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Envelope theory and its application for a forest fire front evolution[1]

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Abstract

In this paper the use of Huygens' principle for the description of forest fire spread in time under variable weather and fuel conditions is investigated. A new procedure for the derivation of elliptical model for two-dimensional fire front evolution is described. It uses classical envelope theory of differential geometry and allows to obtain better insight into the complexity of the studied problem and to better understand advantages and limitations of the model, its implementation and use for the purposes of modeling, simulation and understanding of forest fires.
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1. INTRODUCTION

Mathematical modeling of forest fire spread in time plays a key role in existing decision support systems developed for planning, management and coordination of fire fighting activities and prevention. Forest fires belong to destructive natural phenomena which have been sufficiently described because of their complexity, still inadequate knowledge about the processes determining behaviour of the phenomenon, and huge amount of required data, as well as of serious difficulties with their extraction and gathering. Every year, they cause a large damage of vegetation, property, eco-systems and environment, but they also threaten people's lives and bind significant human resources. Therefore, the development of advanced systems capable to model and simulate the forest fire behaviour has become an urgent challenge for scientists, researchers and developers, as well as for responsible state and regional authorities.

The idea of using the wave approach (Huygens' principle) for prediction of a forest fire front evolution appeared in the literature about thirty years ago. The first method based on the wave principle, the radial fire propagation model (Sanderlin, Sunderson, 1975), was developed in 1975. It used gridded weather inputs and rasterized topography of landscape and fuels to achieve a reasonable approximation of the observed fire growth.

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The concept of Huygens' principle and related terminology was brought into the fire literature in 1982, when a graphical fire prediction technique based on Huygens' principle was developed (Anderson et al., 1982). It assumed that each point on the fire front at time *t* was the ignition point for a small local fire, which in a time interval *dt* burned out an elliptical region (elliptical local model). Assuming that each ellipse was defined by the conditions at its generating point, the fire front at time *t+dt* was plotted by drawing the ellipses and tracing out the new fire front defined by the envelope of all the ellipses. This technique was confirmed to be a suitable fire growth model by comparison of its outputs with experimental test fires. Other fire growth prediction techniques based on Huygens' principle were developed for purposes of fire growth simulation and visualization (see e.g. French, 1992).

In 1990, the use of Huygens' principle for fire front evolution was formulated the first time by G.D.Richards analytically. The fire front growth in time was described by a non-linear system of differential equations of the first order (Richards, 1990). Richards assumed a local elliptical spread of fire from an ignition point on perfectly flat ground with homogeneous fuel and constant wind (elliptical local model).This assumption was ……, because it was observed experimentally that within time *dt* the burning area would have the shape of ellipse with semi-axes *adt* and *bdt*, where the centre of the ellipse would be shifted by *wdt* in the wind direction (Anderson, 1982, Alexander, 1985, Anderson et al., 1982). A starting fire front at time *t*, which bordered the burning area at time *t,* was represented parametrically by a given closed continuous curve $(x(\varphi,t), y(\varphi,t))$, where φ was the parameter of a suitable curve parametrization. Similarly, a new fire front at time *t+dt* was represented by a curve $(x(\varphi,t+dt), y(\varphi,t+dt))$.

In the case that both the curves $(x(\varphi,t), y(\varphi,t))$ and $(x(\varphi,t+dt), y(\varphi, t+td))$ are known, one can calculate the change of position of the fire front in time by limit process as $dt \to 0$. It means that for all values of the parameter φ it is possible to express

$$\lim_{dt\to 0}\frac{x(\varphi,t+dt)-x(\varphi,t)}{dt}=x_t(\varphi,t)\,, \qquad \lim_{dt\to 0}\frac{y(\varphi,t+dt)-y(\varphi,t)}{dt}=y_t(\varphi,t). \qquad (1)$$

By this procedure one can obtain not only the rate of change in time of coordinates of points, which lie on the fire front, but also the system of differential equations which describes this process. To calculate the curve $(x(\varphi,t+dt), y(\varphi, t+td))$ Richards used a linear transform (rotation and re-scalling of axes) to transforming the ellipses into circles and geometrical similarities.

