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Discontinuity of similarity relations*

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Abstract

The main purpose of this paper is to discuss a connection between binary fuzzy equivalence relations called similarity relations and another type of binary fuzzy relations, coherent nearnesses, as well as continuity of similarity relations and their points of discontinuity on the universe of all real numbers.

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1. INTRODUCTION

This paper is intended to discuss some metric properties of one significant type of fuzzy equivalence relations, called the similarity relations. Fuzzy equivalence relations play a significant role in fuzzy set theory, analogical to the role of crisp equivalence relations in classical theory.

Lotfi Zadeh introduced the notion of a fuzzy binary relation in his first paper on fuzzy sets in 1965. The most frequently appearing type of binary fuzzy relations are so called **fuzzy equivalence relations**, playing a significant role in fuzzy set theory. They model in a sense proximity, indistinguishability, or similarity of elements from an arbitrary universe X. In what follows we will pay special attention to one kind of fuzzy equivalence relations, introduced also by Lotfi Zadeh (1971), called **similarity relations**.

We will study similarity relations on the set of all real numbers. We want to discuss two main problems.

First: Whether, in the real case, there is a connection between similarity relations and another type of binary fuzzy relations, modeling indistinguishability of elements, called **coherent nearnesses**.

Second: Whether a similarity relation on the universe of all real numbers $\mathbb{R}$ considered as a real function of two variables is, or if any can be, continuous with respect to the standard metric in E_2

First let us recall several basic necessary concepts.

2. PRELIMINARIES

Definition 1.

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Let $\mathcal{T}$ be a t-norm. A binary fuzzy relation E on a universe X is called a **$\mathcal{T}$-equivalence** (or fuzzy equivalence relation) on X if and only if it is reflexive, symmetric and $\mathcal{T}$-transitive, it means, if and only if for any $x, y, z \in X$:

(E1) $E(x, x) = 1$

(E2) $E(x, y) = E(y, x)$

(E3) $\mathcal{T}(E(x, y), E(y, z)) \leq E(x, z)$.

Let us recall that the triangular norm (briefly, t-norm) $\mathcal{T}$ is a binary operator on $[0, 1]$ that is nondecreasing, associative, commutative and such that $\mathcal{T}(1, x) = x$, for each $x \in [0, 1]$.

Definition 2.
A $\mathcal{T}$-equivalence relation S on a universe X is called a **similarity relation**, if $\mathcal{T} = \mathcal{T}_M$, the minimum t-norm, it means if for any $x, y, z \in X$:

$$\min(S(x, y), S(y, z)) \leq S(x, z)$$

In this article we will restrict ourselves to the case of real similarity relations, if the universe $X = \mathbb{R}$.

From Theorem 5.1. and Proposition 5.4. in [2], it follows

Proposition 1.
A fuzzy relation E on the universe $\mathbb{R}$ is a similarity relation if and only if there exists a fuzzy set $h \in \mathcal{F}(\mathbb{R})$ such that

$$E(x, y) = \begin{cases} \min(h(x), h(y)), & \text{if } x \neq y, \\ 1, & \text{otherwise} \end{cases}$$

Another approach to the fuzzification of proximity is based on the concept of a nearness (see, for example [1], [3], [4]). In [1] a type of nearness, called a **coherent nearness**, modeling equivalence or proximity of two elements in a universe, was introduced as follows.

Definition 3.
Let X be a universe. A binary fuzzy relation N on X is called a coherent nearness on X, if:

(N1) $N(x, x) = 1$, for each $x \in X$

(N2) $N(x, y) = N(y, x)$, for each $x, y \in X$

(N3) For each $\epsilon > 0$ there exists $\delta < 1$ such that

$$N(x, y) > \delta \Longrightarrow |N(x, z) - N(y, z)| < \epsilon, \text{ for each } x, y, z \in X.$$

The motivation for the properties (N1) and (N2), i.e. reflexivity and symmetricity of N, is obvious. The property (N3) substitutes, in a sense, the triangular inequality and it has the following meaning: If two points x and y are sufficiently near one another, then the difference of their nearnesses to any other point z is arbitrarily small.

A trivial verification shows that the property (N3) implies the property

(N3') For each $x, y, z \in X$:

$$N(x,y) = 1 \Longrightarrow N(x,z) = N(y,z)$$

Hence, if N doesn't distinguish two elements from an universe, if they have the maximal possible degree of nearness, then their nearness to any other element of the universe is of the same degree.

For more detailed terminology of fuzzy notions see e.g. [5].

In what follows, for abbreviation, let **similarity** stand for a similarity relation on $\mathbb{R}$ and **nearness** for a coherent nearness on the universe $\mathbb{R}$ as well.

