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On the number of regular vertices and kernel subgraphs in random graphs*

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Abstract

We estimate the number of regular vertices in random subgraphs obtained from a complete graph K_n and an n-ary cube B^n by independent removal of edges. Regular vertices are such vertices in a random graph that are covered in every vertex covering of the graph by cliques or cubes, respectively. Similarly, we estimate the number of kernel cliques or cubes for these types of random graphs. Kernel subgraph are those subgraphs that contain at least one vertex not contained in any other clique or maximal cube. Obtained bounds are useful for designing algorithms for vertex coverings of random subgraphs of K_n or B^n by cliques or cubes, respectively.

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1. INTRODUCTION

The covering problem can be generally stated as follows [1]: Let $M = \{a_1, \dots, a_n\}$ be a finite set, and let $S = \{S_1, \dots, S_m\}$ be a set of some subsets of M, i.e. $S_i \subseteq M$. Any $S' = \{S_{j_1}, \dots, S_{j_k}\} \subseteq S$ such that

$$\bigcup_{i=1}^{k} S_{j_i} = M \tag{1}$$

is called a covering of M. The (minimum) covering problem is the problem of finding (minimum) set S', such that (1) is satisfied.

The covering problem has many applications in graph theory and other fields [1]. Some problems in graph theory can be formulated as covering problems. For example, the simplification of Boolean functions in the class of disjunctive normal forms can be stated equivalently as a covering problem of n-ary cube by (smaller) cubes [4; 3].

There are two natural simplification of the covering problem:

(1) (kernel set) If there exists $a_i \in M$ such that $a_i \in S_k$, and $a_i \notin S_j$ for any $j \neq k$, then S_k must be contained in every covering of M. Hence, the problem can be simplified: $M \leftarrow M \setminus S_k$, $S \leftarrow \{S_1 \setminus S_k, \dots, S_{k-1} \setminus S_k, S_{k+1} \setminus S_k, \dots, S_m \setminus S_k\}$.

(2) (regular member) Let $M_i = \{j \mid a_i \in S_j\}$. If there exist distinct $r, s \in \{1, \dots, n\}$ such that $M_r \subseteq M_s$, then every set from S containing a_r contains a_s as well.

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Hence, the problem can be (again) simplified: $M \leftarrow M \setminus \{a_s\}$, $S \leftarrow \{S_1 \setminus \{a_s\}, \dots, S_m \setminus \{a_s\}\}$.

The paper studies these simplifications for the covering problem of two types of random graphs.

The first type of random graph is obtained from K_n by independent removal of edges with fixed probability $1-p$. We show that there are no (kernel) cliques that appear in every vertex covering of this random graph by cliques. In addition we show that there are no (regular) vertices that are "automatically" covered.

The second type of random graph is obtained from B^n (n-ary cube) by independent removal of edges with fixed probability $1-p$. We show that the number of maximal subcubes contained in every vertex covering of this random graph by cubes is $n^{(1+o(1))\lg\log_{1/p} n}(2(1-p))^n$, for $0 < p < 1$. In addition we show that the number of regular vertices is less than ρ^n, for a constant $1.51 \le \rho < 2$ depending on p.

Precise formulations of our results are in the rest of the paper.

2. PRELIMINARIES

Let $G = (V(G), E(G))$ be a graph with the set of vertices $V(G)$ and the set of edges $E(G)$. A graph G_0 is a subgraph of G, $G_0 \subseteq G$, if $V(G_0) \subseteq V(G)$ and $E(G_0) \subseteq E(G)$. A clique of graph G is a complete subgraph of G that is not contained in any higher order complete subgraph of G. A complete graph of order r is denoted as K_r. Vertex covering of graph G is the set of cliques $\{F_1, \dots, F_m\}$ such that

$$\forall i = 1, \dots, m: \ F_i \subseteq G \quad \wedge \quad \bigcup_{i=1}^{m} V(F_i) = V(G).$$

A graph of n-ary cube is denoted by B^n. Trivially, B^n has 2^n vertices and $n2^{n-1}$ edges. A subcube of order k ($0 \le k \le n$) is a subgraph of B^n, that is a k-ary cube. Let G be a subgraph of B^n. We say that subcube K is a maximal subcube in G, if and only if K is contained in G and there does not exist subcube $L \subseteq G$ such that K is contained in L, and L has higher order than K. Vertex covering of graph G is the set $\{G_1, \dots, G_m\}$ of maximal subcubes contained in G such that

$$\bigcup_{i=1}^{m} V(G_i) = V(G).$$

The value m is the size of vertex cover.

