Journal of Applied Mathematics, Statistics and Informatics (JAMSI), **2** (2006), No. 1

On the Existence and Nonuniqueness of an Evolution System Solution

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Abstract

Many authors study generic properties of solutions for nonlinear ordinary and partial differential problems (see for example [2], [3], [7]-[13]. The study of quantitative and qualitative properties of solution sets is well-founded if the given problem possesses more than one solution.

The presented paper deals with existence theorems for bounded and unbounded nonlinearities in the observed equation. There are solved some problems with the infinite number of cassis solutions.

Mathematics Subject Classification 2000: 35G30, 35K55, 37L05, 47J35

Additional Key Words and Phrases: Evolution systems, an initial - boundary value problem, Hölder space, Green matrix, mild solution

1. INTRODUCTION

With respect to the theoretical and practical point of view, there is very often necessary to study quantitative and qualitative properties of set solution for nonlinear problems. That's the reason why we look for sufficient existence and no uniqueness conditions for some differential systems in this paper.

The Peano phenomenon of the existence of a solution continuum of the initial value problem for ordinary differential systems is well - known. This phenomenon has been studied by many autors in [2], [3], [4], [5], [17], [21]. The structure of solution sets of higher order partial differential equations was observed in papers [7] - [12] .

In this paper we study questions of a solvability of quasilinear initial - boundary value problems for evolutions systems of an even order with the continuous nonlinearities and the general boundary value conditions. Several initial-boundary value problems with a continuum of smooth solutions are implemented in this paper.

The present results allow us to observe different problems describing dynamics of mechanical processes (bendding, vibration), phisycal - heating processes, reaction - diffusion processes in chemical and biological technologies or in the ecology.

2. THE FORMULATION OF PROBLEM, ASSUMPTIONS AND SPACES

The set $\Omega \subset \mathbb{R}^n$ for $n \in N$ means a bounded domain with the boundary $\partial\Omega$. The real number T will be positive and $Q := (0,T] \times \Omega$, $\Gamma := (0,T] \times \partial\Omega$. If we put the multiindex $k = (k_1, \ldots, k_n)$ with the module $|k| = \sum_{i=1}^{n} k_i$, then we use the notation D_x^k for the differential operator $\frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}$ and D_t for $\frac{\partial}{\partial t}$. If the module $|k| = 0$

then D_x^k means an identity mapping. The symbol $cl\, M$ means the closure of the set M in $\mathbb{R}^n$.

In this paper we consider the general system of $p \geq 1$ nonlinear differential equations (parabolic or non-parabolic type) of an arbitrary even order $2b$ (b is a positive integer) with p unknown functions in the column vector form $(u_1, \dots, u_p)^T = u : cl\, Q \to \mathbb{R}^p$. Its matrix form is given as follows:

$$A(t,x,D_t,D_x)u + f(t,x,\bar{D}_x^\gamma u) = g(t,x) \text{ for } (t,x) \in Q\,, \tag{2.1}$$

where

$$A(t,x,D_t,D_x)u := D_t u - \sum_{|k|=2b} a_k(t,x) D_x^k u - \sum_{0\leq|k|\leq 2b-1} a_k(t,x) D_x^k u\,,$$

and $\bar{D}_x^\gamma u$ is a vector function whose components are derivatives $D_x^\gamma u_l$ with the different multiindices $0 \leq |\gamma| \leq 2b-1$ for $l = 1, \dots, p$.

The system of boundary conditions is given by the vector equation with the bp components

$$B(t,x,D_x)u|_{cl\,\Gamma} := (B_1(t,x,D_x)u, \dots, B_{bp}(t,x,D_x)u)^T\big|_{cl\,\Gamma} = 0 \tag{2.2}$$

in which

$$B_j(t,x,D_x)u := \sum_{0\leq|k|\leq r_j} b_{jk}(t,x) D_x^k u$$

for an integer $0 \leq r_j \leq 2b-1$ and $j = 1, \dots, bp$.

Further the initial value homogeneous condition

$$u(0,x) = 0 \text{ for } x \in \bar{\Omega} \tag{2.3}$$

is considered.

Here the given functions are following mappings:
$a_k := \left(a_k^{hl}\right)_{h,l=1}^p : \mathrm{cl}\, Q \to \mathbb{R}^{p^2}$ for $0 \leq |k| \leq 2b$ are $(p \times p)$-matrix functions;
$b_{jk} = (b_{jk}^1, \dots, b_{jk}^p) : cl\,\Gamma \to \mathbb{R}^p$ for $0 \leq |k| \leq r_j$, $j = 1, \dots, bp$ are row vector functions;
$f = (f_1, \dots, f_p)^T : cl\, Q \times \mathbb{R}^\kappa \to \mathbb{R}^p$ and $g = (g_1, \dots, g_p)^T : cl\, Q \to \mathbb{R}^p$ are column vector functions, where κ is a positive integer given by the inequality

$$\kappa \leq \left[\binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots \right.$$
$$\left. \cdots + \binom{n+|\gamma|-2}{|\gamma|-1} + \binom{n+|\gamma|-1}{|\gamma|}\right] p.$$

Under several supplementary assumptions, linear problem (2.1) - (2.3) with $f = 0$ defines homeomorphism between some Hölder spaces. These suppositions are :

(P) A $\delta-$uniform parabolic condition holds for system (2.1) in the sence of J. G. Petrovskii, $\delta > 0$.