In this paper, unlike the paper (Richards 1990), we use classical envelope theory of differential geometry to obtain better insight into the complexity of the studied problem and to detect a hidden coherence present in it. Such analysis allows us to show, for which assumptions the fire front evolution model leads to the system of differential equations known from (Richards, 1990). The paper is organized as follows. In Section 2 the Richards´ approach is briefly described. In Section 3 we derive the model by the use of classical envelope theory of differential geometry. This procedure can help to better understand some assumptions and restrictions of the studied model and their implications for its discrete implementation and practical use for the purposes of forest fire simulation and modeling. Section 4 shows some simple illustrations of possible forest fire front evolution based on envelopes of ellipses.

2. ELLIPTICAL FIRE GROWTH MODEL DERIVED BY G. D. RICHARDS

As was mentioned at introduction the Richards model is based on two principles.

a) A local elliptical model of a fire spread,
b) An application of Huygens´ principle of a wave propagation on the problem of forest fire spread.

The application of these principles can be illustrated by the following Fig.1. Each point on the primary fire front at time *"t"* (Fig.1) can be considered as the starting point of a local fire of a secondary elliptical shape. The new fire front at time *"t+dt"* is defined by the envelope of all secondary ellipses. We need to find it analytically.

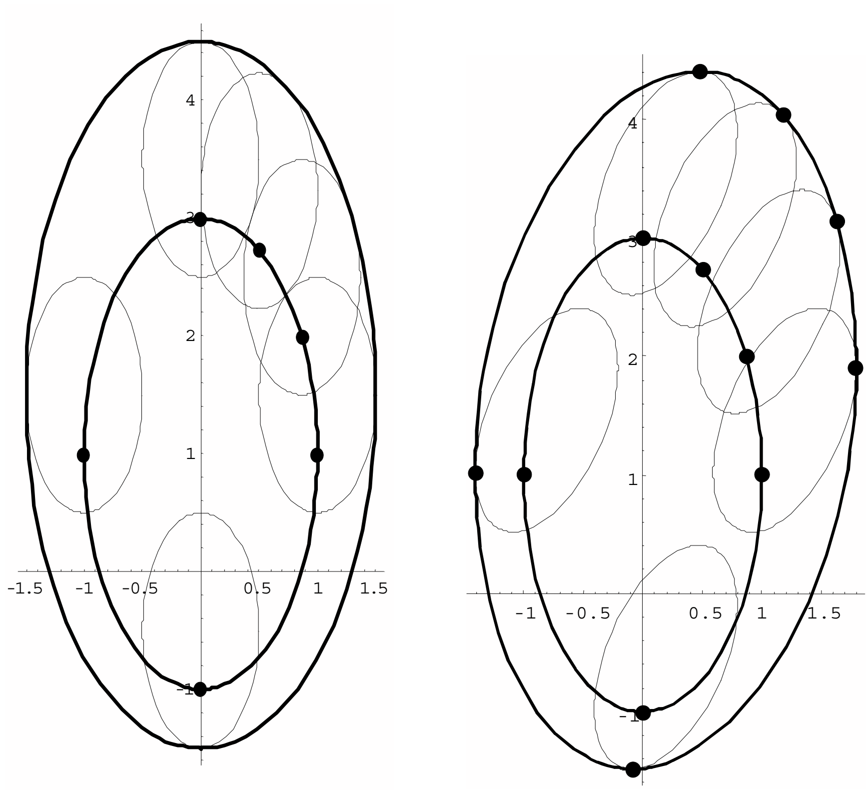


Figure 1: Fire propagation in time: starting fire front (ellipse) and new fire represented by the envelope of the secondary ellipses; the case of (a) constant wind and constant fuel (b) variable wind and constant fuel.

In order to utilize geometrical properties of points lying on a common tangent line of two circles, Richards introduced the following transform (Richards 1990),