We study two types of random graphs. Let G^n be a set of all graphs with n (numbered) vertices. The first type of random graphs are members of G^n – the edges are chosen independently and probability of every edge is p, for constant $0 \le p \le 1$. In other words, a random graph is obtained from K_n by independent removal of edges with probability $1-p$. Random graphs form a probabilistic space (G^n, P_p), where $P_p : G^n \to \langle 0, 1 \rangle$ is a probabilistic function defined as follows:

$$\forall G_0 \in G^n: \ P_p(G_0) = p^{|E(G_0)|} \cdot (1-p)^{\binom{n}{2} - |E(G_0)|}.$$

We call arbitrary subset $Q \subseteq G^n$ a property of graphs of order n. We say that random graph has property Q, if $\lim_{n\to\infty} P_p(Q) = 1$, where the probabilistic function P_p is extended to subsets of G^n: $P_p(Q) = \sum_{G \in Q} P_p(G)$. Similarly, we say that random graph does not have property Q, if $\lim_{n\to\infty} P_p(Q) = 0$.

The second type of random graphs is constructed from n-ary cube B^n by independent removal of edges with probability $1-p$. The set of all these random graphs is denoted by $\tilde{G}^n$. The vertices of B^n are numbered, and the probabilistic function $\tilde{P}_p$ is defined as follows:

$$\forall G_0 \in \tilde{G}:\ \tilde{P}_p(G_0) = p^{|E(G_0)|} \cdot (1-p)^{n2^{n-1} - |E(G_0)|}.$$

We define a property of random graph similarly to previous case.

Properties of random graphs are usually defined as real-valued random variables. Random variable $X : (G^n, P_p) \to R$ or $X : (\tilde{G}^n, \tilde{P}_p) \to R$ is a function on probabilistic space. All random variables in the paper are non-negative, i.e. they attain only non-negative values. The expected value of random variable X is denoted by $\mathrm{E}(X)$.

Let X be a non-negative random variable with expected value $\mathrm{E}(X)$, and let $t > 0$. Then we have (Markov's inequality) [2]:

$$\Pr[X \geq t \cdot \mathrm{E}(X)] \leq \frac{1}{t}.$$

The symbol $\lg x$ denotes binary logarithm of x.

3. RANDOM GRAPHS OBTAINED FROM K_N

3.1 Kernel cliques

Kernel cliques are those cliques that must be contained in every vertex covering of a graph by cliques.

Definition 3.1. A clique K of graph G is a kernel clique if there exists a vertex v such that $v \in V(K)$, and v is not contained in any other clique of G. A set of all kernel cliques in G is called kernel of graph G.

We estimate the number of kernel cliques in a random graph. Let V_r be a random variable denoting the number of kernel cliques of order r in graph G.

LEMMA 3.2. *Let $p \in \langle 0, 1\rangle$, and $1 \leq r \leq n$. Then*

$$\mathrm{E}(V_r) = \binom{n}{r} \cdot p^{\binom{r}{2}} \cdot \left(1 - (1 - (1-p)^{n-r})^r\right).$$

PROOF. The number of cliques of order r in a graph with n vertices is $\binom{n}{r}$. Since the probability of being kernel clique is the same for every clique, we have

$$\mathrm{E}(V_r) = \binom{n}{r} P_K, \tag{2}$$

where P_K denotes the probability that clique K is a kernel clique. Let $V(K) = \{1, \ldots, r\}$. Let A_i, for $i = 1, \ldots, r$, be a random event that vertex i is not connected with any vertex from $G \setminus K$. Then $P_K = P(A_1 \cup \cdots \cup A_r)$, and we obtain:

$$P_K = P(A_1 \cup \cdots \cup A_r) = \sum_{l=1}^{r} (-1)^{l+1} \binom{r}{l} (1-p)^{l(n-r)} p^{\binom{r}{2}}.$$

We simplify the sum through binomial theorem:

$$P_K = p^{\binom{r}{2}}\left(1-(1-(1-p)^{n-r})^r\right)$$

Substitution into (2) yields the lemma. $\square$

For completeness, let us recall that $\binom{1}{2}=0$.

The order of a kernel clique is from 1 to n. The following theorem analyzes expected number of kernel cliques in a random graph.

