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# On Geometric Properties of Some Fuzzy Relations*

DOBRAKOVOVÁ JANA

**Abstract**

This paper is proposed firstly to discuss connection of two types of real binary fuzzy relations, the shift-invariant and the coherent nearnesses, and secondly to investigate shift-invariant nearnesses and their basic properties from the geometric point of view.

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**Additional Key Words and Phrases:** real binary fuzzy relation, shift-invariant nearness, uniform nearness, coherent nearness

## 1. INTRODUCTION

The notion of a fuzzy set originated from the observation made by Zadeh (1965) that "more often than not, the classes of objects encountered in the real physical word do not have precisely defined criteria of membership".

The specificity of fuzzy sets is to capture the idea of partial membership. Of course, the novelty of the ideas carried out by the development of fuzzy sets have raised many doubts by opponents and skeptics. Nevertheless, these ideas have been from the beginning encountering great interest and gaining more and more achievements and applications.

This paper is devoted to some aspects of one important notion of the fuzzy sets theory, namely the binary fuzzy relations.

Let us recall, that, in general, a binary fuzzy relation on a universe $X$ is a fuzzy subset of the Cartesian product $X \times X$. For more details see, for example, [5].

In what follows we will largely pay attention to a special family of real binary fuzzy relations, modelling in a sense proximity, similarity, or indistinguishability of real numbers, which are called the **nearness.**The nearnessess in some sense measure the degree to which two points of a universe are close one another and they are to be a natural fuzzification of the metric, or distance, in the space of real numbers.

In many works ([1] - [7]) several types of nearnesses were discussed. In these papers mutual correspondence and other properties of different types of binary fuzzy relations, defined on a universe, modelling in some way nearness or equivalence of its elements, and other relevant problems, were studied.

If we consider real binary fuzzy relations, it means binary fuzzy relations on $\mathbb{R}$,

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from the geometric point of view, they are represented by real functions of two real variables, with the values from the interval $\langle 0, 1 \rangle$, possessing some significant properties, depending on the type of the relation. The main purpose of this article is to study just the geometric aspects of these fuzzy relations, mainly a special type of nearnesses, called the shift-invariant nearnesses.

## 2. PRELIMINARIES

The **shift-invariant nearness** is called after the fifth property in the following definition. It is a real binary fuzzy relation, satisfying some natural properties, corresponding to the algebraic, topological and lattice structure of real axis.

**Definition 1.**
A binary fuzzy relation $f$ on $\mathbb{R}$ is called the shift-invariant nearness if

(1) $f(x, x) = 1$ for each $x \in \mathbb{R}$,

(2) $f(x, y) = f(y, x)$ for each $x, y \in \mathbb{R}$,

(3) $f(x, z) \leq \min(f(x, y), f(y, z))$ for each $x, y, z \in \mathbb{R}$, such that $x \leq y \leq z$,

(4) $\lim\limits_{n \to \infty} x_n = \infty \Rightarrow \lim\limits_{n \to \infty} f(x_n, x_0) = 0$ for each $x_0 \in \mathbb{R}$,

(5) $f(x, y) = f(x + z, y + z)$ for each $x, y, z \in \mathbb{R}$.

If we consider a nearness in an arbitrary universe, without any structure, then we are not restricted by any requirements of compatibility with original structures and the definition of a nearness should comply only with an intuitive idea of a nearness. This approach to the notion of a nearness is utilized in the next definition.

**Definition 2.**
Let $X$ be a set. A binary fuzzy relation $f$ on $X$ is called the coherent nearness on $X$, if:

(N1) $f(x, x) = 1$, for each $x \in X$,

(N2) $f(x, y) = f(y, x)$, for each $x, y \in X$,

(N3) for each $\epsilon > 0$ there exists $\delta < 1$ such that

$$f(x, y) > \delta \Longrightarrow |f(x, z) - f(y, z)| < \epsilon, \text{ for each } x, y, z \in X.$$

The properties (N1) and (N2) , it means the **reflexivity** and the **symmetricity** of $f$ are identical with properties (1) and (2) of a shift-invariant nearness. The property (N3), the **coherence**, substitutes, in a sense, the triangular inequality and it has the following meaning:

If two points $x$ and $y$ from the universe $X$ are sufficiently near one another, then the difference of their nearnesses to any other point $z \in X$ is arbitrarily small.