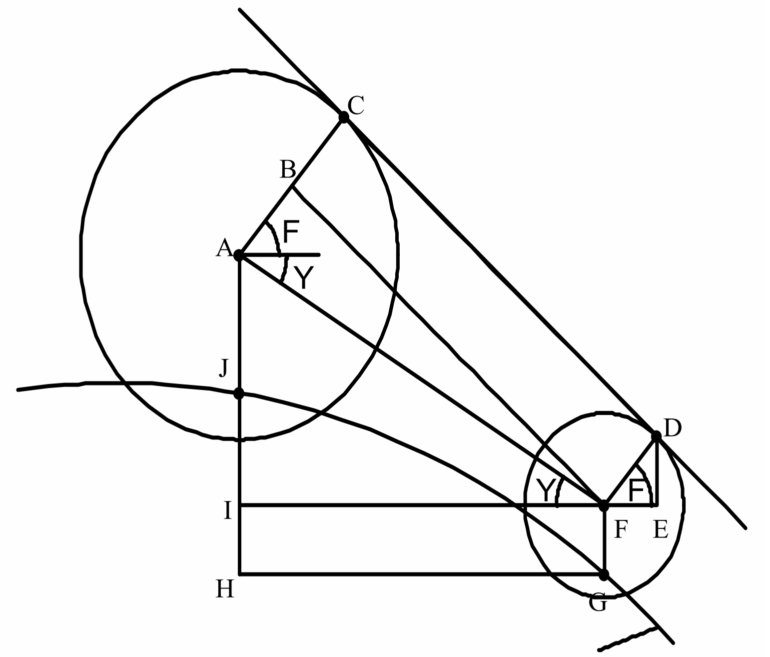
$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} b/a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix}, \quad (2)$$

where X,Y and x,y are coordinates in the transformed and original coordinate system, respectively, and C = cos θ and S = sin θ. The transform T rotates the axes of coordinate system by the angle θ and re-scale the axis by the quotient b/a. By such a procedure, the same direction of the wind and the Y-axis, as well as the transform of secondary ellipses into circles is achieved. Thus, in the transformed coordinate system (*X,Y*), one can find the envelope of circles instead of that of ellipses, and then get the envelope of ellipses by the corresponding inverse transform of T.

In this section we will show the derivation of the system of differential equations for the fire spread description by Richards (1990), where the envelope of circles is expressed utilizing geometric properties of the corresponding points lying at the transformed fire front and the envelope of circles.

Let us analyse the case of local fire spread from two points G and J lying at a fire front shown on Fig. 2, which correspond to the parameters (φ, t), $(\varphi + d\varphi, t)$. For points and distances on Fig. 2 it holds:

$$
\begin{aligned}
G &= (X(\varphi,t) , Y(\varphi,t)) \\
J &= (X(\varphi,t) + d\varphi\, X_\varphi(\varphi,t) , Y(\varphi,t) + d\varphi\, Y_\varphi(\varphi,t)) \\
AB &= dt\, d\varphi\, b_\varphi(\varphi, t) , \quad DF = dt\, b(\varphi,t) \\
FG &= dt\, w(\varphi,t) , \qquad FI = - d\varphi\, X_\varphi(\varphi, t) , \\
AJ &= dt\, w(\varphi+d\varphi, t) = dt\, w(\varphi,t) + dt\, d\varphi\, w_\varphi(\varphi,t) , \\
AC &= dt\, b(\varphi+d\varphi, t) = dt\, b(\varphi,t) + dt\, d\varphi\, b_\varphi(\varphi,t) , \\
AF &= (AI^2 + FI^2)^{1/2} = d\varphi\, [(dt\, w_\varphi(\varphi,t) + Y_\varphi(\varphi, t))^2 + X_\varphi^{2}(\varphi, t)]^{1/2} ,
\end{aligned}
$$



fire front at time t

Figure 2: Fire propagation in time: fire front at time t and two circles in new coordinate system. The circles correspond to two local ellipses at points G and J defined by the parameters φ and $\varphi + d\varphi$.

The corresponding circles (in the transformed coordinate system) represent the local fire fronts at time $t+dt$. It follows from Fig. 2 that (Richards, 1990):

$$X[D] = X(\varphi, t+dt) = X(\varphi, t) + EF = X(\varphi, t) + dt\, b(\varphi, t)\, cos(F + Y - Y) =$$
$$= X(\varphi, t) + dt\, b(\varphi, t)\, [\, cos\,(F + Y)\, cos\, Y + sin\,(F + Y)\, sin\, Y\,] \tag{3}$$

$$Y[D] = Y(\varphi, t+dt) = Y(\varphi, t) + DE + FG = Y(\varphi, t) + dt\, b(\varphi, t)\, sin\,(F + Y - Y) + dt\, w(\varphi, t) =$$
$$= Y(\varphi,t) + dt\, b(\varphi, t)[\, sin\,(F + Y)\, cos\, Y - cos\,(F + Y)\, sin\, Y\,] + dtw(\varphi, t) \tag{4}$$

