0}^{\tau_i^0(s)} \chi_I(\tau_j(\xi)) P_j(\xi) x(\tau_j^0(\xi))\,\mathrm{d}\xi.$$

First suppose that the matrix Λ_1 given by (6) is nonsingular and $r(S_1) < 1$, where the matrix S_1 is defined by (6_1). Then from (2_0), by virtue of (6) and (11_1), we get

$$0 = \sum_{k=1}^{\nu} A_k \left[c + \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s \right] = \Lambda_1 c + \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s$$

and thus

$$c = -{\Lambda_1}^{-1} \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s.$$

Therefore, (11_1) implies

$$x(t) = \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s - \Lambda_1^{-1} \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s$$

and, in view of (5), we get

$$\begin{aligned}
|x(t)| &\le \left| \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s \right| + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} |A_k| \left| \int_{t_0}^{t_k} p(x)(s)\,\mathrm{d}s \right| \le \\
&\le \Bigg[\sum_{i=1}^{m} \left| \int_{t_0}^{t} \chi_I(\tau_i(s)) |P_i(s)|\,\mathrm{d}s \right| + \\
&\qquad + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} |A_k| \sum_{i=1}^{m} \int_{t_0}^{t_k} \chi_I(\tau_i(s)) |P_i(s)|\,\mathrm{d}s \Bigg] |x|_C \le \\
&\le \Bigg[\sum_{i=1}^{m} \int_{t_0}^{T} B_i \chi_I(\tau_i(s)) \tau_i'(s)\,\mathrm{d}s + \\
&\qquad + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} |A_k| \sum_{i=1}^{m} \int_{t_0}^{t_k} B_i \chi_I(\tau_i(s)) \tau_i'(s)\,\mathrm{d}s \Bigg] |x|_C \le \\
&\le \left[\sum_{i=1}^{m} B_i \tau_i^0(T) + |\Lambda_1^{-1}| \sum_{k=1}^{\nu} \sum_{i=1}^{m} |A_k| B_i (\tau_i^0(t_k) - \tau_i^0(t_0)) \right] |x|_C,
\end{aligned}$$

i.e.,

$$|x|_C \le S_1 |x|_C.$$

Whence, together with the assumption $r(S_1) < 1$, we get

$$(E - S_1)|x|_C \le 0 \Rightarrow |x|_C \le (E - S_1)^{-1} 0 = 0.$$

Therefore $x(t) \equiv 0$.

Now suppose that $\Lambda_1 = \Theta$, the matrix Λ_2 given by (7) is nonsingular, and $r(S_2) < 1$, where the matrix S_2 is defined by (7_1). From (2_0), (11_2), (7), and the assumption $\Lambda_1 = \Theta$ we get

$$\begin{aligned}0 &= \sum_{k=1}^{\nu} A_k \left[E + \int_{t_0}^{t_k} p(E)(s)\,\mathrm{d}s\right] c + \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s = \\ &= \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p(E)(s)\,\mathrm{d}s\; c + \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s = \\ &= \Lambda_2\, c + \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s\end{aligned}$$

and thus

$$c = -\Lambda_2^{-1} \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s.$$

Therefore (11_1) yields

$$x(t) = \int_{t_0}^{t} p(x)(s)\,\mathrm{d}s - \Lambda_2^{-1} \sum_{k=1}^{\nu} A_k \int_{t_0}^{t_k} p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s.$$

Hence, on account of (5) and the notation $\widetilde{P}_i(t) = \chi_I(\tau_i(t))P_i(t), \widetilde{\tau}_i'(t) = \chi_I(\tau_i(t))\tau_i'(t)$, we get

$$
\begin{aligned}
|x(t)| \leq & \left|\int_{t_0}^{t} p(x)(s)\,\mathrm{d}s\right| + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} |A_k| \left|\int_{t_0}^{t_k} p\Big(\int_{t_0}^{\cdot} p(x)(\xi)\,\mathrm{d}\xi\Big)(s)\,\mathrm{d}s\right| \leq \\
\leq & \sum_{i=1}^{m} \left|\int_{t_0}^{t} \widetilde{P}_i(s) x(\tau_i^0(s))\,\mathrm{d}s\right| + \\
& + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} |A_k| \sum_{i=1}^{m}\sum_{j=1}^{m} \left|\int_{t_0}^{t_k} \widetilde{P}_i(s) \int_{t_0}^{\tau_i^0(s)} \widetilde{P}_j(\xi) x(\tau_j^0(\xi))\,\mathrm{d}\xi\,\mathrm{d}s\right| \leq \\
\leq & \sum_{i=1}^{m} B_i \tau_i^0(T) |x|_C + \\
& + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} |A_k| \sum_{i=1}^{m}\sum_{j=1}^{m} \int_{t_0}^{t_k} |\widetilde{P}_i(s)| \left|\int_{t_0}^{\tau_i^0(s)} |\widetilde{P}_j(\xi)||x(\tau_j^0(\xi))|\,\mathrm{d}\xi\right| \mathrm{d}s \leq \\
\leq & \sum_{i=1}^{m} B_i \tau_i^0(T) |x|_C + \\
& + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} |A_k| \sum_{i=1}^{m}\sum_{j=1}^{m} B_i B_j \int_{t_0}^{t_k} \widetilde{\tau}_i'(s) \int_{0}^{s} \widetilde{\tau}_j'(\xi)\,\mathrm{d}\xi\,\mathrm{d}s |x|_C \leq \\
\leq & \sum_{i=1}^{m} B_i \tau_i^0(T) |x|_C + \\
& + |\Lambda_2^{-1}| \sum_{k=1}^{\nu} |A_k| \sum_{i=1}^{m}\sum_{j=1}^{m} B_i B_j (\tau_i^0(t_k) - \tau_i^0(t_0)) \tau_j^0(t_k) |x|_C = \\
= & \left[\sum_{i=1}^{m} B_i \tau_i^0(T) + |\Lambda_2^{-1}| \sum_{k=1}^{\nu}\sum_{i=1}^{m}\sum_{j=1}^{m} |A_k| B_i B_j (\tau_i^0(t_k) - \tau_i^0(t_0)) \tau_j^0(t_k)\right] |x|_C = \\
= & S_2 |x|_C.
\end{aligned}
$$

Whence, together with the assumption $r(S_2) < 1$, we get $x(t) \equiv 0$. □

Proof of Corollary 3.3. The validity of corollary follows immediately from Theorem 3.2 with $\nu = 1, A_1 = E$, and $t_1 = t_0$ because, in this case, the matrix Λ_1 given by (6) is the unit matrix. □

Proof of Corollary 3.4. The validity of corollary follows immediately from Theorem 3.2 with $\nu = 2, A_1 = -E, A_2 = E, t_1 = 0$, and $t_2 = T$, because, in this case, $t_0 = 0$ and the matrices Λ_1 and Λ_2 given by (6) and (7), respectively, satisfy $\Lambda_1 = \Theta$ and $\Lambda_2 = \Lambda$. □

Proof of Corollary 3.5 - 3.8. The validity of corollaries follows from Theorem 3.2 when, for $t \in I$, $\tau_i(t) = q_i t - \Delta_i, \tau_i(t) = t - \Delta_i$ and $\tau_i(t) = q_i t$, respectively. □

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