Some questions about the relationship between these two types of nearnesses, it means the shift-invariant and the coherent nearness, in the case if the universe $X$ is the set of all real numbers $\mathbb{R}$, were already studied in [2] and [3].

The main aim of this paper is to proceed in investigating this relationship and to look more closely at the geometric properties of these nearnesses, especially properties of their graphs.

For abbreviation, in what follows, let a fuzzy relation stand for a real binary fuzzy relation.

## 3. COHERENT NEARNESSES NEED NOT BE SHIFT-INVARIANT AND CONVERSELY

Although under some additional conditions every shift-invariant nearness possesses the property (N3), in general it is not so. There exist shift-invariant nearnesses that do not satisfy the property (N3) and on the other hand, there are coherent nearnesses, not being shift-invariant.

**Remark 1.**

It can be proved ([3]), that a fuzzy relation $f$ is the shift-invariant nearness if and only if it is **uniform**, what means that $f$ is expressible in the form

$$f(x, y) = b(|x - y|) \text{ for each } x, y \in \mathbb{R},$$

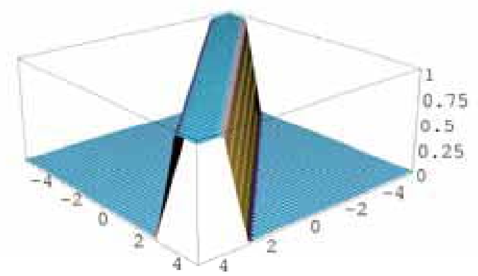
for a **nearness-generating** non-increasing function $b : [0, \infty] \to [0, 1]$, such that $b(0) = 1$, $\lim_{x \to \infty} b(x) = 0$.

In applications of this result we denote the nearness generating function always by $b$.

**Example 1.**

Let us define a fuzzy relation $f$ by

$$f(x, y) = \begin{cases} 1, & \text{for } |x - y| < 1, \\ 2 - |x - y|, & \text{for } 1 \leq |x - y| \leq 2, \\ 0, & \text{else.} \end{cases}$$



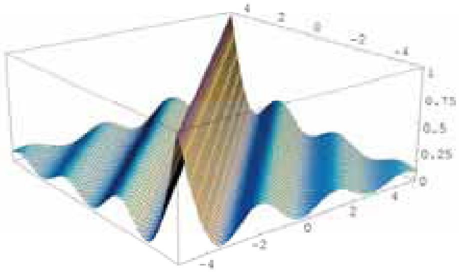

Is is simple to see, that this fuzzy relation is, according to the previous remark, a shift-invariant nearness. But it clearly does not satisfy the property (N3). For example, $f(0, 1) = 1$, but

$$\left| f\left(0, \frac{3}{2}\right) - f\left(\frac{3}{2}, 1\right) \right| = \frac{1}{2}.$$

**Example 2.**

Let us define a fuzzy relation $f$ by

$$f(x,y) = \frac{\cos^2(|x-y|)}{1+|x-y|}.$$



The fuzzy relation $f$ is not any shift-invariant nearness, since it does not satisfy the property (3) from the Definition 1. This follows from the fact, that the nearness generating function of our nearness is the function $b(x) = \frac{\cos^2 x}{1+x}$, which is not monotone on $(0,\infty)$. Really, if $x = 0,\ y = \frac{\pi}{2},\ z = \pi$, then $x \le y \le z$, but

$$f(x,z) = \frac{1}{1+\pi} > 0 = \min(f(x,y), f(y,z)),$$

which contradicts the property (3).

On the other hand, it can be shown, that $f$ is the coherent nearness. Questionable is only the property (N3). Let us suppose, that $f$ does not satisfy (N3). Then there is an $\epsilon > 0$ such that for all natural $n$ there exist $x_n, y_n, z_n \in \mathbb{R}$ such that

$$f(x_n,y_n) = \frac{\cos^2(|x_n-y_n|)}{1+|x_n-y_n|} > 1 - \frac{1}{n}, \text{ but } |f(x_n,z_n) - f(y_n,z_n)| \ge \epsilon.$$

It means, that :

$$\lim_{n\to\infty} b(|x_n-y_n|) = \lim_{n\to\infty} f(x_n,y_n) = 1.$$

But, from properties of the function $b$ it follows, that $\lim_{n\to\infty} |x_n - y_n| = 0$, and this, together with the continuity of the function $f$, leads to the contradiction with the assumption.

