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Oscillation and comparison theorems for higher order neutral differential equations[1]

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Abstract

The object of this paper is to study asymptotic properties of solutions of the $n-th$ order neutral differential equations

$$\Big(x(t)-p(t)x[\tau(t)]\Big)^{(n)}+q(t)x[\sigma(t)]=0.$$

As a consequence of the obtained results a new comparison theorem for $n-th$ order neutral differential equations is presented.

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Introduction

In this paper we are concerned with the problem of oscillatory properties of solutions of the of n-th order neutral differential equations

$$\Big(x(t)-p(t)x[\tau(t)]\Big)^{(n)}+q(t)x[\sigma(t)]=0,\quad n\geq 2. \tag{1}$$

Throughout this paper the following hypothesis (H) are assumed to hold.

(H1) $\tau(t)\in C[t_0,\infty)$, $\tau(t)\leq t$ and $\lim_{t\to\infty}\tau(t)=\infty$;
(H2) $p(t)\in C[t_0,\infty)$, $0\leq p(t)<1$;
(H3) $q(t)\in C[t_0,\infty)$, $q(t)>0$,
(H4) $\sigma(t)\in C^1[t_0,\infty)$, $\sigma'(t)>0$, $\sigma(t)\leq t$ and $\lim_{t\to\infty}\sigma(t)=\infty$;

By a solution of (1) we mean a function x on $[T_x,\infty)$ for some $T_x>t_0$ such that $\{x(t)-p(t)x[\tau(t)]\}$ is n-times continuously differentiable and (1) is satisfied for $t\geq T_x$. Such a nontrivial solution of Eq.(1) is called oscillatory if it has arbitrarily large zeros, and otherwise it is said to be nonoscillatory. Eq.(1) is said to be oscillatory if all its solutions are oscillatory.

There have been obtained (see [1]-[8], [11]-[13]) sufficient conditions for oscillation of (1) or at least for all nonoscillatory solutions of (1) to tend to zero as $t\to\infty$.

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In this paper we present some results of this kind, that involve easily verifiable conditions. Our results here generalize and extend a number of existing criteria.

As is customary, all functional inequalities presented in this paper are assumed to hold eventually, that is they are satisfied for all sufficiently large t.

Main Results

The following identity is a generalization of Taylor's formula and holds for any n-times differentiable function $z(t)$, where $0 \le i \le k \le n-1$ and $t, s \in (t_0, \infty)$:

$$z^{(i)}(t) = \sum_{j=i}^{k} (-1)^{j-i}(s-t)^{j-i} z^{(j)}(s) + (-1)^{k-i+1} \int_t^s \frac{(u-t)^{k-i}}{(k-i)!} z^{(k+1)}(u)\, du. \tag{2}$$

This identity can be easily obtained by repeating integration of $z^{(k+1)}(t) = z^{(k+1)}(t)$. $\blacksquare$

At first we shall assume that for all $\ell \in \{1, 2, \cdots, n-3\}$ the following integrals are convergent

$$\int_{t_0}^{\infty} (u-t_0)^{n-\ell-2} q(u)\, du < \infty.$$

So the following functions are well defined

$$b_{n-1}(t) = q(t)$$
$$b_\ell(t) = \int_t^\infty \frac{(u-t)^{n-\ell-2}}{(n-\ell-2)!} q(u)\, du,$$

for all $\ell \in \{1, 2, \cdots, n-3\}$.

For further references let us denote

$$z(t) = x(t) - p(t)x[\tau(t)]. \tag{3}$$

Theorem 1. *Let $0 \le p(t) \le p < 1$. Assume that for every $\ell \in \{1, 2, \cdots, n-1\}$ such that $n + \ell$ is odd and for some $\lambda_\ell > 1$*

$$\int^\infty \left(\sigma^\ell(t) b_\ell(t) - \frac{\lambda_\ell \ell^2 (\ell-1)! \sigma'(t)}{4\sigma(t)} \right) dt = \infty. \tag{4_ℓ}$$

Then for every positive solution $x(t)$ of (1), we have:

(i) *for n even*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} z^{(j)}(t) = 0, \quad (-1)^{j+1} z^{(j)}(t) > 0, \quad j = 0, 1, \ldots, n-1\ ,$$

(ii) *for n odd*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} z^{(j)}(t) = 0, \quad (-1)^{j} z^{(j)}(t) > 0, \quad j = 0, 1, \ldots, n-1. \tag{5}$$

Proof. Let $x(t)$ be an eventually positive solution of Eq.(1). Then $z(t) \leq x(t)$ and

$$z^{(n)}(t) + q(t)x[\sigma(t)] = 0. \tag{6}$$

Thus $z^{(n)}(t) < 0$ and consequently $z'(t), z''(t), \cdots, z^{(n-1)}(t)$ are of constant signs in some neighborhood of infinity.

