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Properties of stochastic representation of Poisson equation solution

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Abstract

This paper studies the regularity properties of stochastic representation of the solution of Dirichlet problem for Poisson equation. We consider the representation introduced in [Stehlík 05]. When the correlation structure is exponential, we relate the representation process to the Ornstein-Uhlenbeck one. Following [Stehlík 06] we derive some properties of blow-up identification by the singular points of the correlation structure.

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1. INTRODUCTION

Here we consider a Poisson equation, often written as $\Delta u = f$, or $\nabla^2 u = f$. Poisson equation has a broad utility in electrostatics, mechanical engineering and theoretical physics.

The Dirichlet problem for Poisson equation consists in finding a solution $u \in H_0^1(G)$ on some domain G such that for $f \in L^2(G)$ and $\Gamma = \partial G$

$$-\Delta u = f, \text{ in } G, \tag{1}$$

$$u = g, \text{ in } \Gamma. \tag{2}$$

We say, that solution to (1) and (2) is regular, if $u \in C^2(G) \cap C(G \cup \Gamma)$.

Now let G be a bounded domain in R^m and let u be a sufficiently regular solution of the equation

$$\frac{1}{2}\Delta u + f(x) = 0$$

in G, let $u \in C(\bar{G})$ and $u = g$ on ∂G. Then the following representation for $x \in G$ holds

$$u(x) = E[\int_0^{\tau(x)} f(x + W_t)dt + g(x + W_{\tau(x)})], \tag{3}$$

where $\tau(x) = \inf\{t : x + W_t \in G\}$. This relation between the Wiener process W_t and the Laplace operator has been obtained by the employing of probabilistic construction of the solution of the heat equation (see [Prokhorov and Shiryaev 98]).

We may say that such a representation is local, in the sense, that we are computing the value of the solution at a given point of the domain. In [Stehlík 05] we have derived a global probability representation of the Dirichlet boundary problem solution. Therein we consider an isotropic Gaussian random field $Y(x) \in \mathcal{Y} \subset \mathbb{R}^k$ with parametrized covariance function $\mathrm{cov}(Y(s), Y(t)) = c(||s-t||, r)$, measured on some compact design space $X \subset \mathbb{R}^s$, and parameter $r \in G \subset \mathbb{R}^m$. Such a field is in [Stehlík 05] related to the D-optimality problem when the errors are correlated (see e.g. [Müller and Stehlík 04]). Let $h(y, r)$ be the density of the Gaussian random field $Y(x)$. We have defined the abstract energy $E(r) = -\int_{\mathcal{Y}} |\nabla_r u|^2 e^u dy$, where $d\mu = e^u dy$ is some (unit) mass distribution over $\mathcal{Y}$. We say abstract energy, because the integration can be employed over space of dimension k, but the operator (Laplacian) is according to m-dimensional cartesian coordinates $(r_1, \ldots, r_m)$ of the parameter r. This can correspond to the "classicalënergy $-\int_{\mathcal{Y}} |\nabla_{r(y)} u|^2 d\mu$ under $m = k$, due to some diffeomorphism $r = r(y)$. Some interpretations of this abstract energies can be found in theoretical physics. This abstract energy is from the Statistical point of view the Fisher information on the parameter r.

In [Stehlík 05] is given that for any Gaussian random field there exists the Dirichlet problem for Poisson equation, e.g. there exist sufficiently regular f and ϕ so that $\ln h$ is the solution of (1,2). Furthermore, there exists the abstract energy $E(r)$ which is equal to the Fisher information. In [Stehlík 06] we have studied the blow up identification by the singular points of the correlation structure. Based on [Uciński 05] we have provided a deterministic interpretation of abstract energy assuming some regularity conditions on an inverse problem operator Φ given in [Chung and Kravaris 88]. We related the abstract energy to Turing's measure of conditioning (see [Walter and Pronzato 90]).

In the present paper, we show that for the Gaussian representant with parametrized covariance function, the limiting case $r \to 0+$ together with $\sigma = \sqrt{2r}$ of Ornstein-Uhlenbeck process represents the blow up of the limit of the regular solutions of the sequence of Dirichlet problems for the Poisson equation.