THEOREM 3.3. *It holds*

$$\lim_{n\to\infty}\left(\sum_{r=1}^{n}\mathrm{E}(V_r)\right)=\begin{cases}n, & p=0,\\ 1, & p=1,\\ 0, & 0<p<1.\end{cases}$$

PROOF. The cases $p=0$ and $p=1$ trivially follow from Lemma 3.2. From now, let $0<p<1$. The sum is expanded according Lemma 3.2:

$$\sum_{r=1}^{n}\mathrm{E}(V_r)=\sum_{r=1}^{n}\binom{n}{r}\cdot p^{\binom{r}{2}}\cdot\left(1-(1-(1-p)^{n-r})^r\right) \tag{3}$$

We estimate the upper bound of the sum. Let $x=-(1-p)^{n-r}$. Bernoulli inequality $(1+x)^m \geq (1+mx)$ holds for all $x>-1$ and $m\geq 0$. The inequality holds also for $x=-1$ and $m\geq 1$. We use this inequality for $m=r$:

$$\begin{aligned}\sum_{r=1}^{n}\mathrm{E}(V_r) &\leq \sum_{r=1}^{n}\binom{n}{r}\cdot p^{\binom{r}{2}}\cdot r\cdot(1-p)^{n-r}\\ &= n\cdot\sum_{r=0}^{n-1}\binom{n-1}{r}\cdot\left(p^{\frac{r+1}{2}}\right)^r\cdot(1-p)^{n-1-r}\end{aligned}$$

Further simplification uses the fact $p^{3/2}\geq p^{(r+1)/2}$, for $r\geq 2$:

$$\begin{aligned}\sum_{r=1}^{n}\mathrm{E}(V_r)\leq n\cdot\Bigg[&\underbrace{(1-p)^{n-1}}_{A_n}+\underbrace{(n-1)p(1-p)^{n-2}}_{B_n}+\\ &+\sum_{r=2}^{n-1}\binom{n-1}{r}\cdot\left(p^{3/2}\right)^r\cdot(1-p)^{n-1-r}\Bigg]\end{aligned}$$

It is easily seen that $\lim_{n\to\infty}A_n=0$ a $\lim_{n\to\infty}B_n=0$. Thus,

$$\lim_{n\to\infty}\sum_{r=1}^{n}\mathrm{E}(V_r)\leq\lim_{n\to\infty}n\cdot\left[\sum_{r=0}^{n-1}\binom{n-1}{r}\cdot\left(p^{3/2}\right)^r\cdot(1-p)^{n-1-r}-A'_n-B'_n\right],$$

where $A'_n=(1-p)^{n-1}$ and $B'_n=(n-1)p^{3/2}(1-p)^{n-2}$. Again, $\lim_{n\to\infty}A'_n=0$ and $\lim_{n\to\infty}B_n=0$. Using binomial theorem we obtain:

$$\lim_{n\to\infty}\sum_{r=1}^{n}\mathrm{E}(V_r)\leq\lim_{n\to\infty}n(p^{3/2}+1-p)^{n-1}.$$

Since $p<1$, it holds $p^{3/2}+1-p<1$. Thus, $\lim_{n\to\infty}\sum_{r=1}^{n}\mathrm{E}(V_r)\leq 0$. Random variable V_r is non-negative. Combination of both properties yields the theorem. $\square$

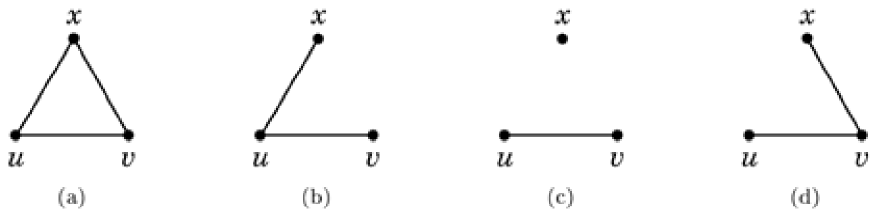

Fig. 1. Possible interconnections of vertices u, v, and x

Corollary 3.4 for random variable $\sum_{r=1}^{n} \mathrm{E}(V_r)$ easily follows from Theorem 3.3 and Markov's inequality.

Corollary 3.4. *A random graph has no kernel clique, i.e. the kernel of a random graph is empty.*

Kernel cliques are important for vertex covering of a graph by cliques since they must be present in any covering. However, kernel cliques are useless for searching covering of a random graph since, according Corollary 3.4, such cliques "do not exist" in random graph.

3.2 Regular vertices

Regular vertices are those vertices of a graph, that can be ignored in vertex covering since they will be covered "automatically" by covering other vertices.