We claim that $x(t)$ is bounded. To prove it assume, on the contrary, that $x(t)$ is unbounded. Hence there exists a sequence $\{t_m\}$ such that $\lim_{m\to\infty} t_m = \infty$ moreover $\lim_{m\to\infty} x(t_m) = \infty$ and $x(t_m) = \max\{x(s); t_0 \leq s \leq t_m\}$. Since $\tau(t) \to \infty$ as $t \to \infty$, we can choose large m such that $\tau(t_m) > t_0$. As $\tau(t) \leq t$, we have

$$\begin{aligned} x(\tau(t_m)) &\leq \max\{x(s);\, t_0 \leq s \leq \tau(t_m)\} \\ &\leq \max\{x(s);\, t_0 \leq s \leq t_m\} \\ &= x(t_m) \end{aligned}$$

Therefore for all large m

$$z(t_m) \geq x(t_m) - px[\tau(t_m)] \geq (1-p)x(t_m).$$

Thus $z(t_m) \to \infty$ as $m \to \infty$. Since $z(t)$, $z'(t)$ are of constant signs this yields $z(t) > 0$, $z'(t) > 0$. By well known lemma of Kiguradze, it follows from $z(t) > 0$ and $z^{(n)}(t) < 0$ that there exists $\ell \in \{1, 2, \cdots, n-1\}$ such that $n + \ell$ is odd and

$$\begin{aligned} z^{(i)}(t) &> 0 \quad \text{for} \quad 0 \leq i \leq \ell, \\ (-1)^{\ell+i} z^{(i)}(t) &> 0 \quad \text{for} \quad \ell \leq i \leq n-1. \end{aligned} \tag{7}$$

Moreover in view of (6), we have

$$z^{(n)}(t) + q(t)z[\sigma(t)] \leq 0. \tag{8}$$

Assume that $\ell < n-1$. Setting $i = \ell + 1$, $k = n-1$ and $s > t$ in (2) and using (7) and (8), we have

$$z^{(\ell+1)}(t) \leq -\int_t^s \frac{(u-t)^{n-\ell-2}}{(n-\ell-2)!} q(u) z[\sigma(u)]\, du.$$

Taking into account the monotonicity of $z[\sigma(t)]$ and letting $s \to \infty$, we obtain

$$z^{(\ell+1)}(t) + b_\ell(t) z[\sigma(t)] \leq 0. \tag{9}$$

From (8) it is easy to see that (9) is true also for $\ell = n-1$. Define

$$w_\ell(t) = \sigma^\ell(t) \frac{z^{(\ell)}(t)}{z[\sigma(t)]}. \tag{10}$$

Then $w_\ell(t) > 0$ and further

$$\begin{aligned} w'_\ell(t) &= \ell\sigma^{\ell-1}(t)\sigma'(t)\frac{z^{(\ell)}(t)}{z[\sigma(t)]} + \sigma^\ell(t)\frac{z^{(\ell+1)}(t)}{z[\sigma(t)]} \\ &\quad - \sigma^\ell(t)\frac{z^{(\ell)}(t)}{z^2[\sigma(t)]}z'[\sigma(t)]\sigma'(t). \end{aligned} \tag{11}$$