The system (2.1) and boundary conditon (2.2) are connected by

(C) a δ^+ - uniform complementary condition with $\delta^+ > 0$ and

(Q) a compatibility condition.

The coefficients of the operator $A(t,x,D_t,D_x)$ from (2.1) and of $B(t,x,D_x)$ from (2.2) and the boundary $\partial\Omega$ satisfy

($S^{l+\alpha}$) a smoothness condition for a nonnegative integer l and a number $\alpha \in (0,1)$.

We shall be employed with the Banach spaces of continuously differentiable functions $C^l_x(cl\,Q, \mathbb{R}^p)$ and $C^{l/2b,l}_{t,x}(cl\,Q, \mathbb{R}^p)$ and the Hölder spaces $C^{l+\alpha}_x(cl\,Q, \mathbb{R}^p)$, $C^{(l+\alpha)/2b,l+\alpha}_{t,x}(cl\,Q, \mathbb{R}^p)$ for a nonnegative integer l and $\alpha \in (0,1)$.

For the exact definition of conditions (P), (C), (Q), ($S^{l+\alpha}$) see [18, pp. 12 - 21] or for $p = 1$ [13, pp. 315 - 319] . For the definition of spaces see [18, pp. 8 - 12] or [6] and for $p = 1$ [13].

The homeomorphism result for (2.1) - (2.3) can be formulated as follows:

Proposition 2.1. (See [18, p. 21] and [15, pp. 182 - 183].) *Let the conditions* (P), (C) *and* (S^α) *be satisfied for* $\alpha \in (0,1)$. *Necessary and sufficient conditions for the existence and uniqueness of the solution*

$$u \in C^{(2b+\alpha)/2b,2b+\alpha}_{t,x}(cl\,Q, \mathbb{R}^p)$$

of linear problem (2.1) - (2.3) *for* $f = 0$ *is*

$$g \in C^{\alpha/2b,\alpha}_{t,x}(cl\,Q, \mathbb{R}^p)$$

and the compatibility condition (Q).

Moreover, there exists a constant $c > 0$ *independent of* g *such that*

$$c^{-1}||g||_{\alpha/2b,\alpha,Q,p} \le ||u||_{(2b+\alpha)/2b,2b+\alpha,Q,p} \le c||g||_{\alpha/2b,\alpha,Q,p}$$

3. THE SOLVABILITY OF NONLINEAR PROBLEMS

Using results on the Green matrix for linear problem (2.1) - (2.3) (with $f = 0$) we shall study the existence of the given nonlinear problem from the Chapter 2.

To prove the existence theorem we shall use estimations of a Green $p \times p-$matrix and its derivatives from [15, pp. 182 - 183] under the assumptions (P), (C), (S^α). Hence we get the following lemma.

Lemma 3.1. *Let the asumptions* (P), (C), (S^α) *be satisfy for* $\alpha \in (0,1)$. *Then we have for the Green matrix* G *of linear problem* (2.1) - (2.3) *with* $f = 0$

$$|D^{k_0}_t D^k_x G(t,x;\tau,\xi)| \le c(t-\tau)^{-\mu}||x-\xi||^{2b\mu-(n+2bk_0+|k|)}_{\mathbb{R}^n}\mathbb{E} \qquad (3.1)$$

for $0 \le 2bk_0 + |k| \le 2b$ *and* $\mu \le (n + 2bk_0 + |k|)/2b$, *thereby* $0 \le \tau < t \le T$ *and* $x, \xi \in cl\,\Omega$, $x \ne \xi$. *The positive constant* c *does not depend on* t, x, τ, ξ *and* $\mathbb{E}$ *means the* $p \times p-$*matrix compositing only from units,* $r = 2b/(2b-1)$.

Proof. From the estimation (see [15, pp. 182 - 183])

$$|D^{k_0}_t D^k_x G(t,x;\tau,\xi)| \le c_1(t-\tau)^{-\frac{n+2bk_0+|k|}{2b}} \exp\{-c_2\frac{||x-\xi||^r_{\mathbb{R}^n}}{(t-\tau)^{1/(2b-1)}}\} \le$$

$$\le c_1(t-\tau)^{-\mu}||x-\xi||^{2b\mu-(n+2bk_0+|k|)}_{\mathbb{R}^n}\times$$

$$\times[\,||x-\xi||^{2b}_{\mathbb{R}^n}/(t-\tau)\,]^{(n+2bk_0+|k|-2b\mu)/2b}\exp\{-c_2[\,||x-\xi||^{2b}_{\mathbb{R}^n}/(t-\tau)\,]^{1/(2b-1)}\}\mathbb{E}$$

