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Homoclinic bifurcations in discontinuous differential equations

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Abstract

We survey our recent results on bifurcations of homoclinic orbits in time-perturbed discontinuous systems. First we study weakly discontinuous equations. Then we consider cases when homoclinic solutions of unperturbed equations cross transversally discontinuity levels. Finally we deal with opposite cases when homoclinic solutions of unperturbed equations slide on discontinuity levels. Concrete examples are presented to illustrate the theory.

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1. INTRODUCTION

One of the most powerful method which is used in nonlinear smooth mechanics is the perturbation approach. A typical such a problem where it is used is a periodically forced nonlinear oscillator like a perturbed Duffing equation

$$\ddot{x} - x + 2x^3 = \varepsilon \cos \omega t\,, \tag{1}$$

which is chaotic for any $\varepsilon \neq 0$ small. This is shown in Section 2. An experimental apparatus is a slender steel beam clamped to a rigid framework which supports two magnets, when x is the beam's tip displacement (see Figure 1). The apparatus is periodically forced using electromagnetic vibration generator.

This perturbation approach is by now known as the Melnikov method for the persistence/bifurcation of either periodics or homoclinics/heteroclinics [11]. We know for the homoclinic bifurcation that the existence of a simple zero of the corresponding Melnikov function implies the existence of a transversal homoclinic point of the period map of the perturbed differential equation. This yields Smale's horseshoe for this period map with the associated chaotic behavior of the solutions to the perturbed differential equation [11; 18] (see also Section 2).

Thus, bifurcation from homoclinic orbits in periodically perturbed smooth differential equations is well developed for hyperbolic equilibria [3; 10; 18].

Recently several papers appeared to extend these results to differential equations with discontinuous right-hand sides. Non-smooth differential equations occur in various situations like in mechanical systems with dry frictions or with impacts. They appear also in control theory, electronics, economics, medicine and biology (see [4; 5; 14; 15] for more references).

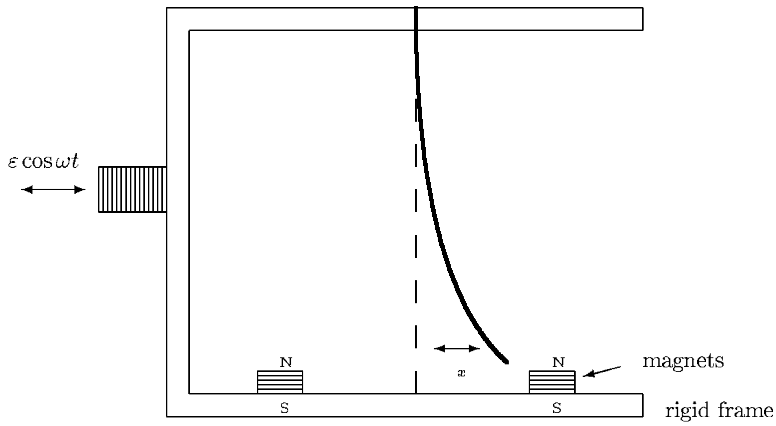

Fig. 1. The forced magneto-elastic beam.

As an example [4; 9] let us consider the following problem: a dry friction force acting on a moving particle due to its contact to a wall has in certain situation the form $\mu_1 \mathrm{sgn}\,\dot{x}$, where x is a displacement from the rest, $\dot{x}$ is the velocity, μ_1 is a positive constant, and $\mathrm{sgn}\, r = r/|r|$ for $r \in \mathbb{R} \setminus \{0\}$, see [6; 12]. So dry friction is modeled by Coulomb's law [9, p. 7] expressed with the discontinuous function $\mathrm{sgn}\, r$. Including also restoring and external forces, the following equation is studied (see Figure 2)

$$\ddot{x} + g(x) + \mu_1 \,\mathrm{sgn}\, \dot{x} = \mu_2 \psi(t)\,, \tag{2}$$

where g is smooth and μ_1, μ_2 are parameters. We assume that ψ is almost periodic and continuous. For $\mu_1 = 0$, μ_2 small and ψ periodic, we get the periodically forced case mentioned above like (1). Clearly, (2) is a weakly discontinuous differential equation with $\mu_{1,2}$ small.