Definition 3.5. Vertex $v \in V(G)$ is called regular, if there exists a vertex $u \neq v$ such that for every clique K of graph G we have: $u \in K \Rightarrow v \in K$. In such a case we say that vertex u induces v.

The following theorem shows that regular vertices are rare in random graphs. Let r_V be a random variable denoting the number of regular vertices in a graph.

Theorem 3.6. *Let* $p \in \langle 0, 1 \rangle$. *Then*

$$\lim_{n \to \infty} \mathrm{E}(r_V) = 0.$$

Proof. We give an upper bound on $\mathrm{E}(r_V)$. For an arbitrary, fixed vertex v we have:

$$\mathrm{E}(r_V) \leq n \cdot \Pr[v \text{ is regular vertex}].$$

Vertex v can be induced by any vertex $u \in V(G) \setminus \{v\}$. We distinguish four cases of interconnection of vertices u, v, and an arbitrary other vertex (we denote it x).

If vertex u induces v, these vertices must be adjacent. Therefore, there are four possible interconnections of vertices u, v, and x, see Figure 1. Further, the variant (d) is invalid, since in this case u does not induce v. Hence, the probability can be bounded as follows:

$$\Pr[v \text{ is regular vertex}] \leq (n-1)p\left(p^2 + p(1-p) + (1-p)^2\right)^{n-2},$$

where $n-1$ is the number of ways how to choose vertex u, p is the probability of edge (u, v), and the expression in brackets covers cases (a)–(c). After simplification we get

$$E(r_V) \leq n(n-1)p(p^2-p+1)^{n-2}.$$

Since $p < 1$ we have $\lim_{n\to\infty} E(r_V) \leq 0$. Taking into account non-negativity of $E(r_V)$ the theorem follows. □

The following corollary for random variable r_V follows from Theorem 3.6 and Markov's inequality.

COROLLARY 3.7. *A random graph has no regular vertex.*

4. RANDOM GRAPHS OBTAINED FROM B^N

4.1 Kernel cubes

Kernel cubes are those subcubes that must be contained in every vertex covering of a graph by maximal subcubes.

Definition 4.1. A maximal subcube K contained in a graph G is a kernel cube if there exists a vertex v such that $v \in V(K)$, and v is not contained in any other maximal subcube contained in G. A set of all kernel cubes in G is called kernel of graph G.

Let c_G denotes the number of kernel cubes contained in a random graph G, and $E(c_G)$ be the expected value of this parameter. The following lemmas estimate the value of $E(c_G)$.

LEMMA 4.2. *Let* $p \in \langle 0, 1\rangle$. *Then*

$$E(c_G) = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} p^{k2^{k-1}} (1-(1-(1-p)^{n-k})^{2^k})$$

PROOF. Let $P_{n,k}$ be a probability that fixed subcube of order k is a kernel cube of random graph G. Then

$$E(c_G) = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} P_{n,k}. \tag{4}$$

The probability $P_{n,k}$ is the same for all subcubes of fixed order. Thus, it is sufficient to compute $P_{n,k}$ for one subcube. The probability that a subcube is contained in a random graph G is $p^{k2^{k-1}}$. The probability that there exists at least one vertex from this subcube not appearing in any maximal subcube in G is $1-(1-(1-p)^{n-k})^{2^k}$. This vertex guarantees the maximality of our fixed subcubes. Thus, $P_{n,k} = p^{k2^{k-1}}(1-(1-(1-p)^{n-k})^{2^k})$. Substitution into (4) yields the lemma. □

The following lemma estimates $E(c_G)$.

LEMMA 4.3. *Let* $\{\gamma_n\}_{n\geq 0}$ *be a sequence such that* $\lim_{n\to\infty} \gamma_n = 0$. *Then*

$$\lim_{n\to\infty} \Pr\left[E(c_G) = n^{(1+\gamma_n)\lg\log_{1/p} n}(2(1-p))^n\right] = 1.$$

PROOF. We divide the sum from Lemma 4.2 into two separate sums S_1 (for $k > \lg n + l$) and S_2 (for $k \le \lg n + l$). Let us estimate S_1:

$$\begin{aligned} S_1 &= \sum_{k > \lg n + l} \binom{n}{k} 2^{n-k} p^{k 2^{k-1}} (1 - (1 - (1-p)^{n-k})^{2^k}) \\ &\le \sum_{k > \lg n + l} \binom{n}{k} 2^{n-k} \le p^{2^{\lg n + l}} \sum_{k \ge 0} \binom{n}{k} 2^{n-k} \\ &= p^{n \cdot 2^l} 3^n = (3p^{2^l})^n \end{aligned}$$

We choose l in a such way that $p^{2^l} < 1/3$. Then $\lim_{n\to\infty} S_1 = 0$.