For $n > 2$ we let $i = 1$, $k = \ell - 1$, $s = t_0 < t$ in (2) and then for any $\lambda_\ell > 1$

$$\begin{aligned} z'(t) \geq \int_{t_0}^t \frac{(t-u)^{\ell-2}}{(\ell-2)!} z^{(\ell)}(u)\,du &\geq z^{(\ell)}(t)\frac{(t-t_0)^{\ell-1}}{(\ell-1)!} \\ &\geq \frac{1}{\lambda_\ell(\ell-1)!}t^{\ell-1}z^{(\ell)}(t), \end{aligned} \tag{12}$$

holds eventually. Note that (12) is satisfied also for $n = 2$. In this case $\ell = 1$ and $\lambda_\ell = 1$. It follows from (12) that

$$z'[\sigma(t)] \geq \frac{1}{\lambda_\ell(\ell-1)!}\sigma^{\ell-1}(t)z^{(\ell)}[\sigma(t)] \geq \frac{1}{\lambda_\ell(\ell-1)!}\sigma^{\ell-1}(t)z^{(\ell)}(t),$$

which in view of (11) and (9) leads to

$$\begin{aligned} w'_\ell(t) &\leq -\sigma^\ell(t)b_\ell(t) - \frac{\sigma^{2\ell-1}(t)\sigma'(t)}{\lambda_\ell(\ell-1)!}\left(\frac{z^{(\ell)}(t)}{z[\sigma(t)]}\right)^2 \\ &\quad + \ell\sigma^{\ell-1}(t)\sigma'(t)\frac{z^{(\ell)}(t)}{z[\sigma(t)]} \\ &= -\sigma^\ell(t)b_\ell(t) + \frac{\ell^2\lambda_\ell(\ell-1)!\sigma'(t)}{4\sigma(t)} \\ &\quad - \frac{\sigma^{2\ell-1}(t)\sigma'(t)}{\lambda_\ell(\ell-1)!}\left(\frac{z^{(\ell)}(t)}{z[\sigma(t)]} - \frac{\ell\lambda_\ell(\ell-1)!}{2\sigma^\ell(t)}\right)^2 \\ &\leq -\sigma^\ell(t)b_\ell(t) + \frac{\ell^2\lambda_\ell(\ell-1)!\sigma'(t)}{4\sigma(t)}. \end{aligned}$$

Integrating from t_1 to t, we get

$$w_\ell(t) \leq w_\ell(t_1) - \int_{t_1}^t \left[\sigma^\ell(s)b_\ell(s) - \frac{\ell^2\lambda_\ell(\ell-1)!\sigma'(s)}{4\sigma(s)}\right] ds.$$

Letting $t \to \infty$ we get in view of (4_ℓ) that $w_\ell(t) \to -\infty$. This contradicts to positivity of $w_\ell(t)$ and we conclude that $x(t)$ is bounded.

Consequently, in view of (3) it is easy to see that $z(t)$ is bounded and hence Kiguradze's lemma gives

$$(-1)^{n+j}z^{(j)}(t) < 0, \quad \text{for } j = 1, 2, \cdots, n-1. \tag{13}$$

We shall discuss the following two cases.

Case 1. Let $z(t) > 0$. Then for n even (13) implies $z'(t) > 0$ and this situation has been arrived to a contradiction with (4_ℓ) above.
For n odd, (13) implies that $\ell = 0$ in (7). Thus $z(t)$ is positive and decreasing, therefore there exists a finite $\lim_{t\to\infty} z(t) = c \geq 0$. If we admit $c > 0$, then (8) is satisfied and setting $i = 0$, $k = n-1$ and $s > t = t_1$ in (2) we get

$$z(t_1) \geq -\int_{t_1}^{s} \frac{(u-t_1)^{n-1}}{(n-1)!} z^{(n)}(u)\, du. \tag{14}$$

Substituting (8) into (14), using $z[\sigma(t)] \geq c$ and then letting $s \to \infty$, we obtain

$$z(t_1) \geq c \int_{t_1}^{\infty} \frac{(u-t_1)^{n-1}}{(n-1)!} q(u)\, du,$$

which implies

$$\int_{t_1}^{\infty} u^{n-1} q(u)\, du < \infty. \tag{15}$$

But in view of (4_{n-1}), we have

$$\infty = \int_{t_1}^{\infty} \sigma^{n-1}(u) q(u)\, du \leq \int_{t_1}^{\infty} u^{n-1} q(u)\, du,$$

which contradicts (15). Consequently, $\lim_{t\to\infty} z(t) = 0$. On the other hand, the boundedness of $x(t)$ yields $\lim_{t\to\infty} \sup x(t) = a$, $0 \leq a < \infty$. Then there exists a sequence $\{t_k\}$ such that $\lim_{k\to\infty} t_k = \infty$, $\lim_{k\to\infty} x(t_k) = a$. If $a > 0$, choosing $\epsilon = a(1-p)/(2p)$ we see that $x[\tau(t)] < a + \epsilon$, eventually. Moreover

$$0 = \lim_{k\to\infty} z(t_k) \geq \lim_{k\to\infty} \big(x(t_k) - p(a+\epsilon)\big) = \frac{a}{2}(1-p) > 0. \tag{16}$$