In this paper we survey our recent results on homoclinic bifurcations for nonsmooth systems [1; 2; 7]. Recently several other papers appeared in this direction. For example 2-dimensional *time-perturbed* discontinuous differential equations are investigated in [13; 14; 20] using a geometric approach. The existence is assumed of a homoclinic solution to a hyperbolic equilibrium for the unperturbed problem that crosses transversally the discontinuity lines. Note an overview of all codimension 1 bifurcations in generic planar discontinuous piecewise smooth *autonomous* differential equations is given in [15].

A piecewise linear 3-dimensional autonomous discontinuous differential equation of the form

$$\dot{x} = \begin{cases} A_- x + \widetilde{b} & \text{for } x_1 \le 0, \\ A_+ x + \widetilde{b} & \text{for } x_1 > 0 \end{cases} \tag{3}$$

is investigated in [17]. In this last paper $x = (x_1, x_2, x_3)$ and the matrices $A_\pm$ depend on 3 parameters. Conditions for those parameters are found so that (3) has a saddle-focus equilibrium $e_- \in \{x \in \mathbb{R}^3 \mid x_1 < 0\}$ with a homoclinic orbit crossing twice the discontinuity level $\{x \in \mathbb{R}^3 \mid x_1 = 0\}$ and possessing the corresponding

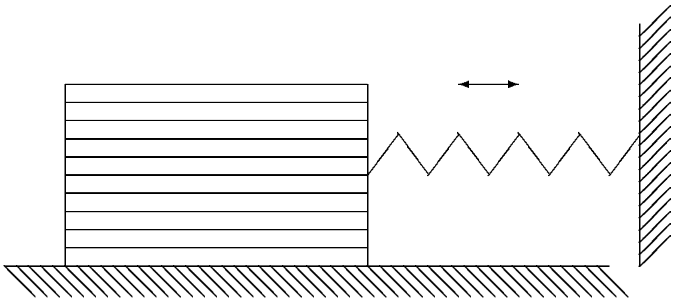

Fig. 2. A dry friction oscillator.

horseshoe dynamics. So the result of [17] is *Shilnikov-type.*

The plan of this note is as follows. In Section 2 we study a weakly discontinuous equation (2) in a context of general theory of [7]. In Section 3 we consider cases when homoclinic solutions of unperturbed equations cross transversally discontinuity levels. Finally Section 4 deals with opposite cases when homoclinic solutions of unperturbed equations slide on discontinuity levels.

2. WEAKLY DISCONTINUOUS SYSTEMS

Since the general theory for weakly discontinuous systems in [7] is rather technical, here we focus on (2). We assume that $g \in C^2(\mathbb{R},\mathbb{R})$, $g(0)=0$ and $g'(0)<0$. Then we suppose the existence of a homoclinic solution γ of $\ddot{x}+g(x)=0$ such that $\lim_{t\to\pm\infty}\gamma(t)=0$ and $\gamma(0)>0$ (see Figure 3). The following result is obvious [8].

LEMMA 2.1. *There is a unique $t_0 \in \mathbb{R}$ satisfying $\dot{\gamma}(t_0)=0$. Consequently, $\dot{\gamma}(t)>0, \forall\, t<t_0$ and $\dot{\gamma}(t)<0, \forall\, t>t_0$.*

The corresponding Melnikov function for the problem (2) is as follows [7; 8]

$$M_\mu(\alpha)=\mu_2\int_{-\infty}^{\infty}\dot{\gamma}(s)\psi(s+\alpha)\,ds-2\gamma(t_0)\mu_1\,,$$

where $\mu:=(\mu_1,\mu_2)$. Note M_μ is C^1-smooth and almost periodic in α. As a consequence of [7, Theorem 4.2] we get the following result.