3. CONNECTION BETWEEN SIMILARITIES AND NEARNESSES

Proposition 2.

Suppose S is a real reflexive and symmetric binary fuzzy relation. Then S is a similarity if and only if for arbitrary three real numbers x, y, z it holds:
$S(x,y) = S(y,z) \leq S(x,z)$, *or* $S(y,x) = S(x,z) \leq S(y,z)$, *or* $S(x,z) = S(z,y) \leq S(x,y)$.

Proof. It is clear, that any reflexive and symmetric fuzzy relation satisfying the property above, is $\mathcal{T}_M$-transitive.

Now let S be a similarity and the property is not fulfilled. Then there must be three real numbers x, y, z such that $S(x,y) > \min(S(y,z), S(x,z))$. But this contradicts the T_M-transitivity of S.

Proposition 3.

Any similarity is a nearness.

Proof. Since each similarity is reflexive and symmetrical, it is sufficient to show, that it satisfies the property (N3).

To obtain a contradiction, let S be a similarity, not satisfying the property (N3). Hence there is an $\epsilon > 0$ such that for all natural n there exist $x_n, y_n, z_n \in \mathbb{R}$ such that $S(x_n, y_n) > 1 - \frac{1}{n}$, but $|S(x_n, z_n) - S(y_n, z_n)| \geq \epsilon$.

It follows, that for each n is $S(x_n, z_n) \neq S(y_n, z_n)$. According to Proposition 2 then

$$S(x_n, y_n) = \min(S(x_n, z_n), S(y_n, z_n)) < \max(S(x_n, z_n), S(y_n, z_n)).$$

It means, that both values $S(x_n, z_n)$ and $S(y_n, z_n)$ belong to the interval $(1 - \frac{1}{n}, 1\rangle$, thus

$$|S(x_n, z_n) - S(y_n, z_n)| < \frac{1}{n},$$

a contradiction.

As the following example shows, opposite is not true. A nearness can be, but need not be a similarity.

Example 1.

Let $N(x, y) = e^{-|x-y|}$ be a fuzzy relation, defined for $x, y \in \mathbb{R}$ (see its graph in Figure 1). It is clearly reflexive and symmetrical. As follows from Theorem 3 in [1], the property (N3) is also fulfilled. Therefore, N is a nearness. But it is easily seen, that it is not similarity, since it is not $\mathcal{T}_M$-transitive:

Put, for example, $x = 1, y = 2, z = 4$. Then

$$N(x, y) = \frac{1}{e}, \quad N(x, z) = \frac{1}{e^3}, \quad N(y, z) = \frac{1}{e^2} \ .$$

It follows, that

$$\min(N(x, y), N(y, z)) = \frac{1}{e^2} > \frac{1}{e^3} = N(x, z).$$

As follows from this example, despite of the fact, that both discussed fuzzy relations, similarities as well as nearnesses represent a type of fuzzification of the same properties, like distance, proximity, or indistinguishability, both are reflexive and symmetrical, there is rather significant difference what about their third properties, $\mathcal{T}_M$-transitivity and the property (N3). While nearnesses can be, as in the previous example, but need not be continuous, similarities are never continuous (excepting a trivial case).

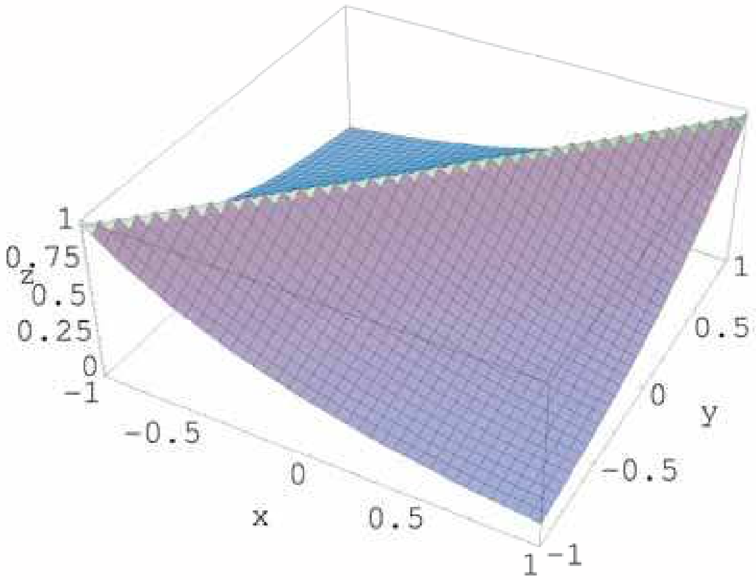

Figure 1.

