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Some examples of goal-minimally 3-diametric graphs*

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Abstract

A graph G with diameter k is said to be goal-minimally k-diametric if for every edge uv of G distance $d_{G-uv}(x,y) > k$ if and only if $\{x,y\} = \{u,v\}$. It is rather difficult to construct such graphs. In this paper we give some relations between the maximum degree, the minimum degree, and the order of such graphs. Several goal-minimally 3-diametric graphs are presented.

Mathematics Subject Classification 2000: 05C12

Additional Key Words and Phrases: graph, diameter, edge deletion, goal-minimal graph

1. INTRODUCTION

For graph-theoretical terminology and notation not defined here we follow Chartrand and Lesniak [6]. We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. The number of vertices [edges] of G is often referred to as the order [size] of G and denoted by n [m, respectively]. The degree of a vertex u is denoted by $deg(u)$ and the maximum [minimum] degree by Δ [δ, respectively]. If G is a connected graph, then the distance $d(u,v)$ between two vertices u and v is defined as the length of a u-v geodesic (a shortest path from u to v). The eccentricity $ec(u)$ of a vertex u is the distance to a farthest vertex from u. For any integer i we denote $D_i(u) = \{x \in V(G) | d(u,x) = i\}$. Thus $D_0(u) = \{u\}$ and the neighborhood $N(u) = D_1(u)$. Clearly, the sets $D_0(u), \ldots, D_{ec(u)}(u)$ form the distance decomposition of $V(G)$ from u. The diameter $d(G)$ [radius $r(G)$] is the maximum [minimum, respectively] eccentricity among the vertices of G. The girth of G is the length of a shortest cycle in G. A graph G with diameter k is also called a k-diametric graph. It is said to be minimal with respect to diameter or, more precisely, minimally k-diametric, if for any edge $e \in E(G)$ we have $d(G-e) > k$. The distance function in $G-e$ is allowed to exceed k in an arbitrary pair of vertices. If we restrict this to the ends of the edge e, then we get the following special class of minimally k-diametric graphs. A graph G with diameter k is said to be goal-minimal with respect to diameter or, more precisely, goal-minimally k-diametric (k-GMD for short), if for each edge uv of G the inequality $d_{G-uv}(x,y) > k$ holds if and only if $\{x,y\} = \{u,v\}$. Clearly, the complete graphs are the 1-GMD graphs, but already the case $k = 2$ is interesting.

The minimal graphs with respect to diameter were studied under various names

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(e.g. graphs without superfluous edges, or critical, or edge-critical, or edge-diameter critical, or diameter-minimal) by many authors (see e.g. [2], [3], [4], [5], [7], [8], [9], [10], [11], [12], [13], [14], [16], [17], [18]). In this paper we deal with k-GMD graphs and mainly with 3-GMD graphs. The goal-minimal graphs with respect to diameter were introduced by Kyš in 1980 [15] which called them "diameter strongly critical graphs". To our knowledge, no other paper has been published on this subject up to now. Kyš [15] presented several results and a few of them will be summarized below. He also observed that any k-GMD graph is 2-connected and has girth at least $k+2$. The latter assertion can be strengthened as follows.

LEMMA 1. *The girth of a k-GMD graph G of order at least 3 is $k+2$ and every edge of G lies in a cycle of length $k+2$.*

Proof: We can assume that $k \geq 2$. If there exists a shorter cycle, then for an edge uv of this cycle we have $d_{G-uv}(u,v) \leq k$, a contradiction. Now consider any edge $xy \in E(G)$ and a path $P = xyz$. In the graph $G - xy$ there exists an x-z path Q of length at most k. It cannot pass through vertex y (because a cycle of length at most $k+1$ would exist) and thus the union of P and Q gives a cycle of length at most $k+2$. By the preceding argument its length must be precisely $k+2$. ■

LEMMA 2. (Kyš [15]) For any two non-adjacent vertices u and v of a k-GMD graph there are at least two internally disjoint u-v paths of length not exceeding k.