Let us estimate S_2:

$$\begin{aligned} S_2 &= \sum_{k \le \lg n + l} \binom{n}{k} 2^{n-k} p^{k 2^{k-1}} (1 - (1 - (1-p)^{n-k})^{2^k}) \\ &\le \sum_{k \le \lg n + l} \binom{n}{k} 2^{n-k} p^{k 2^{k-1}} (1 - (1 - 2^k (1-p)^{n-k})) \\ &= \sum_{k \le \lg n + l} \binom{n}{k} 2^{n-k} p^{k 2^{k-1}} 2^k (1-p)^{n-k} \\ &\le 2^n \sum_{k \le \lg n + l} \binom{n}{k} p^{2^{k-1}} (1-p)^{n-k} \end{aligned}$$

First inequality uses fact that $(1 - nx) \le (1-x)^n$ for any non-negative integer n, and $x \in \langle 0, 1 \rangle$. Let us denote $b_k = \binom{n}{k} p^{2^{k-1}} (1-p)^{n-k}$. The quotient

$$\frac{b_{k+1}}{b_k} = \frac{(n-k) p^{2^{k-1}}}{(k+1)(1-p)}$$

is less than 1 for $k > \lg \log_{1/p} n + 1$, and greater than 1 for $k \le \lg \log_{1/p} n + 1$. Thus, maximal value of b_k is attained either for $k = \lambda$ or for $k = \lambda + 1$, where $\lambda = \lfloor \lg \log_{1/p} n + 1 \rfloor$. The sum S_2 can be bounded as follows:

$$\begin{aligned} S_2 &\le 2^n (\lg n + l + 1) \max_{k \le \lg n + l} b_k \\ &\le 2^n (\lg n + l + 1) \binom{n}{\lambda + 1} (1-p)^{n-(\lambda+1)} p^{2^\lambda} \\ &\le \binom{n}{\lambda+1} \frac{c_1 \lg n}{n} (2(1-p))^n, \end{aligned}$$

where c_1 is a constant (depending on p).

The lower bound of sum S_2 can be estimated as follows:

$$\begin{aligned} S_2 &\geq \binom{n}{\lambda} 2^{n-\lambda} p^{2^{\lambda-1}} \left(1 - (1-(1-p)^{n-\lambda})^{2^\lambda}\right) \\ &\geq \binom{n}{\lambda} 2^{n-\lambda} p^{2^{\lambda-1}} \left(1 - (1 - 2^\lambda (1-p)^{n-\lambda} + (2^\lambda (1-p)^{n-\lambda})^2)\right) \\ &= \binom{n}{\lambda} 2^{n-\lambda} p^{2^{\lambda-1}} 2^\lambda (1-p)^{n-\lambda} \left(1 - 2^\lambda (1-p)^{n-\lambda}\right). \end{aligned}$$

Since $\lim_{n\to\infty} 1 - 2^\lambda (1-p)^{n-\lambda} = 1$, we get for sufficiently large n:

$$S_2 \geq \binom{n}{\lambda} (2(1-p))^n \cdot \frac{c_2 \lg n}{n},$$

where c_2 is a constant. Putting these estimates together we get

$$\binom{n}{\lambda} (2(1-p))^n \cdot \frac{c_2 \lg n}{n} \leq \mathrm{E}(c_G) \leq \binom{n}{\lambda+1} (2(1-p))^n \cdot \frac{c_1 \lg n}{n} + (3p^{2^l})^n.$$

The lemma easily follows from these inequalities. □

Remark 4.4. We make use of the following inequality when estimating sum S_2 in the proof of Lemma 4.3:

$$(1 - nx) \leq (1-x)^n \leq (1 - nx + n^2 x^2),$$

for any integer $n \geq 0$ and $0 \leq x \leq 1$.

The following theorem follows Lemma 4.3 and Markov's inequality.

THEOREM 4.5. *The number c_G of kernel cubes contained in a random graph G is*

$$c_G = n^{(1+o(1)) \lg \log_{1/p} n} (2(1-p))^n.$$

4.2 Regular vertices

Definition 4.6. Let $G \in \tilde{G}^n$. A vertex $v \in V(G)$ is called regular, if there exists a vertex $u \neq v$ such that for every maximal subcube K contained in G one has: $u \in K \Rightarrow v \in K$. In such a case we say that vertex u induces v.