4. SIMILARITY RELATIONS POINTS OF DISCONTINUITY

Since the property (N3) is a kind of generalization, or fuzzification of the triangular inequality, it is natural to expect, that nearnesses, as a special type of real functions of two real variables are in some sense continuous, or coherent w.r.t. the standard metric in E_2.

On the other hand, $\mathcal{T}_M$-transitivity is a little questionable fuzzification of the ordinary transitivity of crisp equivalences and in fact, it has none relationship to the metric in given universe. Contrary, as the following proposition states, each similarity relation, except for trivial cases, has infinitely many points of discontinuity w.r.t. the standard metric in E_2.

Proposition 4.

Let S be a similarity, defined by means of a fuzzy function h, according the formula (1) in Proposition 1. Let x_0 be a real number, such that the function h is continuous at x_0 and $h(x_0) < 1$. Then S is discontinuous at the point $[x_0, y_0]$.

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers, both converging to x_0. It means, that in E_2 the sequence $\{[x_n, y_n]\}$ is converging to the point $[x_0, y_0]$. Further, continuity of the function h implies, that

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} h(y_n) = h(x_0).$$

But then

$$\lim_{n\to\infty} S(x_n, y_n) = \lim_{n\to\infty} \min(h(x_n), h(y_n)) = \min(h(x_0), h(x_0)) = h(x_0) < 1.$$

Therefore

$$\lim_{n\to\infty} S(x_n, y_n) \neq S(x_0, x_0),$$

S is not continuous at the point $[x_0, x_0]$.

Let us consider a similarity, constructed by means of the simplest fuzzy set, a triangular fuzzy number. That is a continuous fuzzy set, so according the foregoing proposition, all points but one lying on the line $y = x$, are points of discontinuity of the corresponding similarity.

Example 2.

Let h be a triangular fuzzy number äpproximately zero":

$$h(x) = \begin{cases} x+1, & \text{if } x \in (-1, 0\rangle, \\ 1-x, & \text{if } x \in (0, 1), \\ 0, & \text{otherwise} \end{cases}$$

Then the similarity, defined by means of this fuzzy number, according the formula (1), looks like this:

$$S(x,y)=\min(h(x),h(y))=\begin{cases} x+1, & \text{if } -1<x\leq -|y|,\ x\neq y,\\ 1-x, & \text{if } |y|\leq x<1,\ x\neq y,\\ y+1, & \text{if } -1<y\leq -|x|,\ x\neq y,\\ 1-y, & \text{if } |x|\leq y<1,\ x\neq y,\\ 1, & \text{if } x=y,\\ 0, & \text{otherwise}\end{cases}$$

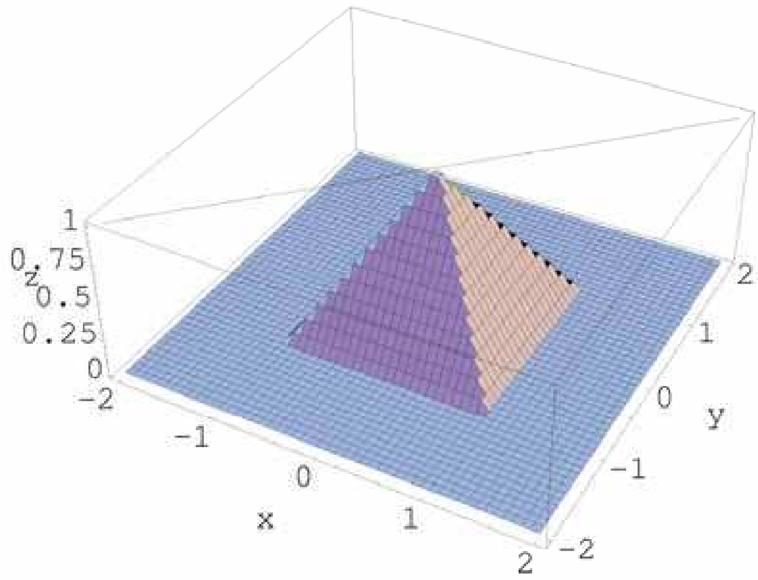

Figure 2.

5. CONCLUSION

Also the last example confirms that the definition of similarities, representing in fact the first attempt to fuzzify the notion of equivalence of two elements, in a universe, equipped with a metric, has only a very weak or even no connection with the distance of these elements in the universe.

Really, in the case of the universe of all real numbers and a continuous "similarity generating"fuzzy set h, excepting the trivial crisp case $h(x) = 1$ for each element x from the universe, corresponding similarity cannot be a continuous function w.r.t. the standard metric.

It seems to be a little strange and it can be interpreted in such a way, that the idea of similarity of two real numbers, according this definition, has a meaning different from the usual apprehension of proximity, or nearness.

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