LEMMA 3. (Kyš [15]) Let G be a graph without 3-cycles. Then there is a 2-GMD graph containing G as an induced subgraph.

The proof of Lemma 3 consists of two steps. First a 2-diametric triangle-free supergraph H of G is constructed such that G is an induced subgraph of H. Then each vertex u of H is multiplied by an arbitrary integer $s(u) \geq 2$.

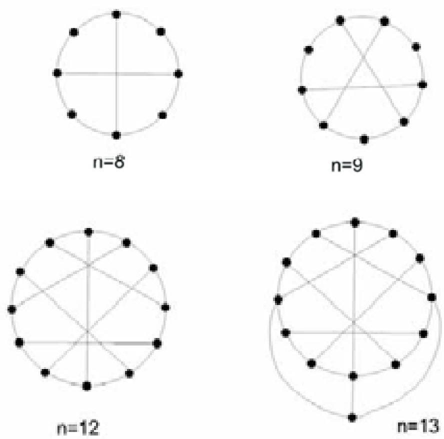

Fig. 1. The known 3-GMD graphs of order at most 14

Kyš also found two simple constructions giving infinitely many 4-GMD graphs. Moreover, he presented two k-GMD graphs for other k, one with $k = 3$ and $n = 12$

(see our Fig. 1) and the second with $k = 6$. He raised the conjecture that for every integer $k \geq 1$ there exists a k-GMD graph. As we have seen, this conjecture was proved by himself only for five values of k. The main purpose of this paper is to present several examples of 3-GMD graphs. We have used a computer search based on some theoretical bounds.

2. BOUNDING THE NUMBER OF VERTICES

In this section we derive some bounds on the number of vertices.

2.1 Lower bound

Since by Lemma 1 in any k-GMD graph all cycle lengths $3, 4, \ldots, k+1$ are forbidden, the following result is applicable.

LEMMA 4. (Alon et al. [1]) If G is a graph of order n without cycles of length at most $2q$ for some integer q, then its size

$$m \leq \frac{1}{2}(n^{1+1/q} + n)$$

However, our class of graphs admits to derive another bound, which appeared to be more useful for our aims.

THEOREM 1. *Let G be a k-GMD graph with $k \geq 3$, minimum degree δ and order n. Then for any vertex $u \in V(G)$ we have:*

(a) If k is odd, then

$$n \geq \begin{cases} deg(u)(k+1)/2 + \max\{2ec(u) - k - 1, 1\} & if \quad \delta = 2 \\ deg(u)[(\delta-1)^{(k+1)/2} - 1]/(\delta - 2) + \max\{2ec(u) - k - 1, 1\} & if \quad \delta > 2 \end{cases}$$

(b) If k is even, then

$$n \geq \begin{cases} deg(u)(k/2+1) + 2ec(u) - k & if \quad \delta = 2 \\ deg(u)[(\delta-1)^{k/2} - 1]/(\delta - 2) + 2ec(u) - k & if \quad \delta > 2 \end{cases}$$

Proof:

(a) As the sets $D_i(u)$, $0 \leq i \leq ec(u)$, form a decomposition of V(G), we have

$$n = |D_0(u)| + |D_1(u)| + \cdots + |D_{ec(u)}(u)| \tag{1}$$

Clearly $|D_0(u)| = 1$ and $|D_1(u)| = deg(u)$. Since the diameter of a connected graph does not exceed twice the eccentricity of any vertex, we see that

$$\frac{k+1}{2} \leq ec(u) \tag{2}$$

Now, if $1 \leq i \leq (k+1)/2 - 1$, then for each $x \in D_i(u)$ the set $S(x) := N(x) \cap D_{i+1}(u)$ contains at least $\delta - 1$ vertices and for distinct vertices $x, y \in D_i(u)$ the sets $S(x)$ and $S(y)$ are disjoint, because G is without cycles of length $c \leq d+1$. Moreover, each set $D_{(k+1)/2+1}(u), \ldots, D_{ec(u)-1}(u)$ contains at least two vertices, because G is 2-connected. If $ec(u) > (k+1)/2$, then $|D_{ec(u)}(u)| \geq 1$ and by (1) we get