Since $n + 2bk_0 + |k| - 2b\mu \ge 0$ and $||x-\xi||_{\mathbb{R}^n} < \operatorname{diam}\Omega$ so for $0 < \delta \le t - \tau \le T$

the estimation (3.1) is true. If $0 < t - \tau < \delta$ such with respect to

$$\lim_{y \to +\infty} y^u \exp\{-cy^v\} = 0$$

for every u, $v \in \mathbb{R}$ and $c > 0$, we get the estimation (3.1) too. □

REMARK 3.1. For any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the inequalities

$$c_n \sum_{i=1}^{n} |x_i| \le ||x||_{\mathbb{R}^n} \le \sum_{i=1}^{n} |x_i| \tag{3.2}$$

holds for $c_n \in (0, 1/(\sqrt{2})^{n-1})$, $n \in N$ does not depend on x.

The aim of this part is to show that nonlinear problem (2.1) - (2.3) has at least one mild solution (see the proof of Theorem 3.1) $u \in C_x^{2b-1}(cl\, Q, \mathbb{R}^p)$ for continuous functions f and g. Then we formulate examples of nonuniquely solvable problems.

THEOREM 3.1. (The existence theorem.) *Let the hypotheses* (P), (C), (Q), (S^α) *for* $\alpha \in (0,1)$ *be satisfied and* $g : cl\, Q \to \mathbb{R}^p$ *be continuous function at* $cl\, Q$. *Let* $f : cl\, Q \times (-\infty, \infty)^\kappa \to \mathbb{R}^p$ *be continuous and bounded function at* $cl\, Q \times (-\infty, \infty)^\kappa$, *where* κ *is the positive integer given in the formulation of the problem* (2.1) - (2.3). *Then there is at least one mild solution* $u \in C_x^{|\gamma|}(cl\, Q, \mathbb{R}^p)$ *for* $0 \le |\gamma| \le 2b - 1$ *of* (2.1) - (2.3).

Proof. We use Lerray - Schauder fixed point theorem from [23, p. 56].

First the mild solution $u \in C_x^{|\gamma|}(cl\, Q, \mathbb{R}^p)$ of the problem (2.1) - (2.3) satisfies the column vector integro-differential equation

$$\begin{aligned} u(t,x) \ &= \int_0^t d\tau \int_\Omega G(t,x;\tau,\xi)\,[\,g(\tau,\xi) - f(\tau,\xi,\overline{D}^\gamma u(\tau,\xi))\,]\,d\xi =: \\ &=: (Su)(t,x) \quad \text{for } (t,x) \in cl\, Q \end{aligned} \tag{3.3}$$

and on the contrary the solution $v \in C_x^{|\gamma|}(cl\, Q, \mathbb{R}^p)$ satisfying (3.3) is called a mild solution of (2.1) - (2.3).

Let us take an arbitrary $u \in C_x^{|\gamma|}(cl\, Q, \mathbb{R}^p)$ where $0 \le |\gamma| \le 2b - 1$. Then there is a constant $M > 0$ such that the vector inequality

$$|g(t,x) - f(t,x,\overline{D}_x^\gamma u(t,x))| \le M\mathbb{J}$$

holds for all $(t,x) \in cl\, Q$ and the column unit vector $\mathbb{J}$. Put the estimation (3.1) into (3.3) and embed $cl\, \Omega$ into the ball

$$B(x,R) := \{\xi \in \mathbb{R}^n; ||x - \xi||_{\mathbb{R}^n} \le R, R > 0\}.$$

for every $x \in cl\, \Omega$.

Then we have the vector inequalities

$$|(D_x^k Su)(t,x)| \le \frac{Mc}{1-\mu} T^{1-\mu} \int_\Omega ||x - \xi||_{\mathbb{R}^n}^{2b\mu-(n+|k|)}\, d\xi \mathbb{J} \le$$

$$\leq \frac{Mc}{1-\mu} T^{1-\mu} \int\limits_{B(x,R)} ||x-\xi||_{\mathbb{R}^n}^{2b\mu-(n+|k|)} \, d\xi \rrbracket.$$

Hence, putting $x = (x_1, \ldots, x_n)$, $\xi = (\xi_1, \ldots, \xi_n)$ and using the spherical transformation

$$\begin{aligned}
\xi_1 &= x_1 + r\cos\varphi_1 \\
\xi_2 &= x_2 + r\sin\varphi_1\cos\varphi_2 \\
&\vdots \\
\xi_{n-1} &= x_{n-1} + r\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\cos\varphi_{n-1} \\
\xi_n &= x_n + r\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{n-2}\sin\varphi_{n-1}
\end{aligned}$$