Since the line CD is the tangent line to both circles, if $d\varphi \to 0$, then $C \to D$ and the line CD approaches the envelope of both circles and the coordinates of point D approach the coordinates $(X(\varphi,t+dt), Y(\varphi,t+dt))$. Therefore, after the substitution of the relations

$$\cos(F + Y) = AB/AF, \qquad \sin(F + Y) = (AF^2 - AB^2)^{1/2}/AF,$$
$$\cos Y = FI/AF, \qquad \sin Y = (AF^2 - FI^2)^{1/2}/AF, \tag{5}$$

which follow from Fig. 2, into (3)-(4), we can calculate as $d\varphi \to 0$:

$$\begin{bmatrix} X(\varphi,t+dt) \\ Y(\varphi,t+dt) \end{bmatrix} = \begin{bmatrix} X(\varphi,t) + P(\varphi,t,dt) \\ Y(\varphi,t) + Q(\varphi,t,dt) \end{bmatrix}, \tag{6}$$

where

$$P(\varphi,t,dt) = dtb\frac{-X_\varphi dtb_\varphi + (dtw_\varphi + Y_\varphi)((dtw_\varphi + Y_\varphi)^2 + X_\varphi^2 - dt^2 b_\varphi^2)^{1/2}}{(dtw_\varphi + Y_\varphi)^2 + X_\varphi^2} \tag{7}$$

$$Q(\varphi,t,dt) = dtb\frac{-((dtw_\varphi + Y_\varphi)^2 + X_\varphi^2 - dt^2 b_\varphi^2)^{1/2} X_\varphi - (dtw_\varphi + Y_\varphi)dtb_\varphi}{(dtw_\varphi + Y_\varphi)^2 + X_\varphi^2} + wdt. \tag{8}$$

All functions on RHS of (7)-(8) are evaluated at point (φ,t). These equations hold in the coordinate system (X,Y), therefore now we need to transform them in the coordinate system (x,y) by the inverse of T :

$$T^{-1}\begin{bmatrix} X(\varphi,t+dt) \\ Y(\varphi,t+dt) \end{bmatrix} = T^{-1}\begin{bmatrix} X(\varphi,t) + P(\varphi,t,dt) \\ Y(\varphi,t) + Q(\varphi,t,dt) \end{bmatrix} =$$
$$= \begin{bmatrix} x(\varphi,t) \\ y(\varphi,t) \end{bmatrix} + \begin{bmatrix} C & S \\ -S & C \end{bmatrix}\begin{bmatrix} a/b & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} P(\varphi,t,dt) \\ Q(\varphi,t,dt) \end{bmatrix}$$

i.e.

$$\begin{bmatrix} x(\varphi,t+dt) \\ y(\varphi,t+dt) \end{bmatrix} - \begin{bmatrix} x(\varphi,t) \\ y(\varphi,t) \end{bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix}\begin{bmatrix} (a/b)P(\varphi,t,dt) \\ Q(\varphi,t,dt) \end{bmatrix}.$$

If we now apply the limit process as $dt \to 0$ to both sides of the equation,

$$\begin{bmatrix} x_t(\varphi,t) \\ y_t(\varphi,t) \end{bmatrix} = \lim_{dt\to 0} \frac{1}{dt} \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} (a/b)P(\varphi,t,dt) \\ Q(\varphi,t,dt) \end{bmatrix}$$

and compute the partial derivatives with respect to φ under the assumption that b/a is not a function of parameter φ

$$\begin{bmatrix} X_\varphi \\ Y_\varphi \end{bmatrix} = \begin{bmatrix} b/a & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} x_\varphi \\ y_\varphi \end{bmatrix} = \begin{bmatrix} (b/a)\left(Cx_\varphi - Sy_\varphi\right) \\ Sx_\varphi + Cy_\varphi \end{bmatrix},$$

we obtain the following system of differential equations:

$$x_t(\varphi,t) = \frac{a^2 C\left(x_\varphi S + y_\varphi C\right) - b^2 S\left(x_\varphi C - y_\varphi S\right)}{\left[b^2\left(x_\varphi C - y_\varphi S\right)^2 + a^2\left(x_\varphi S + y_\varphi C\right)^2\right]^{1/2}} + wS \qquad (9)$$

$$y_t(\varphi,t) = \frac{-a^2 S\left(x_\varphi S + y_\varphi C\right) - b^2 C\left(x_\varphi C - y_\varphi S\right)}{\left[b^2\left(x_\varphi C - y_\varphi S\right)^2 + a^2\left(x_\varphi S + y_\varphi C\right)^2\right]^{1/2}} + wC \quad . \qquad (10)$$

These equations represent the system of differential equations which describes the spread of forest fire on perfectly flat terrain. The model has been generalized for slopy terrain and became part of several useful fire management tools.