Thus $a = 0$ and that is $\lim_{t\to\infty} x(t) = 0$.

Case 2. Let $z(t) < 0$. For n even, it follows from (13) that $z'(t) > 0$ which implies that $\lim_{t\to\infty} z(t) = c \leq 0$. Denote $\limsup_{t\to\infty} x(t) = a$. If $a > 0$ then considering a sequence $\{t_k\}$ as above and proceeding exactly as above we are led to

$$0 \geq c = \lim_{k\to\infty} z(t_k) \geq \lim_{k\to\infty} \big(x(t_k) - p(a+\epsilon)\big) = \frac{a}{2}(1-p) > 0.$$

Then $a = 0$ and $\lim_{t\to\infty} x(t) = 0$ and moreover (3) implies $\lim_{t\to\infty} z(t) = 0$.

For n odd we have $z'(t) < 0$ which yields $\lim_{t\to\infty} z(t) = -c < 0$.

This again yields $\lim_{t\to\infty} x(t) = 0$ and on the other hand, it follows from the inequality $z(t) \geq x(t) - px(\tau(t))$ that $\lim_{t\to\infty} z(t) \geq 0$, a contradiction. The proof is complete now.

Remark 1. The assertion of Theorem 1 can be formulated as that every nonoscillatory solution $x(t)$ of Eq.(1) tends to zero as $t \to \infty$.

Remark 2. It is evidently from the proof of Theorem 1 that we can let $\lambda_1 = 1$ in (4_1).

In the following corollary conditions (4_ℓ) are replaced by easily verifiable ones.

Corollary 1. *Let $0 \le p(t) \le p < 1$. Assume that for every $\ell \in \{1, 2, \cdots, n-1\}$ such that $n + \ell$ is odd*

$$\liminf_{t\to\infty} \frac{\sigma^{\ell+1}(t) b_\ell(t)}{\sigma'(t)} > \frac{\ell^2(\ell-1)!}{4}. \tag{17_ℓ}$$

Then every solution $x(t)$ of Eq.(1) oscillates or tends to zero as $t \to \infty$.

Proof. We will show that(17_ℓ) implies (4_ℓ). It is easy to see from (17_ℓ) that there exist an $\varepsilon > 0$, $\lambda_\ell > 1$ and t_1 such that for all $t \ge t_1$, we have

$$\frac{\sigma^{\ell+1}(t) b_\ell(t)}{\sigma'(t)} > \frac{\lambda_\ell \ell^2(\ell-1)!}{4} + \varepsilon$$

Then

$$\sigma^\ell(t) b_\ell(t) - \frac{\lambda_\ell \ell^2(\ell-1)!\sigma'(t)}{4\sigma(t)} \ge \varepsilon \frac{\sigma'(t)}{\sigma(t)}.$$

Now it is easy to verify that (4_ℓ) holds.

Example 1. Consider the second order equation

$$\Big(x(t) - px(t-\tau)\Big)'' + q(t)x[\sigma(t)] = 0. \tag{18}$$

Then by Theorem 1 and Corollary 1 every nonoscillatory solution of (18) tends to zero as $t \to \infty$ provided that either

$$\liminf_{t\to\infty} \frac{\sigma^2(t)}{\sigma'(t)} q(t) > \frac{1}{4} \qquad \text{or}$$
$$\int^\infty \left(\sigma(t)q(t) - \frac{\sigma'(t)}{4\sigma(t)}\right) dt = \infty. \tag{19}$$

Note that Theorem 2.2 from [4] can be apply to (18) only when $\int^\infty q(t)\, dt = \infty$. On the other hand, we have improved for (18) the result of Lemma 4.4.2 in [3].