for $r \in (0, R\rangle$, $\varphi_i \in (0,\pi), i = 1,\ldots,n-2$ and $\varphi_{n-1} \in (0, 2\pi)$ in the last integral, we get the estimation (the Jacobi determinant of this transformation is $r^{n-1}\sin^{n-2}\varphi_1\sin^{n-3}\varphi_2\ldots\sin\varphi_{n-2} \neq 0$) :

$$|(D_x^k Su)(t,x)| \leq 2\pi^{n-1} T^{1-\mu} R^{2b\mu-|k|} Mc/(2b\mu - |k|)(1-\mu)\rrbracket := d^k \rrbracket$$

for $(t,x) \in cl\,Q$ and $|k|/2b < \mu < 1$, where $|k| = 0, 1, \ldots, 2b-1$. This consideration implies the inclusion

$$S(G(0,d)) \subset G(0,d), \quad d \leq \sum_{0\leq|k|\leq 2b-1} d^k, \tag{3.4}$$

where

$$G(0,d) := \{v \in C_x^{|\gamma|}(cl\,Q, \mathbb{R}^p); ||v||_{C_x^{|\gamma|}(cl\,Q,\mathbb{R}^p)} \leq d,\ 0 \leq \gamma \leq 2b-1\}.$$

To prove the relative compactnes of the $S(G(0,d))$ we apply the Ascoli - Arzelà theorem [22, p. 85]. The equi - boundedness of $S(G(0,d))$ follows from (3.4). For the equi - continuous of $S(G(0,d))$, observe the difference
$((t,x),(s,y) \in cl\,Q,\ t < s,\ y = (y_1,\ldots,y_n))$

$$\begin{aligned}
|(D_x^\gamma Su)(t,x) \ &- \ (D_x^\gamma Su)(s,y)| \leq \\
&\leq M \int_0^t d\tau \int_\Omega |D_x^\gamma G(t,x;\tau,\xi) - D_x^\gamma G(t,y;\tau,\xi)| \, d\xi \rrbracket + \\
&+ M \int_0^t d\tau \int_\Omega |D_x^\gamma G(t,y;\tau,\xi) - D_x^\gamma G(s,y;\tau,\xi)| \, d\xi \rrbracket + \\
&+ M \int_t^s d\tau \int_\Omega |D_x^\gamma G(s,y;\tau,\xi)| \, d\xi \rrbracket
\end{aligned} \tag{3.5}$$

To estimate the first integral of (3.5) we use the mean value theorem, the estimation (3.1) from Lemma 3.1 and the inequalities (3.2) for the difference

$$|D_x^\gamma G(t,x;\tau,\xi) - D_x^\gamma G(t,y;\tau,\xi)| \le \sum_{i=1}^{n} |x_i - y_i| \left| D_x^{\gamma(i)} G(t,x_i^*,\tau,\xi) \right| \le$$

$$\le (c/c_n)||x-y||_{\mathbb{R}^n}(t-\tau)^{-\mu} \sum_{i=1}^{n} ||x_i^* - \xi||_{\mathbb{R}^n}^{2b\mu-(n+|\gamma(i)|)}\mathbb{E}. \qquad (3.6)$$

Here the multiindex $\gamma(i) = (\gamma_1,\dots,\gamma_{i-1},\gamma_i+1,\gamma_{i+1},\dots,\gamma_n) \in \mathbb{R}^n$ and $x_i^* = (y_1,\dots,y_{i-1},z_i,x_{i+1},\dots,x_n) \in \mathbb{R}^n$. The point z_i lies between the numbers x_i and y_i, $|\gamma(i)|/2b \le \mu < 1$ and $||x-y||_{\mathbb{R}^n} > ||x-x_i^*||_{\mathbb{R}^n}$. By the last inequality we obtain for $|\gamma| = 0,1,\dots,2b-2$

$$J_{1,|\gamma|} := \int_0^t d\tau \int_\Omega |D_x^\gamma G(t,x;\tau,\xi) - D_x^\gamma G(t,y;\tau,\xi)|\, d\xi \le d_1 ||x-y||_{\mathbb{R}^n}\mathbb{E}, \qquad (3.7)$$

where the constant $d_1 > 0$ does not depend of t, x, y.

In the case $|\gamma| = 2b-1$, we take the points x, y, $\xi \in cl\,\Omega$ satisfying the inequality $2||x-y||_{\mathbb{R}_n} < ||\xi-x||_{\mathbb{R}_n}$. Then, it is obvious that $||x-y||_{\mathbb{R}_n} < ||x_i^*-\xi||_{\mathbb{R}_n}$, whence $||x-\xi||_{\mathbb{R}_n} \le ||x-x_i^*||_{\mathbb{R}_n} + ||x_i^*-\xi||_{\mathbb{R}_n} < ||x-y||_{\mathbb{R}_n} + ||x_i^*-\xi||_{\mathbb{R}_n} < 2||x_i^*-\xi||_{\mathbb{R}_n}$. From the estimation (3.6) we obtain the inequality

$$\begin{aligned} |D_x^\gamma G(t,x;\tau,\xi) \; &- \; D_x^\gamma G(t,y;\tau,\xi)| \le \\ &\le (c/c_n)\cdot ||x-y||_{\mathbb{R}_n}(t-\tau)^{-\mu} n\,(2^{-1}||x-\xi||_{\mathbb{R}_n})^{2b\mu-(n+2b)}\mathbb{E}. \end{aligned}$$