THEOREM 2.2. *If there are $a<b$, μ_0, $|\mu_0|=1$ such that $M_{\mu_0}(a)M_{\mu_0}(b)<0$ then there is a constant $K>0$ and for any sufficiently small $s\neq 0$ there is an increasing sequence $\{t_j\}_{j\in\mathbb{Z}}$, $t_{j+1}-t_j\geq 2/\sqrt{|s|}$, $\forall j\in\mathbb{Z}$ such that for $\mu=s\mu_0$ and any infinite sequence $E=\{e_j\}_{j\in\mathbb{Z}}\in\{0,1\}^{\mathbb{Z}}$, the weakly discontinuous differential*

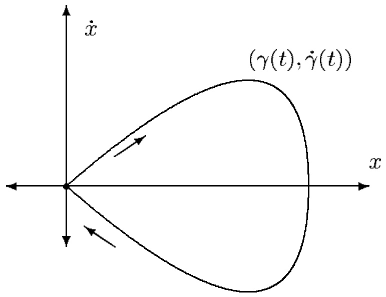

Fig. 3. The homoclinic structure of $\ddot{x} + g(x) = 0$.

equation (2) *possesses a solution* $x_{E,s}$ *satisfying*

$$\sup_{t_j \le t \le t_{j+1}} \left| x_{E,s}(t) - \gamma(t - \bar{\alpha}_{j,E,s}) \right| \le K|s| \quad when \quad e_j = 1\,,$$

$$\sup_{t_j \le t \le t_{j+1}} \left| x_{E,s}(t) \right| \le K|s| \quad when \quad e_j = 0\,,$$

where $\bar{\alpha}_{j,E,s} \in (t_j, t_{j+1})$ *for any* $j \in \mathbb{Z}$ *with* $e_j = 1$. *The mapping* $E \to z_{E,s}$ *is injective.*

Let $\mathcal{E} := \{0,1\}^{\mathbb{Z}}$ be the set of doubly infinite sequences of 0 and 1. $\mathcal{E}$ endowed with the metric

$$d(\{e_{n \in \mathbb{Z}}\}, \{e'_{n \in \mathbb{Z}}\}) := \sum_{n \in \mathbb{Z}} \frac{|e_n - e'_n|}{2^{|n|+1}}$$

is a compact metric space. Let $J : \mathcal{E} \to \mathcal{E}$ be the Bernoulli shift defined by

$$J\big(\{e_j\}_{j \in \mathbb{Z}}\big) = \{\tilde{e}_j\}_{j \in \mathbb{Z}}, \quad \tilde{e}_j = e_{j+1}\,.$$

The dynamics of J is extremely rich as it is indicated in the next theorem [11; 19].

THEOREM 2.3. *J is a homeomorphism having*

i) a countable infinity of periodic orbits of all possible periods,
ii) an uncountable infinity of nonperiodic orbits, and
iii) a dense orbit.

Now Theorem 2.2 for the periodic case gives the next result.

THEOREM 2.4. *Let* $\psi(t)$ *be* 2*-periodic. If there are* $a < b$, μ_0, $|\mu_0| = 1$ *such that* $M_{\mu_0}(a) M_{\mu_0}(b) < 0$. *Then for any* $\mu = s\mu_0$ *with* $s \ne 0$ *sufficiently small and* $m \in \mathbb{N}$ *sufficiently large,* (2) *possesses a family of solutions* $\{z_{m,E,s}\}_{E \in \mathcal{E}}$ *such that*

(i) $E \to x_{m,E,s}$ *is injective;*
(ii) $x_{m,J(E),s}(t)$ *is orbitally close to* $x_{m,E,s}(t+2m)$*.*