$$n \geq 1 + deg(u) + deg(u)(\delta-1) + \cdots + deg(u)(\delta-1)^{\frac{k+1}{2}-1} + \max\{2ec(u) - k - 2, 0\} \tag{3}$$

which gives the desired inequality.

(b) The proof of this part is similar. Instead of $(k+1)/2$ we must put $k/2$. However, no vertex u with $ec(u) = k/2$ can exist, as one can easily see. Thus only the case $ec(u) > k/2$ is possible. The details are left to the reader. ■

As we can take for u a vertex of maximum degree, we obtain:

COROLLARY 1. *Let G be a k-GMD graph with $k \geq 3$, maximum degree Δ, minimum degree δ and order n. Then we have:*

(a) If k is odd, then

$$n \geq \begin{cases} \Delta(k+1)/2 + 1 & if \quad \delta = 2 \\ \Delta[(\delta-1)^{(k+1)/2} - 1]/(\delta - 2) + 1 & if \quad \delta > 2 \end{cases}$$

(b) If k is even, then

$$n \geq \begin{cases} \Delta(k/2+1) + 2 & if \quad \delta = 2 \\ \Delta[(\delta-1)^{k/2} - 1]/(\delta - 2) + 2 & if \quad \delta > 2 \end{cases}$$

Immediately from Corollary 1 we get

COROLLARY 2. *Let G be a k-GMD graph with $k \geq 3$, maximum degree Δ, minimum degree δ and order n. Then we have:*

(a) If k is odd, then

$$\Delta \leq \begin{cases} \lfloor \frac{2(n-1)}{k+1} \rfloor & if \quad \delta = 2 \\ \lfloor \frac{(n-1)(\delta-2)}{(\delta-1)^{(k+1)/2}-1} \rfloor & if \quad \delta > 2 \end{cases}$$

(b) If k is even, then

$$\Delta \leq \begin{cases} \lfloor \frac{2(n-2)}{k+2} \rfloor & if \quad \delta = 2 \\ \lfloor \frac{(n-2)(\delta-2)}{(\delta-1)^{k/2}-1} \rfloor & if \quad \delta > 2 \end{cases}$$

Note that these bounds are tight at least for $k = 3$ (cf. graphs with $n = 8, 12$ and 13 in Fig. 1 and those with $n = 15$ in Fig. 2).

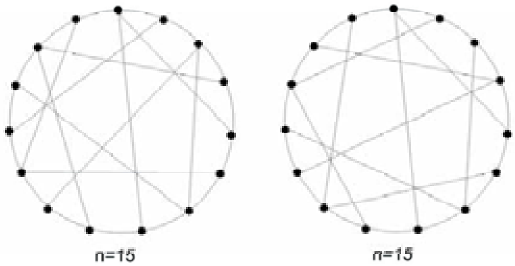

Fig. 2. Two non-isomorphic 3-GMD graphs of order 15

2.2 Upper bound

Now we give an upper bound on n. The well known Moore bound (see e.g. [6], p. 312) is applicable but we can provide a better one.