Let r_G be the number of regular vertices in a random graph G, and $\mathrm{E}(r_G)$ be the expected value of this parameter.

THEOREM 4.7. *The number r_G of regular vertices in a random graph G is*

$$r_G < \rho^n,$$

where $1.51 \leq \rho < 2$ is a constant depending on p.

PROOF. Let $P_n(u)$ be the probability that $u \in V(G)$ is a regular vertex. Since this probability is the same for all vertices, we have $\mathrm{E}(r_G) = 2^n P_n(u)$. For every vertex $v \in V(G)$ we denote by $P_{n,v}(u)$ the probability that u is a regular vertex, and v induces u. This probability is the same for all vertices with the distance k from u and we denote this probability by $P_{n,k}(u)$. Then

$$P_n(u) \leq \sum_{k=1}^{n} \binom{n}{k} P_{n,k}(u).$$

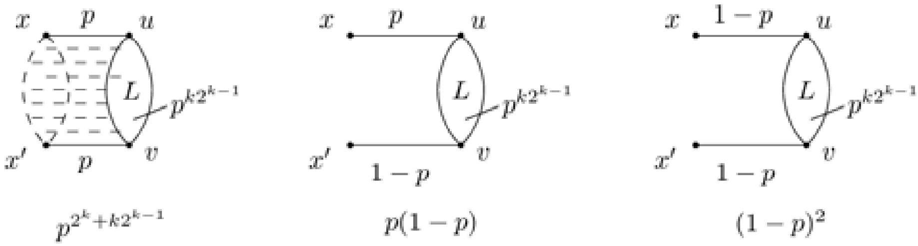

Fig. 2. Interconnections of vertices u, v, and x

We have an inequality, since the regular vertex can be induced by multiple vertices.

Let us estimate $P_{n,k}(u)$. Let v be an arbitrary vertex in the distance k from u. Vertices u and v define a subcube L of order k. Let x be an arbitrary neighboring vertex of u, not belonging to L (there are $n-k$ such vertices). Vertex v induces u, if one of the cases depicted in Figure 2 happens.
Hence, we get

$$\begin{aligned} P_{n,k} &= p^{k2^{k-1}} \left(p(1-p) + (1-p)^2 + p^{2^k+k2^{k-1}}\right)^{n-k} \\ &= p^{k2^{k-1}} \left(1-p+p^{(k+2)2^{k-1}}\right)^{n-k}, \end{aligned}$$

and

$$P_n(u) \le \sum_{k=1}^{n} \binom{n}{k} p^{k2^{k-1}} \left(1-p+p^{(k+2)2^{k-1}}\right)^{n-k}.$$

The expected value $\mathrm{E}(r_G)$ can be estimated as follows:

$$\begin{aligned} \mathrm{E}(r_G) &\le 2^n \sum_{k=1}^{n} \binom{n}{k} p^{k2^{k-1}} \left(1-p+p^{(k+2)2^{k-1}}\right)^{n-k} \\ &= \sum_{k=1}^{n} \binom{n}{k} (2p^{2^{k-1}})^k \left(2(1-p+p^{(k+2)2^{k-1}})\right)^{n-k} \\ &\le 2np(2(1-p+p^3))^{n-1} + \sum_{k=2}^{n} \binom{n}{k} (2p^2)^k (2(1-p+p^8))^{n-k} \\ &\le 2np(2(1-p+p^3))^{n-1} + \sum_{k=0}^{n} \binom{n}{k} (2p^2)^k (2(1-p+p^8))^{n-k} \\ &= 2np(2(1-p+p^3))^{n-1} + (2(1-p+p^2+p^8))^n \\ &\le c_3\rho^n, \end{aligned}$$

where c_3 is a constant, and ρ is a constant depending on p. Let f be a function defined as follows: $f(p) = 2(1-p+p^2+p^8)$. Then f attains minimum $\approx 1,5064$ on interval $[0,1]$. This gives a lower bound on ρ. The upper bound easily follows from the number of vertices in B^n.

Using Markov's inequality we get $r_G < \rho^n$. □

5. CONCLUSION

The results can be interpreted in the following way. In the case of random graphs obtained from K_n, regular vertices and kernel cliques do not exist, which means that these graphs are (in some sense) "dense". On the other hand, the existence of regular vertices and kernel cubes in random graphs obtained from B^n can be used to estimate the lower bound for the length of vertex covering of these graphs by cubes, or it can be used to design algorithms for constructing such vertex coverings.

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