2. BLOW-UP AND ORNSTEIN-UHLENBECK PROCESS

In this section we relate the stochastic representation via the Gaussian field to the linear stochastic differential equation. Let us consider the linear stochastic equation (for more details see [Karatzas and Shreve 91]):

$$dX_t = (A(t)X_t + a(t))dt + \sigma(t)dW_t, 0 \le t < \infty.$$

$$X_0 = \xi.$$

Here W is an r-dimensional Brownian motion independent on the d-dimensional initial vector ξ, and the $d \times d, d \times 1$ and $d \times r$ matrices $A(t), a(t)$ and $\sigma(t)$ are nonrandom, measurable and locally bounded. In the case $d = r = 1, a(t) = 0, A(t) = -\alpha$ and $\sigma(t) = \sigma > 0$ we obtain the oldest example

$$dX_t = -\alpha X_t dt + \sigma(t)dW_t$$

of a stochastic differential equation (see [Karatzas and Shreve 91], p. 358). If the initial random variable X_0 has a normal distribution with mean zero and variance $\frac{\sigma^2}{2\alpha}$, then X is a stationary, zero-mean Gaussian process with covariance function $\rho(s,t) = \frac{\sigma^2}{2\alpha} e^{-\alpha|t-s|}$. If the representing process has exponential correlation of the form $\exp(-rd)$, then it is driven by Ornstein-Uhlenbeck process via

$$dX_t = -rX_t dt + \sqrt{2r} dW_t.$$

The Ornstein-Uhlenbeck process is stationary, Gaussian, Markovian and continuous in probability.

Now let us consider the blow-up problem introduced in [Stehlík 06]. Let us have a system $\{G_n, f_n\}$ where $G_n \subset G_{n+1} \subset R^m$ is the increasing system of sufficiently regular domains and $f_{n+1}/G_n = f_n$ is the system of right sides for Dirichlet problem. The limiting domain is $G_\infty = \cup_{i=1}^{\infty} G_i$ with boundary $\Gamma_\infty = \partial G_\infty$.

If the Dirichlet problem (1,2) is regular for all n, we are interested in whether there is a blow up at Γ_∞. Theorem 2.1 in [Stehlík 05] is saying that if the correlation structure of the representation process is collapsing, then there is a blow-up. Theorem 1 in [Stehlík 06] is giving a dimensionality interpretation of such blow-up at $r^\star$ in the sense of the dimensionality loss of the stochastic representant. In more details, the representing stochastic process Y is loosing the dimension, i.e. there exist a hyperplane L_1 such that $L_1 \subset R^k$ and $\dim L_1 < k$ and there exists a neighborhood $U(r^\star)$ of point $r^\star$ such that for all $r \in U(r^\star) \cap G_\infty$ we have $P(Y \in L_1|r) < 1$ but $P(Y \in L_1|r^\star) = 1$.

The following theorem will provide a different interpretation via Ornstein-Uhlenbeck process. Note that we can represent the whole system $\{G_n, f_n\}$ just by one Gaussian field.

THEOREM 1. *Let $\{G_n, f_n\}_{n=1}^{\infty}$ is the system of domains and right sides given above. Assume that the solution of Dirichlet problem (1,2) is regular for all n and $\det \Sigma(r) \to 0$ for $r \to 0 \in \Gamma_\infty$. Let the representant is Gaussian with exponential covariance $\exp(-rd)$. Then there is a blow-up at $r^\star = 0$ and the systems $\{\ln h(Y(x), r)\}_{r>0}$ and $\{h(Y(x), r)\}_{r>0}$ are stochastically unbounded. Moreover, the related Ornstein-Uhlenbeck process*

$$dX_t = -rX_t dt + \sqrt{2r} dW_t$$

degenerates to the $dX_t = 0, t > 0$.

Proof

It is easy to check that $\det \Sigma \to 0$, for $r \to 0 \in G_\infty$. From continuity of the map $r \to \Sigma(r)$ we have also $\det \Sigma(0) = 0$.

Now let $C > 0$ be arbitrary constant. Then

$$P(\ln h(Y(x), r) > C) = \int_{\{y: y^T \Sigma^{-1} y < -2\ln(2\pi e^C (det\Sigma)^{1/2})\}} d\mu(y).$$

Let U be the nonsingular matrix such that $U^T \Sigma^{-1} U = I$. Then after substitution $w = U^{-1} y$ we employ the polar coordinates substitution to obtain the latter integral in the form $1 - e^{-\sqrt{-2\ln(2\pi e^C (det\Sigma)^{1/2})}}$. Thus we have for $0 < r < -\frac{1}{2d}\ln(1 - \frac{e^{-2C}}{4\pi^2})$ the formula

$$P(\ln h(Y(x), r) > C) = 1 - e^{-\sqrt{-2\ln(2\pi e^C \sqrt{1-e^{-2rd}})}}.$$

So finally $\lim_{r\to 0} P(\ln h(Y(x), r) > C) = 1$. We have proved, that systems

$$\{\ln h(Y(x), r)\}_{r>0}$$

and $\{h(Y(x), r)\}_{r>0}$ are stochastically unbounded.

