Journal of Applied Mathematics, Statistics and Informatics (JAMSI), **1** (2005), No. 2

Existence and Uniqueness of Solutions of Nonlinear Integrodifferential Equations*

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Abstract

This paper deals with the existence and uniquenes of solutions of singular initial value problems of nonlinear integrodifferential equations.

Mathematics Subject Classification 2000: 45J05

Additional Key Words and Phrases: Singular inital value problem, Banach contract principle

1. INTRODUCTION

Singular initial value problems have been considered by many authors (see e.g.[1-5]). The most results concerning of behaviour of singular ordinary and integrodifferential equations were studied by means of modifications of Wazewski's topological method and by fixed point theorems as well.
In this paper we shall also consider a problem of continuous dependence of solutions on a parameter and we give sufficient conditions for existence and uniqueness of solutions of the following singular initial value problem

$$(1) \qquad y'(t) = \mathcal{F}\left(t, y(t), \int_{0+}^{t} K(t,s,y(t),y(s))ds, \mu\right), \quad y(0^+,\mu) = 0,$$

where

(I) $\mathcal{F} : \Omega \to R^n$, $\mathcal{F} \in C^0(\Omega)$,
$\Omega = \{(t,u_1,u_2,\mu) \in J \times R^n \times R^n \times R : |u_1| \le \phi(t),\ |u_2| \le \psi(t)\},\ \ J = (0,t_0]$, $t_0 > 0$, $0 < \phi(t) \in C^0(J)$, $\phi(0^+) = 0$, $0 < \psi(t) \in C^0(J)$, $|\cdot|$ denotes the usual norm in R^n, $|\mathcal{F}(t,\overline{u}_1,\overline{u}_2,\mu) - \mathcal{F}(t,\overline{\overline{u}}_1,\overline{\overline{u}}_2,\mu)| \le M_1|\overline{u}_1 - \overline{\overline{u}}_1| + M_2|\overline{u}_2 - \overline{\overline{u}}_2|$ for all $(t,\overline{u}_1,\overline{u}_2,\mu)$, $(t,\overline{\overline{u}}_1,\overline{\overline{u}}_2,\mu) \in \Omega$, $M_i \ge 0,\ i = 1,2$.

(II) $K : \Omega_1 \to R^n$, $K \in C^0(\Omega_1)$,
$\Omega_1 = \{(t,s,v_1,v_2) \in J \times J \times R^n \times R^n : |v_1| \le \phi(t)\ |v_2| \le \phi(t).\}$, $|K(t,s,\overline{v}_1,\overline{v}_2) - K(t,s,\overline{\overline{v}}_1,\overline{\overline{v}}_1| \le N_1|\overline{v}_1 - \overline{\overline{v}}_1| + N_2|\overline{v}_2 - \overline{\overline{v}}_2|$ for all $(t,s,\overline{v}_1,\overline{v}_2)$, $(t,s,\overline{\overline{v}}_1,\overline{\overline{v}}_2) \in \Omega_1$, $N_i \ge 0,\ i = 1,2$.

*This work has been supported by the Czech Ministry of Education in the frame of MSM002160503 Research Intention MIKROSYN New Trends in Microelectronic Systems and Nanotechnologies.

2. MAIN RESULTS

THEOREM 2.1. *Let the functions* $\mathcal{F}(t,u_1,u_2,\mu)$, $K(t,s,v_1,v_2)$ *satisfy conditions* (I), (II) *and, moreover*

$$|\mathcal{F}| \le g_1(t)|u_1| + g_2(t)|u_2|, \quad 0 < g_i(t) \in C^0(J),\ i = 1,2, \quad \int_{0+}^{t} g_1(s)\phi(s)ds \le \alpha\phi(t),$$

$$\int_{0+}^{t} g_2(s)\psi(s)ds \le \beta\phi(t), \quad \alpha + \beta \le 1,$$

then the problem (1) *has a unique solution* $y(t,\mu)$ *for each* $\mu \in R$, $t \in J$.