In our previous results we have assumed that the integrals in definition of b_ℓ are convergent. The natural question arises what happens if

$$\int_{t_0}^\infty (u - t_0)^{n-\ell-2} q(u)\, du = \infty.$$

The following theorem covers this case.

Theorem 1a. *The conclusions of Theorem 1 hold true if we replace for every $\ell \in \{1, 2, \cdots, n-3\}$ conditions (4_ℓ) with*

$$\int_{t_0}^\infty (u - t_0)^{n-\ell-2} q(u)\, du = \infty.$$

Proof. Assume that $x(t)$ is a positive solution of (1). Following all steps of the proof of Theorem 1 we arrive to (8). For $\ell < n-1$ we set $i = \ell + 1$, $k = n-1$ and $s > t_0$ in (2) and using (7) and (8), we have

$$z^{(\ell+1)}(t_0) \le -\int_{t_0}^{s} \frac{(u-t_0)^{n-\ell-2}}{(n-\ell-2)!} q(u) z[\sigma(u)]\, du.$$

Taking into account the monotonicity of $z[\sigma(t)]$ and letting $s \to \infty$, we obtain

$$\int_{t_0}^{\infty} \frac{(u-t_0)^{n-\ell-2}}{(n-\ell-2)!} q(u)\, du < \infty,$$

which contradicts to assumptions. The case $\ell = n-1$ can be treated with help of (9) as in the proof of Theorem 1. The next steps of our considerations are same as in the proof of Theorem 1.

Oscillation

Employing additional conditions imposed on the coefficient of Eq.(1) the conclusion of Theorem 1 (Corollary 1) can be straightened as follows.

Corollary 2. *Assume that n is even. Let all assumptions of Theorem 1 (Corollary 1) hold. Moreover if $p(t)$ oscillates, then Eq.(1) is oscillatory.*

Proof. Let $x(t)$ be a positive solution of (1), then by Theorem 1, $z(t) < 0$. If $\{t_k\}$ is a sequence of zeros of $p(t)$ then

$$0 > z(t_k) = x(t_k) - p(t_k) x(\tau(t_k)) > 0$$

a contradiction.

Example 2. We consider the second order neutral differential equation

$$\Big(x(t) - \frac{1-\sin t}{3}\, x[\tau(t)]\Big)'' + \frac{a}{t^2}\, x(\beta t) = 0, \quad 0 < \beta < 1. \tag{20}$$

Then by Corollary 2, Eq.(20) is oscillatory provided that

$$a\beta > \frac{1}{4}.$$

On the other hand, Theorem 2.1 in [11] guarantees oscillation of (20) if

$$a\beta > \frac{6}{\mathrm{e}(-\ln\beta)}.$$

Corollary 3. *Assume that n is even. Let all the hypotheses of Theorem 1 (Corollary 1) hold. Furthermore assume that $\tau(t)$ is increasing and*

$$\varphi(t) = \tau^{-1}[\sigma(t)] < t \quad \text{and} \quad \varphi(t) \to \infty \text{ as } t \to \infty,$$

where $\tau^{-1}(t)$ is the inverse function to $\tau(t)$. If for some $k \in \{0, 1, \cdots, n-1\}$

$$\limsup_{t\to\infty} \int_{\varphi(t)}^{t} \Big[s - \varphi(t)\Big]^k \Big[\varphi(t) - \varphi(s)\Big]^{n-k-1} q(s)\, ds > k!(n-k-1)! \tag{21}$$

then Eq.(1) is oscillatory.

Proof. Assume that $x(t)$ is an eventually positive solution of (1), then according to Theorem 1, we have $(-1)^{i+1} z^{(i)}(t) > 0$, $i = 0, 1, \cdots, n-1$, where $z(t)$ is defined by (3). Moreover (3) implies $z(t) \geq -p\, x[\tau(t)]$, that is

$$x(t) \geq \frac{-1}{p}\, z\left[\tau^{-1}(t)\right], \tag{22}$$

eventually. Now (6) together with (22) implies

$$z^{(n)}(t) - \frac{1}{p} q(t) z\left(\tau^{-1}[\sigma(t)]\right) \leq 0.$$

Then $v(t) = -z(t)$ satisfies

$$v^{(n)}(t) - \frac{1}{p} q(t) v(\varphi(t)) \geq 0 \tag{23}$$

and $(-1)^i v^{(i)}(t) > 0$, $i = 0, 1, \cdots, n-1$. On the other hand, Koplatatze and Chanturia in [9] (see also Kusano in [10]) have shown that (21) guarantees that differential inequality (23) has no such solution. This contradiction proves the theorem.