In the following section we will show a new approach based on the classical envelope theory from differential geometry (Glasa, Halada, 2005). From the envelope of ellipses, the system of differential equations will be obtained by limit process.

3. APPLICATION OF ENVELOPE THEORY

Let us consider, that the burning area at time t is represented by a given simply closed planar curve ($x(\varphi,t)$, $y(\varphi,t)$), where φ is parameter dependent on the curve parametrization, for example $\varphi \in <0, 2\pi)$. According to Huygens principle, each point on this curve is ignition point with the given burning conditions defined by the values $a(\varphi,t)$, $b(\varphi,t)$, $w(\varphi,t)$. The new fire front at time $t+dt$ is defined by the envelope of system of secondary ellipses generated by the mentioned functions $a(\varphi,t)$, $b(\varphi,t)$, $w(\varphi,t)$. This system of ellipses, which correspond to different values of the parameter φ can be expressed in the form

$$\begin{bmatrix} \tilde{x}(\varphi,t+dt) \\ \tilde{y}(\varphi,t+dt) \end{bmatrix} = \begin{bmatrix} x(\varphi,t) \\ y(\varphi,t) \end{bmatrix} + \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} a(\varphi,t)dt\cos\alpha \\ b(\varphi,t)dt\sin\alpha + w(\varphi,t)dt \end{bmatrix}, \qquad (11a)$$

respectively,

$$\begin{bmatrix} \tilde{x}(\varphi,t+dt) - x(\varphi,t) \\ \tilde{y}(\varphi,t+dt) - y(\varphi,t) \end{bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{bmatrix} a(\varphi,t)dt\cos\alpha \\ b(\varphi,t)dt\sin\alpha + w(\varphi,t)dt \end{bmatrix}, \qquad (11b)$$

where $\tilde{x}$, $\tilde{y}$ and $\alpha \in <0, 2\pi>$ are variables of secomdary ellipses.

This system of ellipses is represented by 2 parameters φ and α. Unfortunately, we need only one parameter expression. It can be obtained by the following way. For the brevity, let us make the substitution

$$\Delta x = \tilde{x}(\varphi,t+dt) - x(\varphi,t)$$
$$\Delta y = \tilde{y}(\varphi,t+dt) - y(\varphi,t)$$

and let the independent variables φ and t of the functions a, b, c be omitted. By the matrix inversion we obtain from (11b)

$$\begin{bmatrix} C & -S \\ S & C \end{bmatrix} \begin{bmatrix} \Delta x(\varphi,t) \\ \Delta y(\varphi,t) \end{bmatrix} = \begin{bmatrix} a(\varphi,t)dt\cos\alpha \\ b(\varphi,t)dt\sin\alpha + w(\varphi,t)dt \end{bmatrix} \qquad (12a)$$

and briefly, we can (12a) re-write as follows

$$C\Delta x - S\Delta y = a\,dt\cos\alpha$$
$$S\Delta x + C\Delta y - w\,dt = b\,dt\sin\alpha \qquad (12b)$$

Now, if we make

1. multiplication of the first and second equation with the value b and a, respectively
2. raise to both equation to second power
3. add both equations

we have

$$\left(\frac{C\,\Delta x - S\,\Delta y}{a\,dt}\right)^2 + \left(\frac{S\,\Delta x + C\,\Delta y - w\,dt}{b\,dt}\right)^2 - 1 = 0 \qquad (13)$$