This result extends the deterministic chaos to (2) as follows. When $\mu_1 = 0$, then (2) is a regular system like (1) of the form

$$\ddot{x} + g(x) = \mu_2 \psi(t)\,. \tag{4}$$

Under the existence of a simple zero α_0 of

$$M(\alpha) = \int_{-\infty}^{\infty} \dot{\gamma}(s)\psi(s+\alpha)\,ds\,,$$

i.e. $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$, the assumption $M(a)M(b) < 0$ of Theorem 2.4 is satisfied for $a = \alpha_0 - \delta$ and $b = \alpha_0 + \delta$ with $\delta > 0$ small. Note $M(\alpha) = M_{(0,1)}(\alpha)$ in Theorem 2.4. But since the existence of a simple zero of M is stronger than the above assumption of Theorem 2.4, when $\mu_2 \neq 0$ is small in (4), the property (ii) of this theorem has the form (cf. [18; 19])

$$x_{m,J(E)}(t) = x_{m,E}(t+2m)\,. \tag{5}$$

Put

$$\Sigma := \{(x_{m,E}(0), \dot{x}_{m,E}(0)) \mid E \in \mathcal{E}\} \subset \mathbb{R}^2\,.$$

Let F_{μ_2} be the time map of the flow of the first order system of (4), i.e. $F_{\mu_2}(x_0, y_0) := (x(2), y(2))$, where $x(t)$ and $y(t)$ solves the Cauchy problem

$$\dot{x}(t) = y(t), \quad \dot{y}(t) = \mu_2\psi(t) - g(x(t)), \quad x(0) = x_0, \quad y(0) = y_0\,.$$

Then form (5) we immediately derive

$$F_{\mu_2}^m\left((x_{m,E}(0), \dot{x}_{m,E}(0))\right) = (x_{m,J(E)}(0), \dot{x}_{m,J(E)}(0))\,.$$

Hence

$$F_{\mu_2}^m : \Sigma \to \Sigma$$

and

$$F_{\mu_2}^m \circ \mathcal{J} = \mathcal{J} \circ J\,, \tag{6}$$

where $\mathcal{J} : \mathcal{E} \to \Sigma$ is defined as follows

$$\mathcal{J}(E) = (x_{m,E}(0), \dot{x}_{m,E}(0))\,.$$

Then (6) means that $F_{\mu_2}^m$ has the same dynamics on Σ as the Bernoulli shift J on $\mathcal{E}$. So by Theorem 2.3, the time map $F_{\mu_2}^m$ is chaotic on Σ. Moreover, it is possible to show a sensitive dependence on initial conditions of $F_{\mu_2}^m$ on Σ in the sense that there is an $\varepsilon_0 > 0$ such that for any $(x,y) \in \Sigma$ and any neighborhood U of (x,y), there exists $(u,z) \in U \cap \Sigma$ and an integer $q \geq 1$ such that

$$|F_{\mu_2}^{mq}(x,y) - F_{\mu_2}^{mq}(u,z)| > \varepsilon_0\,.$$

Of course, these results are known [18; 19]. Hence (4) is chaotic and sensitive depends on initial conditions as well. The set Σ is Smale's horseshoe of $F_{\mu_2}^m$ and we say that $F_{\mu_2}^m$ has horseshoe dynamics on Σ.

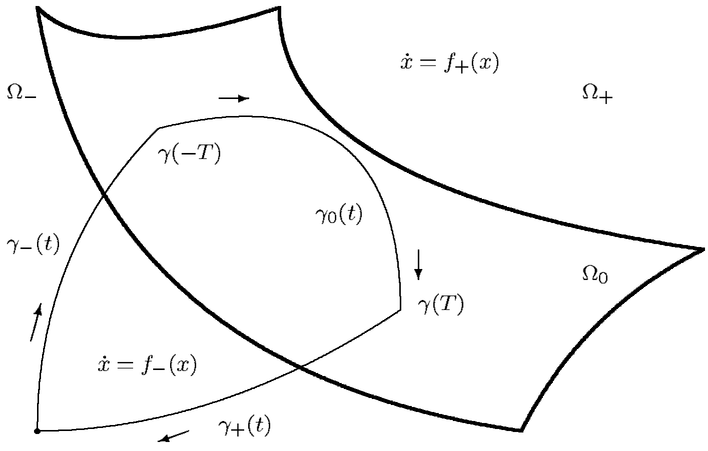

Fig. 4. A transversal homoclinic crossing of the surface of discontinuity.

Summarizing, when $\mu_1 \neq 0$ is small then Theorem 2.4 extends the deterministic chaos of (4) to (2), when in place of (5) we get property (ii) of Theorem 2.4.