PROOF. Denote H the Banach space of continuous vector-valued functions

$$h : J_0 \to R^n,\ J_0 = [0, t_0],\ |h(t)| \le \phi(t)$$

on J with the norm

$$||h||_\lambda = \max_{t \in J_0}\{e^{-\lambda t}|h(t)|\},$$

where $\lambda > 0$ is an arbitrary parameter. The initial value problem (1) is equivalent to the system of integral equations

$$y(t) = \int_{0+}^{t} \mathcal{F}\left(s, y(s), \int_{0+}^{s} K(s,w,y(s),y(w))dw, \mu\right) ds. \tag{2}$$

Define the operator T by right-hand side of (2)

$$T(h) = \int_{0+}^{t} \mathcal{F}\left(s, h(s), \int_{0+}^{s} K(s,w,h(s),h(w))dw, \mu\right) ds,$$

where $h \in H$. Let $\mu \in R$ be fixed. The transformation T maps H continuously into itself because

$$|T(h)| \le \int_{0+}^{t} \left|\mathcal{F}\left(s, h(s), \int_{0+}^{s} K(s,w,h(s),h(w))dw, \mu\right)\right| ds \le$$

$$\int_{0+}^{t} \left[g_1(s)|h(s)| + g_2(s)\int_{0+}^{s} |K(s,w,h(s),h(w))|dw\right] ds \le$$

$$\le \int_{0+}^{t} \left(g_1(s)\phi(s) + g_2(s)\psi(s)\right) ds \le (\alpha+\beta)\phi(t) \le \phi(t)$$

for every $h \in H$. We shall prove that

$$||T(h_2) - T(h_1)||_\lambda \le ||h_2 - h_1||_\lambda \left(\frac{M_1 + t_0 M_2 N_1}{\lambda} + \frac{M_2 N_2 + M_2 N_1}{\lambda^2}\right) \tag{3}$$

for all $h_1, h_2 \in H$. Using (I), (II) and the definition $||.||_\lambda$ we have

$$|T(h_2) - T(h_1)| \le \int_{0+}^{t} \left|\mathcal{F}\left(s, h_2(s), \int_{0+}^{s} K(s,w,h_2(s),h_2(w))dw, \mu\right) - \right.$$

$$-\mathcal{F}\left(s, h_1(s), \int_{0+}^{s} K(s,w,h_1(s),h_1(w))dw, \mu\right)\Bigg| ds \leq$$

$$\int_{0+}^{t}\Big(M_1|h_2(s)-h_1(s)|+$$

$$+M_2\int_{0+}^{s}|K(s,w,h_2(s),h_2(w))-K(s,w,h_1(s),h_1(w))|dw\Big)ds \leq$$

$$\int_{0+}^{t}\Big(M_1|h_2(s)-h_1(s)|+$$

$$M_2\int_{0+}^{s}(N_1|h_2(s)-h_1(s)|+N_2|h_2(w)-h_1(w)|)dw\Big)ds \leq$$

$$M_1||h_2-h_1||_\lambda \int_{0+}^{t} e^{\lambda s}ds + M_2N_1||h_2-h_1||_\lambda \int_{0+}^{t} se^{\lambda s}ds+$$

$$M_2N_2||h_2-h_1||_\lambda \int_{0+}^{t}\int_{0+}^{s} e^{(\lambda w)}dwds =$$

$$= ||h_2-h_1||_\lambda\Big(M_1(\frac{e^{\lambda t}}{\lambda}-\frac{1}{\lambda})+M_2N_1(\frac{te^{\lambda t}}{\lambda}-\frac{e^{\lambda t}}{\lambda^2}+\frac{1}{\lambda^2})+$$

$$M_2N_2(\frac{e^{\lambda t}}{\lambda^2}-\frac{1}{\lambda^2}-\frac{t}{\lambda})\Big) <$$

$$< ||h_2-h_1||_\lambda e^{\lambda t}\left(\frac{M_1+t_0M_2N_1}{\lambda}+\frac{M_2N_2+M_2N_1}{\lambda^2}\right).$$

Thus

$$||T(h_2)-T(h_1)||_\lambda = \max_{t\in J_0}\{e^{-\lambda t}|T(h_2)-T(h_1)|\} \leq$$

$$\leq ||h_2-h_1||_\lambda\left(\frac{M_1+t_0M_2N_1}{\lambda}+\frac{M_2N_2+M_2N_1}{\lambda^2}\right).$$