THEOREM 2. *Let G be a k-GMD graph with $k \geq 3$, maximum degree Δ, minimum degree δ and order n. Then we have:*

$$n \leq 1 + \delta\Big[\frac{(\Delta-1)^{k-1}-1}{\Delta-2} + \frac{(\Delta-1)^{k-2}(\Delta-2)}{2}\Big]$$

Proof: Taking a vertex u of degree δ, we see that $|D_1(u)| = \delta$ and $|D_i(u)| \leq |D_{i-1}(u)|(\Delta-1)$ if $2 \leq i \leq k-1$ (because for each vertex $v \in D_{i-1}(u)$ there is at least one edge going from $D_{i-2}(u)$ to v and thus there are at most $\Delta - 1$ edges going from v to $D_i(u)$). Further, by Lemma 2 for each vertex $x \in D_{k-1}(u)$ there are at least two internally disjoint u-x paths of length at most k. These paths cannot intersect $D_k(u)$ and thus at least two edges go to x from $D_{k-2}(u) \bigcup D_{k-1}(u)$. Consequently, at most $\Delta - 2$ edges remain to go from x to $D_k(u)$. But for every vertex $y \in D_k(u)$ there are at least two of these edges ending at y (by Lemma 2). Hence $|D_k(u)| \leq |D_{k-1}(u)|(\Delta-2)/2$. Using (1) we get

$$n \leq 1 + \delta + \delta(\Delta-1) + \cdots + \delta(\Delta-1)^{k-2} + \delta(\Delta-1)^{k-2}(\Delta-2)/2,$$

which gives the desired inequality. ■

2.3 Diameter three

If $k = 3$, then Corollary 1 and Theorem 2 get the following simple form.

COROLLARY 3. *Let G be a 3-GMD graph with maximum degree Δ and minimum degree δ. Then the order n of G fulfills the following inequalities*

$$1 + \delta\Delta \leq n \leq 1 + \delta\Big[\Delta + \frac{(\Delta-1)(\Delta-2)}{2}\Big]$$

3. FINDING 3-GMD GRAPHS

To find several 3-GMD graphs a computer search has been done. Starting from a subgraph H which is necessarily in such a graph, H was successively complemented to a graph G with girth 5 fulfilling prescribed degree bounds based on Corollary 3. Then G was examined on the diameter and the goal-minimality. However many good graphs remain after this sieve. Therefore we needed to perform an isomorphism test. This difficult task was replaced by a simplified test in which the set of cardinalities of the distance sets $D_1(u), D_2(u)$ and $D_3(u)$ was computed for each vertex u. If such systems of two graphs differ, then the graphs cannot be isomorphic although the converse may not be true (but we were unable to prove or disprove this conjecture). Nevertheless, for some orders several non-isomorphic graphs have been found. We have found about 50 non-isomorphic 3-GMD graphs altogether. Their order ranges between 8 and 38. While we know only six 3-GMD graphs of order at most 15 (they are in Figs. 1 and 2), we have found five non-isomorphic 3-GMD graphs of order 17 and eleven such graphs of order 18 (one of them is in Fig. 3).

In Examples 1 to 4 we have presented four examples of 3-GMD graphs of order 25, 26, 36, and 38, respectively. The order is denoted by n and the size by m. In

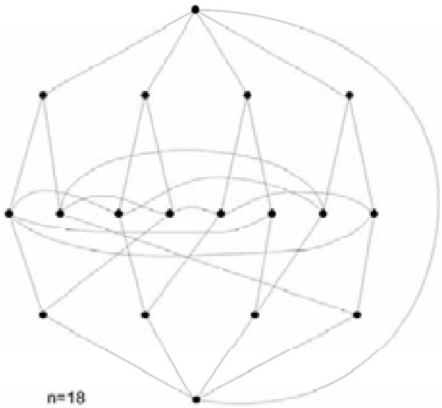

Fig. 3. An example of a 3-GMD graph of order 18

each row we have given a vertex $u \in V(G) = \{1, 2, \ldots, n\}$, its restricted neighborhood $N^{>}(u) = \{v \in N(u) | u < v\}$, the degree, and the eccentricity.