Applying these results to (1), the equation $\ddot{x} - x + 2x^3 = 0$ has the homoclinic solution $\gamma(t) = \operatorname{sech} t$ and $M(\alpha) = \int_{-\infty}^{\infty} \dot{\operatorname{sech}}\, t \cos\omega(t+\alpha)\, dt = \pi\omega \operatorname{sech} \frac{\pi\omega}{2} \sin \alpha\omega$. So $\alpha_0 = 0$ is a simple root of $M(\alpha)$ and (1) is chaotic for $\varepsilon \neq 0$ small. Similarly, a discontinuous version $\ddot{x} - x + 2x^3 + \mu_1 \operatorname{sgn} \dot{x} = \mu_2 \cos t$ of (1) is chaotic for any μ_1, μ_2 sufficiently small satisfying $\frac{\pi}{2} \operatorname{sech} \frac{\pi}{2} \cdot |\mu_2| > |\mu_1|$.

3. TRANSVERSAL HOMOCLINIC CROSSING OF SURFACES OF DISCONTINUITY

In Section 2, we study bifurcation of homoclinic and chaotic solutions for ordinary differential equations with small multivalued perturbations. We assume that ordinary differential equations have homoclinic solutions to hyperbolic equilibria. More recently [1; 2], we have considered bifurcation from non-smooth homoclinics, i.e. we considered parameterized discontinuous ordinary differential equations with homoclinics crossing discontinuity levels. In [2] we investigate the following problem:

Let $G(x)$ be a C^r–function on $\Omega \subset \mathbb{R}^n$, $r \geq 2$ and let

$$\Omega_\pm = \{x \in \Omega \mid \pm G(x) > 0\}, \quad \Omega_0 := \{x \in \Omega \mid G(x) = 0\}.$$

Let $f_\pm(x) \in C^r(\bar{\Omega}_\pm)$ and consider the equation

$$\dot{x} = f_\pm(x) + \varepsilon g(t, x, \varepsilon), \quad x \in \bar{\Omega}_\pm, \tag{7}$$

where $g \in C^r(\mathbb{R} \times \Omega \times \mathbb{R})$ and $\varepsilon \in \mathbb{R}$ is a small parameter. Assume (see Figure 4)

H1) For $\varepsilon = 0$ equation (7) has the hyperbolic equilibrium $x = 0 \in \Omega_-$ and a continuous (not necessarily C^1) homoclinic orbit $\gamma(t)$ to $x = 0$ that consists of three solutions

$$\gamma(t)=\begin{cases}\gamma_-(t) & \text{if } t\le -T\\ \gamma_0(t) & \text{if } -T\le t\le T\\ \gamma_+(t) & \text{if } t\ge T\end{cases}$$

where $\gamma_\pm(t)\in\Omega_-$ for $|t|>T$, $\gamma_0(t)\in\Omega_+$ for $|t|<T$ and

$$\gamma_-(-T)=\gamma_0(-T)\in\Omega_0,\quad \gamma_+(T)=\gamma_0(T)\in\Omega_0.$$

H2) Moreover we also assume that

$$G'(\gamma(-T))f_\pm(\gamma(-T))>0,\quad\text{and}\quad G'(\gamma(T))f_\pm(\gamma(T))<0.$$

We also need a nondegeneracy condition of $\gamma(t)$ with respect to $\dot{x}=f_\pm(x)$, which reduces to the known notion of nondegeneracy in the smooth case [3; 18]. That nondegeneracy condition is more specified in [2]. Note $\gamma(t)$ crosses transversally the discontinuity level Ω_0 in (7) and $\gamma_\pm(t)$, $\gamma_0(t)$ satisfy $\dot{\gamma}_\pm(t)=f_-\left(\gamma_\pm(t)\right)$, $\dot{\gamma}_0(t)=f_+\left(\gamma_0(t)\right)$, respectively.