Now we choose $\lambda > 0$ so that $(\frac{M_1+t_0M_2N_1}{\lambda}+\frac{M_2N_2+M_2N_1}{\lambda^2}) < 1$ and apply the classical Banach contraction principle to T and the distance function $||h_2-h_1||_\lambda$ to complete the proof. □

THEOREM 2.2. *Let the assumptions of Theorem 2.1. be satisfied and let there exist a constant* $L > 0$ *and an integrable function* $\gamma : J_0 \to J_0,\ J_0 = [0, t_0]$ *such that*

$$|\mathcal{F}(t,u_1,u_2,\mu_2)-\mathcal{F}(t,u_1,u_2,\mu_1)| \leq \gamma(t)|\mu_2-\mu_1|,$$

where $(t,u_1,u_2,\mu_1),\ (t,u_1,u_2,\mu_2) \in \Omega$ *and*

$$\max_{t\in J_0}\left\{e^{-\lambda t}\int_{0+}^{t}\gamma(s)ds\right\} \leq L,$$

then the solution $y(t,\mu)$ *of* (1) *is continuous with respect to the variables* $(t,\mu) \in J \times R$.

PROOF. Define as above, for $h \in H$ the transformation $T_\mu(h)$ by means of the right-hand side (2). From (3) we obtain

$$||T_\mu(h) - T_\mu(y)||_\lambda \le \left(\frac{M_1 + t_0 M_2 N_1}{\lambda} + \frac{M_2 N_2 + M_2 N_1}{\lambda^2}\right) ||h - y||_\lambda.$$

By the hypothesis of Theorem 2.2. we get

$$e^{-\lambda t}|T_{\mu_2}(h) - T_{\mu_1}(h)| \le e^{-\lambda t} \int_{0+}^{t} \left|\mathcal{F}(s, h(s), \int_{0+}^{s} K(s, w, h(s), h(w))dw, \mu_2) - \right.$$

$$\left. - \mathcal{F}(s, h(s), \int_{0+}^{s} K(s, w, h(s), h(w))dw, \mu_1)\right| ds \le$$

$$\le e^{-\lambda t} \int_{0+}^{t} \gamma(s)|\mu_2 - \mu_1| ds \le L|\mu_2 - \mu_1|.$$

Hence

$$||T_{\mu_2}(h) - T_{\mu_1}(h)||_\lambda \le L|\mu_2 - \mu_1|.$$

From this and by Theorem 2.1. we obtain

$$||h(t,\mu_2) - h(t,\mu_1)||_\lambda = ||T_{\mu_2}[h(t,\mu_2)] - T_{\mu_2}[h(t,\mu_1)] + T_{\mu_2}[h(t,\mu_1)] -$$

$$- T_{\mu_1}[h(t,\mu_1)]||_\lambda \le$$

$$||T_{\mu_2}[h(t,\mu_2)] - T_{\mu_2}[h(t,\mu_1)]||_\lambda + ||T_{\mu_2}[h(t,\mu_1)] - T_{\mu_1}[h(t,\mu_1)]||_\lambda \le$$

$$\le \left(\frac{M_1 + t_0 M_2 N_1}{\lambda} + \frac{M_2 N_2 + M_2 N_1}{\lambda^2}\right) ||h(t,\mu_2) - h(t,\mu_1)||_\lambda + L|\mu_2 - \mu_1|.$$

Thus

$$||h(t,\mu_2) - h(t,\mu_1)||_\lambda \le \left[1 - \left(\frac{M_1 + t_0 M_2 N_1}{\lambda} + \frac{M_2 N_2 + M_2 N_1}{\lambda^2}\right)\right]^{-1} L|\mu_2 - \mu_1|.$$

Consequently the function $h(t,\mu)$ is uniformly continuous with respect to the variable $\mu \in R$; so $y(t,\mu)$ is also continuous with respect to two variables $(t,\mu) \in J \times R$, which completes the proof. □

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Received October 2005