The representing process satisfies

$$dX_t = -rX_t dt + \sqrt{2r} dW_t$$

and for $r \to 0$ we have $dX_t = 0$. This completes the proof. □

The following example is illustrating such a situation.

EXAMPLE 1. *Let us consider* $m = 1, k = 2$, $G_n = (\frac{1}{n}, \frac{1}{3}), \Gamma_n = \{\frac{1}{n}, \frac{1}{3}\}$, $n > 3, \mathcal{Y} = R^2, Y(x_1) = y_1, Y(x_2) = y_2$, $d = x_2 - x_1 > 0$ *(here* $\{x_1, x_2\}$, $x_1 < x_2$, *is the design in* $X = [-1, 1]$*). Further we have* $f_n = f/G_n$ *where* $f(r) = \exp(-rd)d^2(-2\exp(-rd) + 2\exp(-3rd) + 2y_1^2 \exp(-3rd) + 2y_1^2 \exp(-rd) - y_1y_2 - 6y_1y_2 \exp(-2rd) - y_1y_2 \exp(-4rd) + 2y_2^2 \exp(-3rd) + 2y_2^2 \exp(-rd))/$
$(-1 + \exp(-2rd))^3$.

Let us consider the system of boundary functions $g_n = g/\Gamma_n, g(\Gamma_n) \subset R$ *such that*

$$\lim_{n\to+\infty} g(\frac{1}{n}) = \infty$$

and

$$g(\frac{1}{3}) = -\ln 2\pi - 0.5 \ln \det \Sigma(\frac{1}{3}) + 0.5 \frac{y_1^2 - 2y_1y_2 \exp(-\frac{d}{3}) + y_2^2}{-1 + \exp(-\frac{2}{3}d)},$$

where $\Sigma_{k,l}(r) = \exp(-r|x_k - x_l|)$.

We obtain $d\mu(y) = (2\pi)^{-1}|\Sigma|^{-1/2} \exp(y^T \Sigma^{-1} y) dy_1 dy_2$, *and solution*

$$u(r) = -\ln 2\pi - 0.5 \ln \det \Sigma(r) + 0.5 \frac{h(y_1, y_2, r)}{-1 + \exp(-2rd)}$$

where $h(y_1, y_2, r) = y_1^2 - 2y_1y_2 \exp(-rd) + y_2^2$. *Such a solution* $u(r)$ *is blowing-up for all evolutions* $r \to h(y_1, y_2, r)$ *in* $\mathcal{Y}$ *for which*

$$\lim_{r\to 0} \frac{h(y_1, y_2, r)}{\det \Sigma(r)} < +\infty.$$

3. DISCUSSION

Typically, an interesting case of the Ornstein-Uhlenbeck process is studied, when r approaches zero with σ becomes infinite in such a way, that $r\sigma^2$ approaches a fixed constant (see [Rybicki 94]). Usually it is accomplished by letting $\sigma^2 = \frac{D}{2r}$, where D is a constant, called the diffusivity. This limiting case is often called the Gaussian random walk process. The correlation function $\sigma^2 e^{-rd}$ of the random walk is not defined, since $\sigma \to +\infty$ as $r \to 0+$. However, the variogram defined as

$$\psi(\tau) = E([x(t) - x(t + \tau)]^2) = 2\sigma^2(1 - e^{-r|\tau|})$$

does have meaning in the limit, namely $\psi(\tau) = D|\tau|$.

However, as we have shown in the present paper, also limit $r \to 0+$ together with $\sigma = \sqrt{2r}$ gives an interesting interpretation for the blow up of the limit of the regular solutions of the sequence of Dirichlet problems for the Poisson equation.

The generalization of the Gaussian field representation is through the set probability density function with support on a finite interval. The set of probability functions is a convex subset of L^1 and it does not have a linear structure when using ordinary sum and multiplication by real constants. To overcome these limitations, Aitchinson's ideas on compositional data analysis have been used in [Egozcue et al. 06], generalizing perturbation and power transformation, as well as the Aitchison inner product, to operations on probability density functions with support on a finite interval.

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