If we put $B_1 = \{\xi \in \Omega;\ ||\xi - x||_{\mathbb{R}_n} > 2||x-y||_{\mathbb{R}_n}\}$ and $B_2 = \Omega - B_1$, then we have for $(2b-1+\alpha)/2b \le \mu < 1,\ \alpha \in (0,1)$

$$\begin{aligned} J_{1,2b-1} &\le (n\,c/c_n) 2^{n+2b-2b\mu} \times \\ \times \quad & \left[\int_0^t d\tau \int_{B_1} (t-\tau)^{-\mu} \left(||x-\xi||_{\mathbb{R}_n}^{2b\mu-(n+2b-1)} + ||y-\xi||_{\mathbb{R}_n}^{2b\mu-(n+2b-1)} \right) d\xi + \right. \\ + \quad & \left. \int_0^t d\tau \int_{B_2} (t-\tau)^{-\mu} ||x-y||_{\mathbb{R}_n} ||x-\xi||_{\mathbb{R}_n}^{2b\mu-(n+2b)} d\xi \right] \mathbb{E} \le \\ \le \quad & C_1 ||x-y||_{\mathbb{R}_n}^{2b\mu-(2b-1)}\mathbb{E}, \quad C_1 > 0\,. \end{aligned}$$

Again employing the mean value theorem and (3.1) we find $t^* \in (t,s)$ such that

$$|D_x^\gamma G(t,y;\tau,\xi) - D_x^\gamma G(s,y;\tau,\xi)| = |D_t D_x^\gamma G(t^*,y;\tau,\xi)|(s-t) \le$$

$$c(s-t)(t-\tau)^{-\mu}||y-\xi||_{\mathbb{R}^n}^{2b\mu-(n+2b+|\gamma|)}$$

for $\mu \le (n+2b+|\gamma|)/2b$ $(0 < t-\tau < t^*-\tau)$, $0 \le |\gamma| \le 2b-1$.

Hence, if we put $S_1 = \{\xi \in cl\,\Omega; ||y-\xi||_{\mathbb{R}^n} < (s-t)^{1/2b}\}$ and $S_2 = cl\,\Omega - S_1$, then by the estimate (3.1) we get for the two last integral members of (3.5) ($0 \le |\gamma| \le 2b-1$)

$$
\begin{aligned}
J_{2,|\gamma|} := & \int_0^t d\tau \int_\Omega |D_x^\gamma G(t,y;\tau,\xi) - D_x^\gamma G(s,y;\tau,\xi)| d\xi + \\
+ & \int_t^s d\tau \int_\Omega |D_x^\gamma G(s,y;\tau,\xi)| d\xi \le \\
\le & \int_0^t d\tau \int_{S_1} |D_x^\gamma G(t,y;\tau,\xi)| d\xi + \int_0^s d\tau \int_{S_1} |D_x^\gamma G(s,y;\tau,\xi)| d\xi + \\
+ & \int_0^t d\tau \int_{S_2} |D_x^\gamma G(t,y;\tau,\xi) - D_x^\gamma G(s,y;\tau,\xi)| d\xi + \\
+ & \int_t^s d\tau \int_{S_2} |D_x^\gamma G(s,y;\tau,\xi)| d\xi \le \\
\le & \; c\int_0^t d\tau \int_{S_1} (t-\tau)^{-\lambda} ||y-\xi||_{\mathbb{R}^n}^{2b\lambda-(n+|\gamma|)} d\xi\mathbb{E} + \\
+ & \; c\int_0^s d\tau \int_{S_1} (s-\tau)^{-\nu} ||y-\xi||_{\mathbb{R}^n}^{2b\nu-(n+|\gamma|)} d\xi\mathbb{E} + \\
+ & \; c\int_0^t d\tau \int_{S_2} (s-t)(t-\tau)^{-\mu} ||y-\xi||_{\mathbb{R}^n}^{2b\mu-(n+2b+|\gamma|)} d\xi\mathbb{E} + \\
+ & \; c\int_t^s d\tau \int_{S_2} (s-\tau)^{-\sigma} ||y-\xi||_{\mathbb{R}^n}^{2b\sigma-(n+|\gamma|)} d\xi\mathbb{E}
\end{aligned}
\tag{3.8}
$$

for $0 < \lambda \le (n+|\gamma|)/2b$, $0 < \nu \le (n+|\gamma|)/2b$, $0 < \mu \le (n+2b+|\gamma|)/2b$ and $0 < \sigma \le (n+|\gamma|)/2b$. If apply the spherical transformation for ξ with the center y and the radius $r \in (0,(s-t)^{1/2b})$ in the two integrals over S_1, such for $|\gamma|/2b < \lambda < 1$ and $|\gamma|/2b < \nu < 1$