We have derived a Melnikov bifurcation function to find a solution $x(t,\varepsilon)$ of equation (7) such that

$$\sup_{t\in\mathbb{R}}|x(t,\varepsilon)-\gamma(t-\alpha(\varepsilon))|\to 0,\ \text{as}\ \varepsilon\to 0$$

for some $\alpha(\varepsilon)\in\mathbb{R}$, where $\alpha(0)$ is a simple root of the Melnikov function. Since $x=0\in\Omega_-$ and $f_-\in C^r(\Omega_-)$ we can apply regular perturbation theory to show the existence of a unique bounded solution $x_b(t,\varepsilon)\in\Omega_-$ of equation $\dot{x}=f_-(x)+\varepsilon g(t+\alpha(\varepsilon),x,\varepsilon)$ such that $\sup_{t\in\mathbb{R}}|x(t,\varepsilon)|\to 0$ as $\varepsilon\to 0$. Then, by standard dynamical systems methods (see [18]), one can show that

$$\lim_{|t|\to\infty}|x(t,\varepsilon)-x_b(t,\varepsilon)|=0.$$

Thus $x(t,\varepsilon)$ is homoclinic to $x_b(t,\varepsilon)$. Note that, if $g(t,x,\varepsilon)$ is periodic in t then, by uniqueness it follows that $x_b(t,\varepsilon)$ is a hyperbolic periodic solution of $\dot{x}=f_-(x)+\varepsilon g(t+\alpha(\varepsilon),x,\varepsilon)$.

We applied our general bifurcation result in [2] to the example

$$\dot{x}=\begin{cases}Ax+\varepsilon g\sin\omega t & \text{for } \widetilde{a}\cdot x<d,\\ Ax+b+\varepsilon g\sin\omega t & \text{for } \widetilde{a}\cdot x>d\end{cases}\tag{8}$$

of a periodically perturbed piecewise linear 3-dimensional differential equation. Here $d>0$, $\omega>0$, $\widetilde{a},x,g\in\mathbb{R}^3$, $\widetilde{a}\cdot x$ is the scalar product in $\mathbb{R}^3$. We suppose that A has semisimple eigenvalues. So the unperturbed (8) has a hyperbolic equilibrium $x=0$. We find conditions under which it has a homoclinic solution, and then we show that it persists for ε small under additional conditions. Hence our result for (8) is *Melnikov-type*. Therefore, in some sense, we study a complementary problem to (3) in [17]. Unfortunately, we have not proved the existence of the horseshoe dynamics for (8). The reason was that to find the homoclinic bifurcation conditions for (8) was rather technical itself. Of course the main purpose to find homoclinics for (8) is to show its chaotic dynamics. We intend to carry out this in a forthcoming paper.

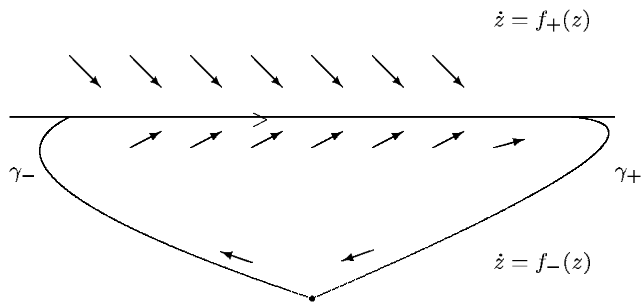

Fig. 5. A planar homoclinic sliding on the line of discontinuity.

Finally, for $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $b = (-1,-1,1)$, $G(x) = \widetilde{a} \cdot x - d = x_2 + x_3 - \frac{3}{4}$, $g = (1,1,1)$, we derived in [2] that $T = \ln 2$ and $\gamma_-(t) = \left(\frac{15}{8}\,\mathrm{e}^{2t}, \frac{3}{2}\,\mathrm{e}^{t}, 0\right)$, $\gamma_0(t) = \left(\frac{1}{8}\left(4 - \mathrm{e}^{2t}\right), \frac{1}{2}\left(2 - \mathrm{e}^{t}\right), \frac{1}{2}\left(2 - \mathrm{e}^{-t}\right)\right)$, $\gamma_+(t) = \left(0, 0, \frac{3}{2}\,\mathrm{e}^{-t}\right)$. If $\omega \neq \frac{\pi n}{\ln 2}$, $\forall n \in \mathbb{N}$ then the homoclinic solution $\gamma(t)$ for such (8) persists for any $\varepsilon \neq 0$ small.