Thus, the equation (13) represents a function only of three independent variables $\tilde{x}$, $\tilde{y}$ and φ. The parameter α has been eliminated and now, the system of all elipsses is expresed by one parameter only. Formally, (13) can be written as

$$F(\tilde{x}, \tilde{y}, \varphi) = 0 .$$

According to the envelope theory and its consequences, the envelope of all ellipses can be find as the solution of the system

$$F(\tilde{x}, \tilde{y}, \varphi) = 0$$

$$F_\varphi(\tilde{x}, \tilde{y}, \varphi) = 0 , \qquad (14)$$

If we make substitution

$$X = \frac{C\Delta x - S\Delta y}{adt}, \qquad Y = \frac{S\Delta x + C\Delta y - wdt}{bdt},$$

then it can be easily verified that the system (14) has the form

$$X^2 + Y^2 - 1 = 0$$

$$X\left(\frac{Cx_\varphi - Sy_\varphi}{adt} + X\frac{a_\varphi}{a}\right) + Y\left(\frac{Sx_\varphi + Cy_\varphi}{bdt} + Y\frac{b_\varphi}{b}\right) = 0 \qquad (15)$$

which has 4 solutions, generally. However, only two solutions are acceptable from the physical point of you, namely the outer and inner envelope of ellipses. It can be proved, that the system (15) has two solutions if

$$\frac{a_\varphi}{a} = \frac{b_\varphi}{b} \qquad \textit{resp.} \qquad \frac{a_\varphi}{b_\varphi} = \frac{a}{b}. \qquad (16)$$

Thus, the explicit form of the envelope of ellipses is

$$\tilde{x}=x+wSdt+dt^2\frac{-a^2bb_\varphi(w_\varphi dtS+x_\varphi)}{a^2(w_\varphi dt+Sx_\varphi+Cy_\varphi)^2+b^2(Cx_\varphi-Sy_\varphi)}\pm$$

$$dt\frac{\left[Ca^2(w_\varphi dt+Sx_\varphi+Cy_\varphi)-Sb^2(Cx_\varphi-Sy_\varphi)\right]\left[b^2(Cx_\varphi-Sy_\varphi)+a^2(w_\varphi dt+Sx_\varphi+Cy_\varphi)^2-a^2b_\varphi^2dt^2\right]^{1/2}}{a^2(w_\varphi dt+Sx_\varphi+Cy_\varphi)^2+b^2(Cx_\varphi-Sy_\varphi)^2}$$

$$\tilde{y}=y+wCdt+dt^2\frac{-a^2bb_\varphi(w_\varphi dtC+y_\varphi)}{a^2(w_\varphi dt+Sx_\varphi+Cy_\varphi)^2+b^2(Cx_\varphi-Sy_\varphi)}\pm$$

$$dt\frac{\left[Sa^2(w_\varphi dt+Sx_\varphi+Cy_\varphi)+Cb^2(Cx_\varphi-Sy_\varphi)\right]\left[b^2(Cx_\varphi-Sy_\varphi)+a^2(w_\varphi dt+Sx_\varphi+Cy_\varphi)^2-a^2b_\varphi^2dt^2\right]^{1/2}}{a^2(w_\varphi dt+Sx_\varphi+Cy_\varphi)^2+b^2(Cx_\varphi-Sy_\varphi)^2}.$$

The derivation of the system of differential equation, which describes the fire growth, is straightforward by limit process as dt $\rightarrow 0$

$$x_t(\varphi,t)=\lim_{dt\to 0}\frac{\tilde{x}-x}{dt}, \qquad y_t(\varphi,t)=\lim_{dt\to 0}\frac{\tilde{y}-y}{dt}.$$

This computation leads to equation derived by Richards under the assumptions (16).

4. Conclusion

In this paper, the application of envelope theory for modeling the steady-state forest fire spread under the assumption of the local elliptical spread model is described. The investigated model is based on Huygens' principle of wave propagation assuming that each point on the starting fire front becomes the ignition source of a local fire of elliptical shape and the new fire front is defined by the envelope of these ellipses. Following the assumptions used by Richards to derive the model, we used the standard procedure for the envelope derivation known in the envelope theory of sets of curves to derive the explicite formulae for the envelope of the set of ellipses which forms the resulting fire front.

The proposed procedure of the model derivation avoids the necessity of transform ofthe original co-ordinate system used by Richards to transform ellipses to circles and to utilize of points lying on common tangent line of two circles. We use the same assumptions as Richards to get the same resulting formulae (but formally without the use of the Richards' transform) and show the equivalence between the presented procedures.

The readers who are engaged in the use of methods and systems developed for the forest

fire spread modeling and simulation can benefit from deeper analysis of fundaments of the mathematical fire spread model investigated. What is very important, the new procedure allows to suggest further generalizations of the model, including e.g. the use of some non-elliptical local spread models.

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