$$
\begin{aligned}
& \int_0^t d\tau \int_{S_1} (t-\tau)^{-\lambda} ||y-\xi||_{\mathbb{R}^n}^{2b\lambda-(n+|\gamma|)} d\xi \le \\
& \quad \le 2\pi^{n-1} T^{1-\lambda} (s-t)^{(2b\lambda-|\gamma|)/2b} / (2b\lambda - |\gamma|)(1-\lambda)
\end{aligned}
\tag{3.9}
$$

and

$$\int_0^t d\tau \int_{S_1} (s-\tau)^{-\nu} \, ||y-\xi||^{2b\nu-(n+|\gamma|)} d\xi \leq$$
$$\leq \; 2\pi^{n-1} T^{1-\nu} (s-t)^{(2b\nu-|\gamma|)/2b} / (2b\nu - |\gamma|)(1-\nu) \tag{3.10}$$

If we embed the set S_2 into the set

$$B(y,(s-t)^{1/2b},R) := \{\xi \in \mathbb{R}^n; (s-t)^{1/2b} \leq ||y-\xi||_{\mathbb{R}^n} \leq R, R > 0\} \supset S_2$$

and we shall use the spherical substitution for ξ with the center y and radius $r \in ((s-t)^{1/2b}, R)$ in the two integrals over S_2, such we get for $|\gamma|/2b < \mu < 1$ and $|\gamma|/2b < \sigma < 1$

$$(s-t) \int_0^t d\tau \int_{S_2} (t-\tau)^{-\mu} \, ||y-\xi||_{\mathbb{R}^n}^{2b\mu-(n+2b+|\gamma|)} d\xi \leq$$
$$\leq \quad 2\pi^{n-1} T^{1-\mu} (s-t)^{(2b\mu-|\gamma|)/2b} / (2b + |\gamma| - 2b\mu)(1-\mu) \tag{3.11}$$

and

$$\int_t^s d\tau \int_{S_2} (s-\tau)^{-\sigma} \, ||y-\xi||_{\mathbb{R}^n}^{2b\sigma-(n+|\gamma|)} d\xi \leq$$
$$\leq \; 2\pi^{n-1} R^{2b\sigma-|\gamma|} (s-t)^{1-\sigma} / (2b\sigma - |\gamma|)(1-\sigma) \tag{3.12}$$

From the inequality (3.5) and the estimations (3.7), (3.8), (3.9), (3.10), (3.11) and (3.12) we can conclude that the operator S is compact. □

The following theorem ensures the existence of solution for problem (2.1)-(2.3) without the boundedness of nonlinearity f.

THEOREM 3.2. *Let the hypotheses* (P), (C), (Q), (S^α) *for* $\alpha \in (0,1)$ *be satisfied and* $g : cl\, Q \to \mathbb{R}^p$ *be continuous function at* $cl\, Q$. *Let* $f : cl\, Q \times (-\infty,\infty)^\kappa \to \mathbb{R}^p$ *be continuous function at* $cl\, Q \times (-\infty,\infty)^\kappa$, *where* κ *is the positive integer given in the formulation of the problem (2.1)-(2.3). Furthermore*

$$f[t,x,\overline{D}^{\gamma}_{x-2x} u(t,x)] = 0 \qquad \text{for } (t,x) \in clQ, \tag{3.13}$$

where $u \in C_x^{|\gamma|}(clQ,\mathbb{R}^p)$ *for* $0 \leq |\gamma| \leq 2b-1$ *is a mild solution of the linear nonhomogenous problem (2.1) with* $f = 0$, *(2.2), (2.3). Then nonlinear problem* (2.1) - (2.3) *possesses at least one mild solution from the class* $C_x^{|\gamma|}(clQ,\mathbb{R}^p)$.

Proof. The solution $u \in C_x^{|\gamma|}(clQ,\mathbb{R}^p)$ of the linear problem (2.1) with $f = 0$ (2.2), (2.3) is a solution of the given nonlinear problem, evidently . □

COROLLARY 3.1. *If instead of assumption (3.13) in Theorem 3.2, we suppose*

$$f(t,x,\overline{0}) = g(t,x) \qquad \text{for } (t,x) \in clQ \tag{3.14}$$

such the statement of this theorem is correct, too. Here $\overline{0} = (0,\ldots,0) \in \mathbb{R}^\kappa$.

Assumption (3.14) enables to study problems with strong nonlinearities f, for example: $f(t,x,u) = g(t,x)u^{\alpha}, \alpha \in (0,\infty)$, or $f(t,x,u) = g(t,x) - e^{u^k} + 1, k > 0$.

4. NONUNIQUENESS EXAMPLES

The following examples ilustrate a nonuniqueness of classical solution of parabolic or nonparabolic problem (2.1) - (2.3).