Despite of the fact, that neither in the real case the shift-invariant and coherent nearnesses do coincide, there is a significant family of shift-invariant nearnesses, having always the property (N3). The following proposition is a straightforward consequence of theorems in [2].

**Proposition 1.**

Any real shift-invariant nearness with a continuous and decreasing nearness-generating function $b$ possesses the property (N3).

## 4. SOME GEOMETRIC ASPECTS OF SHIFT-INVARIANT NEARNESSES

In this part we are going to discuss some basic geometric properties of shift-invariant nearnesses. They are real functions of two real variables, defined on $\mathbb{R} \times \mathbb{R}$, with values from the interval $\langle 0, 1 \rangle$.

First two properties of any such nearness $f$, it means reflexivity and symmetricity (properties (1) and (2) from the Definition 1, have very simple geometric interpretation.

Geometrically speaking, the reflexivity means, that the function $f$ gains at each point of the line $y = x$ the value 1. For simplicity of terminology, in what follows, we will call this line the **line of symmetry**.

Another way of stating the symmetricity is to say that the graph of $f$ is a surface, symmetrical about the plane $y = x$.

Unless otherwise stated in the remainder of this paper we assume any fuzzy relation $f$ to be reflexive and symmetric and for simplicity we will call it briefly fuzzy relation.

Consider now the property (3) from Definition 1. This property yields information about behaviour of the function $f$ on straight lines perpendicular to the line of symmetry.

**Proposition 2.**
Let a fuzzy relation $f$ satisfy the property (3) from Definition 1 and let $a$ be any real number. Then the real function of a real variable $F_a$ defined by

$$F_a(x) = f(x, a - x), \quad \text{for each} \quad x \in \mathbb{R},$$

is non-decreasing on the interval $(-\infty, \frac{a}{2})$ and non-increasing on the interval $(\frac{a}{2}, \infty)$, $F_a(\frac{a}{2}) = 1$ and $F_a(\frac{a}{2} - x) = F_a(\frac{a}{2} + x)$, for each real $x$.

*Proof.* $F_a(\frac{a}{2}) = 1$ and $F_a(\frac{a}{2} - x) = F_a(\frac{a}{2} + x)$, for each real $x$ follows directly from the definition of the function $F_a$ and from the reflexivity and symmetricity of $f$.

We have only to prove that $F_a$ is non-increasing on the interval $(\frac{a}{2}, \infty)$ and then, combining this with the equality $F_a(\frac{a}{2} - x) = F_a(\frac{a}{2} + x)$, it is easily seen that $F_a$ must be at the same time non-decreasing on the interval $(-\infty, \frac{a}{2})$.

Let us suppose, that $x_1$, $x_2$ are such real numbers, that $\frac{a}{2} < x_1 < x_2$. It is clear, that then

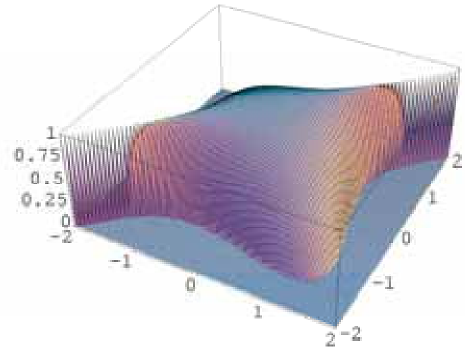
$$a - x_2 < a - x_1 < \frac{a}{2} < x_1 < x_2.$$

From the property (3) it follows that $f(x_2, a - x_2) \leq f(x_2, a - x_1) \leq f(x_1, a - x_1)$ and therefore, $F_a(x_2) \leq F_a(x_1)$.

Opposite is not true. It means that there exist fuzzy relations, not satisfying the property (3), but in spite of that, if we construct for them functions $F_a$ in the same way as in the previous proposition, these functions have the same properties, i.e. they are non-increasing on $(\frac{a}{2}, \infty)$ and $F_a(\frac{a}{2} - x) = F_a(\frac{a}{2} + x)$.