4. PLANAR HOMOCLINIC SLIDING ON LINES OF DISCONTINUITY

On the other hand, in [1], we study a case when a part of homoclinic orbit is sliding on a discontinuity level: Consider the planar discontinuous system

$$\begin{aligned} \dot z &= f_+(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for} \quad y > 1\,, \\ \dot z &= f_-(z) + \varepsilon g(z,t,\varepsilon) \quad \text{for} \quad y < 1\,, \end{aligned} \tag{9}$$

where $z = (x,y) \in \mathbb{R}^2$, $f_\pm$, g are C^3-smooth, $\varepsilon \in \mathbb{R}$ is a small parameter and g is 1-periodic in t. While on $y = 1$ (cf. [16]), we consider the equation

$$\begin{aligned} \dot x = {} & \frac{q_{+2}(x,1,t,\varepsilon)}{q_{+2}(x,1,t,\varepsilon) - q_{-2}(x,1,t,\varepsilon)} q_{+1}(x,1,t,\varepsilon) \\ & + \frac{q_{-2}(x,1,t,\varepsilon)}{q_{-2}(x,1,t,\varepsilon) - q_{+2}(x,1,t,\varepsilon)} q_{-1}(x,1,t,\varepsilon)\,, \end{aligned} \tag{10}$$

where $q_\pm = (q_{\pm 1}, q_{\pm 2})$ and $q_\pm(z,t,\varepsilon) = f_\pm(z) + \varepsilon g(z,t,\varepsilon)$. We suppose the following conditions

i). $f_-(0) = 0$ and $Df_-(0)$ has no eigenvalues on the imaginary axis.

ii). There are two solutions $\gamma_-(s)$, $\gamma_+(s)$ of $\dot z = f_-(z)$, $y \leq 1$ defined on $\mathbb{R}_- = (-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty)$, respectively, such that $\lim_{s \to \pm\infty} \gamma_\pm(s) = 0$ and $\gamma_\pm(s) = (x_\pm(s), y_\pm(s))$ with $y_\pm(0) = 1$, $x_-(0) < x_+(0)$. Moreover, $f_\pm(z) = (f_{\pm 1}(z), f_{\pm 2}(z))$ with $f_{\pm 1}(x,1) > 0$, $f_{+2}(x,1) < 0$ for $x_-(0) \leq x \leq x_+(0)$. Furthermore, $f_{-2}(x,1) > 0$ for $x_-(0) \leq x < x_+(0)$, $f_{-2}(x_+(0),1) = 0$ and $\partial_x f_{-2}(x_+(0),1) < 0$.

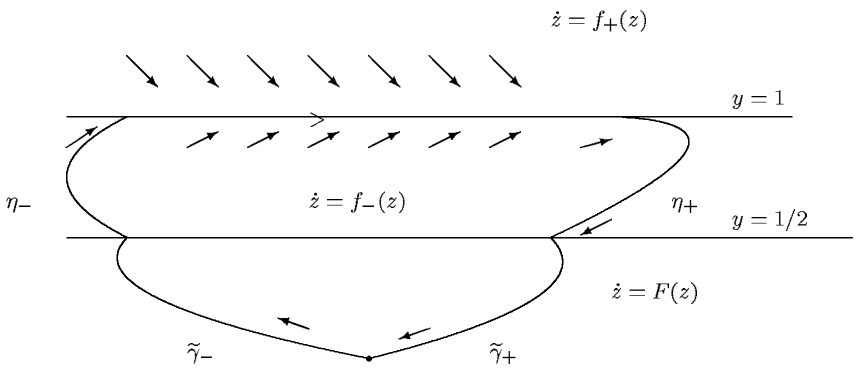

Fig. 6. A planar homoclinic sliding on the line of discontinuity with transversal crossing of another discontinuity line.