EXAMPLE 4.1. Theorem 3.1 can be ilustrated by the initial-boundary value problem for the equation (2.1) with $u = (u_1, u_2)^T$,

$$f(t,x,\overline{D}_x^{\gamma}u) = (\sin(u_1|u_2|^{\alpha}), \arctan(|u_1|^{\beta}u_2))^T$$

for $|\gamma| = 0, \alpha, \beta \in (0,1)$ and $g = (0,0)^T$ at clQ with conditions (2.2), (2.3).

It is evident that this problem posseses the trivial solution $u(t,x) = (0,0)^T$ at clQ. Because nonlinearity f is not locally Lipschitz continuous in $clQ \times R^2$ it is plausible that problem (2.1) - (2.3)is not uniquelly solvable at clQ.

EXAMPLE 4.2. Consider the two Neumann type initial - boundary value problem (parabolic and non - parabolic)

$$\frac{\partial u}{\partial t} = \pm \frac{\partial^2 u}{\partial x^2} + f(t,x,u), \ (t,x) \in (0,T\rangle \times \Omega = Q \subset \mathbb{R}^2 \qquad (4.1^*)$$

$$\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \ t \in \langle 0,T\rangle \qquad (4.2^*)$$

$$u(0,x) = 0, \ x \in \overline{\Omega} \qquad (4.3^*)$$

(i) If $f(t,x,u) = |u|^{\alpha}$, $\alpha \in (0,1)$, the given problem has a continuum of the solutions $u_r \in C^{1,2}_{t,x}(cl\,Q, \mathbb{R})$ for $r \in (0,T)$

$$u_r(t,x) = \begin{cases} 0, & \text{if } (t,x) \in \langle 0,r\rangle \times \overline{\Omega} \\ (1-\alpha)^{1/(1-\alpha)}(t-r)^{1/(1-\alpha)}, & \text{if } (t,x) \in (r,T\rangle \times \overline{\Omega} \end{cases}$$

$u_0(t,x) = (1-\alpha)^{1/(1-\alpha)}t^{1/(1-\alpha)}$ and $u_T(t,x) = 0$ are solutions of (4.1*) - (4.3*), too.

(ii) Similary if $f(t,x,u) = |u|^{1/2} - au, \ a > 0$ we have a continuum of solutions of (4.1*) - (4.3*) for $r \in (0,T)$

$$u_r(t,x) = \begin{cases} 0, & \text{if } (t,x) \in \langle 0,r\rangle \times \overline{\Omega} \\ \frac{1}{a^2}(1 - \exp\{-\frac{a}{2}(t-r)\})^2, & \text{if } (t,x) \in (r,T\rangle \times \overline{\Omega} \end{cases}$$

The functions $u_0(t,x) = \frac{1}{a^2}(1 - \exp\{-at/2\})^2$, $u_T(t,x) = 0$ are solutions of the given problem, too.

(iii) We obtain an analogical situation for $f(t,x,u) = t^{\beta}|u|^{\alpha}$ with $\alpha \in (0,1)$ and $\beta > 0$. Other nonlinearities f can be taken, too.

(iv) Consider the following initial-boundary value problem for the system of p

equations

$$D_t v - \sum_{1\le|k|\le 2b-1} a_k(t,x) D_x^k v = h(t,x,v)\,,\ (t,x)\in(0,T\rangle\times\Omega\subset\mathbb{R}^{n+1}$$

with the p unknown functions $(v_1,\ldots,v_p) =: v$. Here $h = (f,0,\ldots,0) : cl\,Q\times\mathbb{R}\to\mathbb{R}^p$ and f is some of the functions given in (i)-(iii) of this example.

We take the conditions

$$B_j(t,x,D_x)v\,|_{cl\,\Gamma} := \sum_{1\le|k|\le r_j} b_{jk}(t,x)D_x^k v\,|_{cl\,\Gamma} = 0$$

for $j=(1,\ldots,bp)$, $\ 1\le r_j\le 2b-1$ and

$$v(0,x) = 0 \text{ for } x\in cl\,\Omega\ \ .$$

Then there is a continuum of the vector solutions $w_r\in C^{1,2}_{t,x}(cl\,\Omega,\,\mathbb{R}^p)$, where $w_r(t,x) = (u_r(t,x),\,0,\,\ldots,\,0)$ and $u_r\in C^{1,2}_{t,x}(cl\,\Omega,\,\mathbb{R})$ for $r\in\langle 0,\,T\rangle$ is for example some of the functions from (i)-(iii) of this example. □