**Example 3.**
Let us consider a fuzzy relation $f$ defined by

$$f(x,y)=\begin{cases}1-\frac{1}{8}(x-y)^2e^{(x+y)^2}, & \text{for } (x-y)^2e^{(x+y)^2}<8,\\ 0, & \text{else.}\end{cases}$$




From the definition it is clear, that $f$ is reflexive and symmetric. But it does not satisfy the property (3): It can be easily checked that

$$f\left(-\frac{5}{4},\frac{5}{4}\right)=\frac{7}{32}\doteq 0.22,\quad f\left(-\frac{5}{4},0\right)=f\left(0,\frac{5}{4}\right)=1-\frac{25e^{\frac{25}{16}}}{128}\doteq 0.07.$$

This contradicts the property (3), since

$$-\frac{5}{4}<0<\frac{5}{4}\quad\text{and}\quad f\left(-\frac{5}{4},\frac{5}{4}\right)>\min\left(f\left(-\frac{5}{4},0\right),f\left(0,\frac{5}{4}\right)\right).$$

On the other hand, let $a$ be any real number. Consider the function

$$F_a(x)=f(x,a-x)=\begin{cases}1-\frac{1}{8}(2x-a)^2e^{a^2}, & \text{for } |x-\frac{a}{2}|<\sqrt{\frac{2}{e^{a^2}}},\\ 0, & \text{else.}\end{cases}$$

It can be simply shown that the function $F_a$ satisfies all properties from Proposition 2.

**Corollary.**
Let a fuzzy relation $f$ satisfy the property (3) from Definition 1 and let $[x_0,y_0]$, $x_0\neq y_0$ be any ordered couple of two different real numbers (a point not lying on the line of symmetry). Then $f(x,y)\geq f(x_0,y_0)$ for any point $[x,y]$ from the right-angled isosceles triangle with vertices $[x_0,y_0]$, $[x_0,x_0]$, $[y_0,y_0]$.

From the geometric point of view, the property (4) from Definition 1 means, that the value of $f$ at a pair of points tends to 0, if one is fixed and the other tends to infinity along any straight line, parallel to one of coordinate axes. Let us try to characterize this property also by means of the functions $F_a$. In general, for arbitrary fuzzy relation, as the following trivial examples show, there is no significant coincidence between the property (4) and the behaviour of functions $F_a$.

**Example 4.**

$$f_1(x,y) = \begin{cases} 1, & \text{if } x = y, \text{ or } x.y = 0, \\ 0, & \text{else} \end{cases}, \quad f_2(x,y) = \begin{cases} 1, & \text{if } |x| = |y|, \\ 0, & \text{else} \end{cases}$$

It is clear that the fuzzy relation $f_1$ does not satisfy the property (4). For example $\lim_{n\to\infty} f_1(0,n) = 1$. Nevertheless it has the following interesting property: It holds $\lim_{x\to\infty} F_a(x) = 0$, for each real number $a$.

On the other hand, the fuzzy relation $f_2$ satisfies the property (4), but $\lim_{x\to\infty} F_0(x) = 1$.

Of course, as follows immediately from the Corollary, none of the functions $f_1$, $f_2$ satisfies the property (3).

But if a binary fuzzy relation $f$ satisfies (3), then the property (4) implies again similar behaviour of $f$ on all lines perpendicular to the line of symmetry, what means, that the value $f$ at a point tends to 0 if the point tends to infinity along any line with the equation $x + y = a$, for each real $a$.

**Proposition 3.**
Let a fuzzy relation $f$ satisfy the properties (3) and (4) from Definition 1 and let $a$ be any real number. If the real function of a real variable $F_a$ is defined as above, then

$$\lim_{x\to\infty} F_a(x) = \lim_{x\to-\infty} F_a(x) = 0.$$

*Proof.* Let the assertion be not valid. It means, that there exists a real number $a$ such that $\lim_{x\to\infty} F_a(x) \neq 0$. As follows from the Corollary, the function $F_a$ is on the interval $(\frac{a}{2}, \infty)$ non-increasing and as follows from its definition, it is also bounded below.

It implies the existence of a number $a_0 \in (0,1\rangle$ such that $\lim_{x\to\infty} F_a(x) = a_0$. From the definition and properties of $F_a$ then $f(x, a-x) = F_a(x) \geq a_0$, for each real $x$. It can be easily proved, that then also $f(x,y) \geq a_0$, for each couple $[x,y]$.