Assumptions i) and ii) mean (see Figure 5) that (9) for $\varepsilon = 0$ has a sliding homoclinic solution γ to a hyperbolic equilibrium 0, where γ is created by γ_-, γ_+ and the sliding segment $\{(x,1) \in \mathbb{R}^2 \mid x \in [x_-(0), x_+(0)]\}$. Conditions for the bifurcation of γ to bounded solutions on $\mathbb{R}$ of (9) under the perturbation $\varepsilon g(z,t,\varepsilon)$ are derived in [1] for $\varepsilon \neq 0$ small. Functional-analytical methods are used in [1; 2] based on the implicit function theorem.

We have also studied cases when homoclinic orbit $\gamma(s)$ transversally crosses another curves of discontinuity (see Figure 6). For simplicity, we supposed that such a discontinuity in (9) occurs at the level $y = 1/2$, i.e. we considered the system

$$\begin{aligned} \dot{z} &= f_+(z) + \varepsilon g(t) \quad \text{for} \quad y > 1\,, \\ \dot{z} &= f_-(z) + \varepsilon g(t) \quad \text{for} \quad 1/2 < y < 1\,, \\ \dot{z} &= F(z) + \varepsilon g(t) \quad \text{for} \quad y < 1/2\,, \end{aligned} \tag{11}$$

where $z = (x,y) \in \mathbb{R}^2$, $f_\pm$, F, g are C^3-smooth, $\varepsilon \in \mathbb{R}$ is a small parameter and g is 1-periodic in t. To be more concrete, we focus on the following concrete discontinuous forced planar system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= x - 3y + \varepsilon \cos t \quad \text{for} \quad y > 1\,, \\[1ex] \dot{x} &= y, \\ \dot{y} &= 2y - x + \varepsilon \cos t \quad \text{for} \quad 1/2 < y < 1\,, \\[1ex] \dot{x} &= -2ay, \\ \dot{y} &= -\tfrac{1}{2a}x + \varepsilon \cos t \quad \text{for} \quad y < 1/2\,, \end{aligned} \tag{12}$$

where $a = e^{a_+}(2 - a_+) \doteq 2.65554$ and $a_+ \doteq 0.768039$ is the unique (positive) solution of the equation $e^{a_+}(1 - a_+) = 1/2$. Now the homoclinic orbit γ consists from the following parts (see Figure 6)

$$\eta_+(s) = (e^s(2-s), e^s(1-s)) \quad \text{for} \quad s \in [0, a_+],$$

$$\widetilde{\gamma}_+(s) = \big(e^{-s}a, e^{-s}/2\big) \quad \text{for} \quad s \geq 0,$$

$$\widetilde{\gamma}_-(s) = (-e^s a, e^s/2) \quad \text{for} \quad s \leq 0,$$

$$\eta_-(s) = \left(-ae^{s-a_-} + \Big(\frac{1}{2}+a\Big)e^{s-a_-}(s-a_-), \frac{1}{2}e^{s-a_-} + \Big(\frac{1}{2}+a\Big)e^{s-a_-}(s-a_-)\right) \quad \text{for} \quad s \in [a_-, 0],$$

and the sliding segment $\left\{(x,1) \in \mathbb{R}^2 \mid x \in \left[-ae^{-a_-} - \left(\frac{1}{2}+a\right) e^{-a_-}a_-, 2\right]\right\}$,

where $a_- \doteq -0.122043$ is the unique (negative) solution of the equation $e^{a_-} + \Big(\frac{1}{2} + a\Big)a_- = 1/2$. Note $-ae^{-a_-} - \left(\frac{1}{2}+a\right) e^{-a_-}a_- \doteq -2.56514$.

By [1] the Melnikov function takes the form

$$\widetilde{M}(\alpha) = -0.441052\cos\alpha - 1.7501\sin\alpha\,. \tag{13}$$

Since the function (13) has two different simple roots over the period 2π, for $\varepsilon \neq 0$ small, there are two bounded solutions of (12) near to γ which are homoclinic to a small hyperbolic 2π-periodic solution of (12).

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