Example 1 (n=25, m=54)

u	$N^{>}(u)$	$deg(u)$	$ec(u)$
1:	2 6 7 20	4	3
2:	3 8 19 25	5	3
3:	4 9 23	4	3
4:	5 6 10	4	3
5:	7 8 11 22	5	3
6:	12 24	4	3
7:	9 13	4	3
8:	12 14	4	3
9:	12 15	4	3
10:	13 14 15 16	5	3
11:	15 17 23 25	5	3
12:	16 17 18	6	2
13:	17 19 24	5	3
14:	20 23	4	3
15:	21	4	3
16:	22 25	4	3
17:	20	4	3
18:	19 21 23	4	3
19:	22	4	3
20:	21 22	5	3
21:	24	4	3
22:		4	3
23:		4	3

Example 2 (n=26, m=55)

u	$N^{>}(u)$	$deg(u)$	$ec(u)$
1:	2 5 6 9 10 11	6	2
2:	3 12 13 14 15	6	2
3:	4 16 17 18 19	6	2
4:	5 20 21 22 23	6	2
5:	24	3	3
6:	7 16 20	4	3
7:	8 12 17 21 24	6	3
8:	9 13 18 22	5	3
9:	19 23	4	3
10:	17 22 25	4	3
11:	18 21 26	4	3
12:	23	3	3
13:	20	3	3
14:	21 25	3	3
15:	22 24 26	4	3
16:	26	3	3
17:		3	3
18:		3	3
19:	24 25	4	3
20:	25	4	3
21:		4	3
22:		4	3
23:	26	4	3

24:	25	4	3
25:		4	3

Example 3 (n=36, m=93)

u	$N^{>}(u)$	$deg(u)$	$ec(u)$
1:	2 6 7 20 30	5	3
2:	3 8 19 23	5	3
3:	4 9 33 35	5	3
4:	5 6 10 24 25	6	3
5:	7 8 11 22 29	6	3
6:	12 28 32	5	3
7:	9 13 27	5	3
8:	12 14 31	5	3
9:	12 15 26	5	3
10:	13 14 15 16 36	6	3
11:	15 17 30 32	5	3
12:	16 17 18	6	3
13:	17 19 25	5	3
14:	20 33	4	3
15:	21 28	5	3
16:	22 27 30	5	3
17:	20 23	5	3
18:	19 21 24 36	5	3
19:	22 25	5	3
20:	21 22 26	6	3
21:	27 35	5	3
22:	28	5	3
23:	24 27 28 29	6	3
24:	26 30	5	3
25:	27 31	4	3
26:	31 32	5	3
27:	32 33	7	3
28:	31 34	6	3
29:	35 36	4	3
30:	34 35	6	3
31:	35	5	3
32:	36	5	3
33:	34	4	3
34:	36	4	3
35:		6	3
36:		5	3

24:		4	3
25:	26	5	3
26:		5	3

Example 4 (n=38, m=101)

u	$N^{>}(u)$	$deg(u)$	$ec(u)$
1:	2 6 7 20 30 37	6	3
2:	3 8 19 23	5	3
3:	4 9 33 35	5	3
4:	5 6 10 24 25	6	3
5:	7 8 11 22 29	6	3
6:	12 28 32	5	3
7:	9 13 27	5	3
8:	12 14 31	5	3
9:	12 15 26 36	6	3
10:	13 14 15 16	5	3
11:	15 17 30 32	5	3
12:	16 17 18	6	3
13:	17 19 34 38	6	3
14:	20 33 36	5	3
15:	21 28	5	3
16:	22 27 30	5	3
17:	20 23	5	3
18:	19 21 24 37	5	3
19:	22 25	5	3
20:	21 22 26	6	3
21:	27 35	5	3
22:	28	5	3
23:	24 27 28 29	6	3
24:	26 30	5	3
25:	27 31 36	5	3
26:	31 32	5	3
27:	32 33	7	3
28:	31 34	6	3
29:	35 38	4	3
30:	34 35 36	7	3
31:	35 37	6	3
32:	38	5	3
33:	34 37	5	3
34:		4	3
35:		5	3
36:	38	5	3
37:	38	5	3
38:		5	3

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