Example 3. Consider Eq.(18) with $\sigma(t) < t - \tau$. Then according to Corollary 3 Eq.(18) is oscillatory if at least one condition presented in (19) holds and one of the following two conditions is satisfied:

$$\limsup_{t\to\infty} \int_{\sigma(t)+\tau}^{t} \big(\sigma(t) - \sigma(s)\big) q(s)\, ds > 1 \qquad \text{or}$$

$$\limsup_{t\to\infty} \int_{\sigma(t)+\tau}^{t} q(s)\, ds > 1.$$

In what follows we are concerned with the investigation of oscillation of special case of (1) with n odd, that is we shall assume that $\sigma(t) = t - \sigma$, $\tau(t) = t - \tau$, $p(t) = p$, with $\sigma > 0$, $\tau > 0$, $p \in (0, 1)$.

Corollary 4. *Assume that n is odd. Let the hypotheses of Theorem 1 hold. Furthermore assume that*

$$\liminf_{t\to\infty}\int_{t-\sigma}^{t} q(s)(s-t)^{n-1}ds > (1-p)(n-1)!\ . \tag{24}$$

Then Eq.(1) is oscillatory.

Proof. Let $x(t)$ be an eventually positive solutions of (1). Then it follows from Theorem 1 that (5) holds. On the other hand condition (24) (see Theorem 4.1 in [4]) implies that Eq.(1) has no solution satisfying (5). The proof is complete.

Comparison result

It is convenient to deduce the oscillatory character of one equation by comparing it with another suitable equation. We derive result of this type here. We investigate the neutral differential equation of odd order.

Lemma 1. *Assume that $\tau > 0$, n is odd and (4_ℓ) hold for every $\ell \in \{2,4,\cdots,n-1\}$. If $x(t) > 0$ satisfies*

$$x(t) \ge px(t-\tau) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} q(s)x[\sigma(s)]\,ds, \quad t \ge t_0$$

then there exist $y(t) > 0$ such that

$$y(t) = py(t-\tau) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} q(s)y[\sigma(s)]\,ds.$$

Proof. The proof follows the same steps as proof of Theorem 2.2 in [4] and so it can be omitted.

Theorem 2. *Assume that n is odd and (4_ℓ) is satisfied for every $\ell \in \{2,4,\cdots,n-1\}$. Further assume that*

$$0 \le p \le \tilde{p}(t) \le \tilde{p} \le 1, \quad \tilde{q}(t) \ge q(t) > 0, \quad \tau > 0.$$

Then oscillation of

$$\Big(x(t) - px(t-\tau)\Big)^{(n)} + q(t)x[\sigma(t)] = 0 \tag{25}$$

implies the oscillation of

$$\Big(x(t) - \tilde{p}(t)x(t-\tau)\Big)^{(n)} + \tilde{q}(t)x[\sigma(t)] = 0. \tag{26}$$

Proof. Assume the converse. Let $x(t)$ be an eventually positive solution of (26). We set

$$\tilde{z}(t) = x(t) - \tilde{p}(t)x(t-\tau).$$

Then integrating Eq.(26) n-times from t to ∞ we get in view of Corollary 1 that

$$\begin{aligned} x(t) &= \tilde{p}(t)x(t-\tau) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\tilde{q}(s)x[\sigma(s)]\,ds \\ &\geq px(t-\tau) + \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}q(s)x[\sigma(s)]\,ds. \end{aligned}$$

By Lemma 1, it follows that (1) has a positive solution. This contradicts the assumptions of theorem. The proof is complete.

Theorem 2 generalizes and extends Theorem 2.2 in [4] since we have considered more general equation and the assumption $\int^\infty q(s)\,ds = \infty$ imposed in [4] is not required. This result improves Lemma 2.1 in [4].

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