It is trivial, if $[x,y]$ is a point from the line of symmetry or from the line $x+y = a$. Let suppose, that we have a point not lying on these two lines.

First let $[x,y]$ be such a point, that $x > y$ and $x+y > a$, or $x < y$ and $x+y < a$. Then $a-x < y < x$, or $a-x > y > x$ and consequently $f(x,y) \geq f(x,a-x) \geq a_0$.

The other possibility is that $x < y$ and $x+y > a$, or $x > y$ and $x+y < a$. Then $a-y < x < y$, or $a-y > x > y$ and consequently $f(x,y) \geq f(a-y,y) \geq a_0$.

But it is clear, that this result contradicts the property (4).

Again, the opposite implication is not true, convergence of $f$ to zero along each line parallel to one of axes does not imply convergence of $f$ to zero along lines perpendicular to the line of symmetry. This fact illustrates the following example.

**Example 5.**

$$f(x,y) = \begin{cases} 1, & \text{if } x.y \geq 0, \\ 0, & \text{else} \end{cases}$$

It can be simply checked, that $f$ is a binary fuzzy relation, satisfying the properties (1), (2) and (3). It evidently does not satisfy the property (4), for example

$$\lim_{n\to\infty} f(0,n) = 1.$$

On the other hand,it can be easily seen, that if $a$ is any fixed real number, then the function

$$F_a(x) = f(x, a-x) = \begin{cases} 1, & \text{if } x \in \langle \min(0,a), \max(0,a) \rangle, \\ 0, & \text{else,} \end{cases}$$

It follows immediately, that

$$\lim_{x\to\infty} F_a(x) = \lim_{x\to-\infty} F_a(x) = 0$$

The last property is the property (5), so called shift-invariance. Its geometric meaning is treated in the following

**Proposition 4.**
For any fuzzy relation $f$ the following three statements are equivalent:
(a) $f$ satisfies the property (5) from Definition 1,
(b) $f$ is constant on each line parallel to the line of symmetry,
(c) for any real number $a$ : $F_a(x) \equiv F_0(x - \frac{a}{2})$.

*Proof.*
(a)$\Longrightarrow$ (b) Let $l$ be a straight-line: $y = x + q$ and let $A = [x_1, x_1 + q]$ and $B = [x_2, x_2 + q]$ be a couple of its points. Then

$f(A) = f(x_1, x_1+q) = f(x_1+(x_2-x_1), x_1+q+(x_2-x_1)) = f(x_2, x_2+q) = f(B)$

(b)$\Longrightarrow$ (c) Let $a, x_0 \in \mathbb{R}$, then points $A = [x_0, a - x_0]$ and $B = [x_0 - \frac{a}{2}, \frac{a}{2} - x_0]$ both belong to the line $y = x + (a - 2x_0)$. Therefore

$$F_a(x_0) = f(x_0, a - x_0) = f\left(x_0 - \frac{a}{2}, \frac{a}{2} - x_0\right) = F_0\left(x_0 - \frac{a}{2}\right)$$

(c)$\Longrightarrow$ (a) Let $a, x_0, y_0 \in \mathbb{R}$, then

$$f(x_0 + a, y_0 + a) = f(x_0 + a, x_0 + y_0 + 2a - (x_0 + a)) = F_{x_0+y_0+2a}(x_0 + a) =$$

$$= F_0\left(x_0 + a - \frac{x_0 + y_0 + 2a}{2}\right) = F_0\left(\frac{x_0}{2} - \frac{y_0}{2}\right) =$$

$$= F_0\left(x_0 - \frac{x_0 + y_0}{2}\right) = F_{x_0+y_0}(x_0) = f(x_0, (x_0 + y_0) - x_0) = f(x_0, y_0)$$

**Corollary.**

Let a fuzzy relation $f$ satisfy the properties (3) and (5) from Definition 1. Then $f$ is the shift-invariant nearness if and only if for each real number $a$

$$\lim_{x \to \infty} F_a(x) = \lim_{x \to -\infty} F_a(x) = 0.$$

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Jana Dobrakovová,
Department of Mathematics,
Slovak University of Technology,
nám. Slobody 17, 812 31 Bratislava, Slovak Republic
e-mail: jana.dobrakovova@stuba.sk


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