EXAMPLE 4.3. (i) Consider the initial - boundary value problem for the nonlinear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sqrt{\frac{2}{\pi}}\left|\int_0^\pi u(t,y)\sin y\,dy\right|^{1/2}\sin x + {}$$
$$+\sqrt{\frac{2}{\pi}}\left|\int_0^\pi u(t,y)\sin 2y\,dy\right|^{1/2}\sin 2x \qquad (4.1^{**})$$

for $(t,x)\in(0,T\rangle\times(0,\pi)$ with the Dirichlet type boundary value condition

$$u(t,0) = u(t,\pi) = 0, \qquad t\in\langle 0,T\rangle \qquad (4.2^{**})$$

and the initial condition

$$u(0,x) = 0, \quad x\in\langle 0,\pi\rangle \qquad (4.3^{**})$$

A continuum of solutions belonging to $C^{1,2}_{t,x}(cl\,Q,\mathbb{R})$ of this problem represents the set of functions

$$u_r(t,x) = a_r(t)\sin x + b_r(t)\sin 2x, \quad (t,x)\in cl\,Q$$

for $r\in\langle 0,T\rangle$. Here for $r\in(0,T)$

$$a_r(t) = \begin{cases} 0, & \text{if } t\in\langle 0,r\rangle \\ (1-\exp\{-(t-r)/2\})^2, & \text{if } t\in(r,T\rangle \end{cases}$$

and

$$b_r(t) = \begin{cases} 0, & \text{if } t\in\langle 0,r\rangle \\ \frac{1}{16}(1-\exp\{-2(t-r)\})^2, & \text{if } t\in(r,T\rangle \end{cases}$$

Further, $a_0(t) = (1 - \exp\{-t/2\})^2$, $a_T(t) = 0 = b_T(t)$, $b_0(t) = \frac{1}{16}(1 - \exp\{-2t\})^2$.

The function a_r and $b_r : \langle 0, T\rangle \to \mathbb{R}$ are the solutions of the initial value problems

$$\frac{da}{dt} + a = |a|^{1/2},\ t \in (0, T\rangle,\ a(0) = 0$$

and

$$\frac{db}{dt} + 4b = |b|^{1/2},\ t \in (0, T\rangle,\ b(0) = 0$$

respectively.

(ii) The initial - boundary value problem for the nonlinear equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sqrt{\frac{2}{\pi}} \left| \int_0^\pi u(t,y) \cos y \, dy \right|^{1/2} \cos x + $$
$$+ \sqrt{\frac{2}{\pi}} \left| \int_0^\pi u(t,y) \cos 2y \, dy \right|^{1/2} \cos 2x,$$

$(t, x) \in (0, T\rangle \times (0, \pi)$, the mixed conditions

$$\frac{\partial u}{\partial x}(t, 0) = u(t, \pi) = 0, \quad t \in \langle 0, T\rangle$$

and the initial condition

$$u(0, x) = 0, \quad x \in \langle 0, \pi\rangle$$

has a continuum solutions of $C^{1,2}_{t,x}(cl\, Q, \mathbb{R})$ in the form

$$u_r(t,x) = c_r(t) \cos\frac{x}{2} + e_r(t) \cos\frac{3x}{2}, \quad (t,x) \in cl\, Q$$

for $r \in \langle 0, T\rangle$.

Here the functions c_r and $e_r : \langle 0, T\rangle \to \mathbb{R}$ satisfy the Cauchy problems for the ordinary differential equations

$$\frac{dc}{dt} + \frac{1}{4}c = |c|^{1/2},\ t \in (0, T\rangle,\ c(0) = 0$$

and

$$\frac{de}{dt} + \frac{9}{4}e = |e|^{1/2},\ t \in (0, T\rangle,\ e(0) = 0$$

respectively. Both problems have a continuum classical solutions.

(iii) For the fourth order nonlinear differential problem

$$\frac{\partial u}{\partial t} = \frac{\partial^4 u}{\partial x^4} + \sqrt{\frac{2}{\pi}} \left| \int_0^\pi u(t,y) \sin 3y \, dy \right|^{1/2} \sin 3x + $$
$$+ \sqrt{\frac{2}{\pi}} \left| \int_0^\pi u(t,y) \sin 4y \, dy \right|^{1/2} \sin 4x,$$

$$\begin{aligned} u(t,0) &= u(t,\pi) = \frac{\partial^2 u}{\partial x^2}(t,0) = \frac{\partial^2 u}{\partial x^2}(t,\pi) = 0, \\ u(0,x) &= 0 \end{aligned}$$

we have also continuum solutions belonging to $C^{1,4}_{t,x}(cl\,Q,\mathbb{R})$. It is given by the functions $u_r : cl\,\Omega \to \mathbb{R}$, $r \in \langle 0,\, T\rangle$, where

$$u_r(t,x) = g_r(t)\sin 3x + h_r(t)\sin 4x, \quad (t,x) \in cl\,Q\,.$$

The function $g_r, h_r : \langle 0,\, T\rangle \to \mathbb{R}$ for $r \in \langle 0,\, T\rangle$ are classical solutions of the equations

$$\frac{dg}{dt} - 81g = |g|^{1/2},\ t \in (0,T\rangle,\ g(0) = 0, \frac{dh}{dt} - 156h = |h|^{1/2},\ t \in (0,T\rangle,\ h(0) = 0\,.$$

(iv) According to the previous examples one can easily construct initial - boundary value problems for systems of evolutions equations with the property of the non-uniqueness. □

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Received October 2005