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Generalized contact transformations

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Abstract

Point transformations can be applied to any system of differential equations. On the other hand, classical contact transformations concern only the particular case of one dependent variable. The article is devoted to certain new contact–like transformations which can be applied to several dependent variables. They are again defined in terms of appropriate contact conditions which are expressed by the intersection of wave fronts. The "reverse waves" with the variables and the parameters interchanged provide the inverse mapping. As a result, the ancient dream of Lie becomes true: the existence of the higher–order contact transformations is established.

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All investigations of equivalences and symmetries of differential equations are as a rule carried out in a finite order jet space: a certain finite order jet space is transformed into itself. In fact this is a short cut of the problem. There exist invertible mappings which do not preseve the hierarchy of finite order jet spaces and they are completely neglected in this approach. We will mention several instructive examples. They are analogous to the classical contact transformations, for instance a beautiful geometrical interpretation of the results in terms of wave fronts is possible and the inversion can be easily found by using the "reverse waves".

1. ONE INDEPENDENT VARIABLE

We are interested in (local) automorphisms of the family of C^∞–smooth curves in $\mathbb{R}^{m+1}$ $(m \geq 1)$. In explicit terms, we investigate (local) C^∞–smooth invertible transformations given by the equations

$$\bar{x} = F(x, w_0^1, \ldots, w_0^m, \ldots, w_S^1, \ldots, w_S^m),$$

$$\bar{w}_0^j = F_0^j(x, w_0^1, \ldots, w_0^m, \ldots, w_S^1, \ldots, w_S^m) \quad (1)$$

$$\left(w_r^i = \mathrm{d}^r w^i / \mathrm{d}x^r, \bar{w}_s^j = \mathrm{d}^s \bar{w}^j / \mathrm{d}\bar{x}^s\right)$$

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between two families

$$w^i = w^i(x)\ (i = 1, \dots, m), \quad \bar{w}^j = \bar{w}^j(\bar{x})\ (j = 1, \dots, m) \tag{2}$$

of C^∞–smooth curves. (The definition domains are not specified. It would be better to speak of the germs of C^∞–smooth curves.) We suppose

$$XF \neq 0 \qquad (X = \frac{\partial}{\partial x} + \sum w^j_{s+1} \frac{\partial}{\partial w^j_s}) \tag{3}$$

in order to ensure the change of the independent variable. Then formulae (1) can be completed by the recurrences

$$\bar{w}^j_{s+1} = F^j_{s+1}(x, w^1_0, \dots, w^m_0, \dots, w^1_{S+s+1}, \dots, w^m_{S+s+1}) = \frac{XF^j_s}{XF} \tag{4}$$

for the higher–order derivatives. Denoting by

$$\omega^i_r = \mathrm{d}w^i_r - w^i_{r+1}\mathrm{d}x,\ \bar{\omega}^j_s = \mathrm{d}\bar{w}^j_s - \bar{w}^j_{s+1}\mathrm{d}\bar{x}$$

the contact forms, equation (4) is equivalent to the congruence

$$\bar{\omega}^j_s = \mathrm{d}F^j_s - F^j_{s+1}\mathrm{d}F \cong 0 \qquad (\text{mod all forms } \omega^i_r), \tag{5}$$

by easy verification.

The inversion of (1, 4) is expressed by certain formulae

$$\begin{aligned} x &= G(\bar{x}, \bar{w}^1_0, \dots, \bar{w}^m_0, \dots, \bar{w}^1_R, \dots, \bar{w}^m_R), \\ w^i_r &= G^i_r(\bar{x}, \bar{w}^1_0, \dots, \bar{w}^m_0, \dots, \bar{w}^1_{R+r}, \dots, \bar{w}^m_{R+r}) \end{aligned} \tag{6}$$

where the *recurrences*

$$G^i_{r+1} = \frac{\bar{X}G^i_r}{\bar{X}G} \qquad (\bar{X} = \frac{\partial}{\partial \bar{x}} + \sum \bar{w}^j_{s+1} \frac{\partial}{\partial \bar{w}^j_s}) \tag{7}$$

are automatically satisfied. (Hint: Formulae (6) provide a certain development

$$\omega^i_r = \mathrm{d}G^i_r - G^i_{r+1}\mathrm{d}G = g\mathrm{d}\bar{x} + \sum g^j_s \bar{\omega}^j_s \qquad (g^j_s = \frac{\partial g}{\partial w^j_s})$$

for every form ω^i_r. Here $\mathrm{d}\bar{x} = XF\mathrm{d}x + \sum \partial F/\partial w^j_s \cdot \omega^j_s$ may be substituted and assuming (4), hence (5), we obtain $gXF = 0$ hence $g = 0$ identically. It follows that $\omega^i_r \cong 0$ (mod all forms $\bar{\omega}^j_s$) which is equivalent to (7).)

In general, the inversion (6) of the infinite system (1, 4) cannot be easily determined and the invertibility criterion is represented by a lengthy algorithm. However, *we will find a large class of transformations* (1, 4) *where the inversions* (6) *follow by a formal interchange of variables with bars and without bars in the definition formulae.*

2. A TECHNICAL NOTE

We will use two infinite series of (infinite order jet) coordinates

$$x, w^i_r\ (i = 1, \dots, m;\ r = 0, 1, \dots), \quad \bar{x}, \bar{w}^j_s\ (j = 1, \dots, m;\ s = 0, 1, \dots) \tag{8}$$

and consider C^∞–smooth functions

$$f(x,\ldots,w^i_r,\ldots,\bar{x},\ldots,\bar{w}^j_s,\ldots)=0$$

of a *finite* number of variables (8). Denoting

$$Df=\frac{\partial f}{\partial x}+\sum w^i_{r+1}\frac{\partial f}{\partial w^i_r},\ \bar{D}f=\frac{\partial f}{\partial \bar{x}}+\sum \bar{w}^j_{s+1}\frac{\partial f}{\partial \bar{w}^j_s}$$

then the identities

$$\mathrm{d}f=Df\mathrm{d}x+\sum\frac{\partial f}{\partial w^i_r}\omega^i_r+\bar{D}f\mathrm{d}\bar{x}+\sum\frac{\partial f}{\partial \bar{w}^j_s}\bar{\omega}^j_s \tag{9}$$

hold true. Here (8) are regarded as *independent variables.*

On the contrary, let us suppose the interrelations (1, 4) or, equivalently, interrelations (6) among variables (8). Then we shall use another notation, namely (3, 4, 7). In more detail, assuming (1, 4) then f may be regarded as a function of variables (8_1),

$$f(x,\ldots,w^i_r,\ldots,F,\ldots,F^j_s,\ldots)=\mathcal{F}(x,\ldots,w^i_r,\ldots)$$

and we denote $Xf=D\mathcal{F}$. (Roughly: Applying the operator X means substitution (1, 4) followed by the operator D. Alternatively, in the reverse order:

X means $D+DF\cdot\bar{D}$ followed by substitution (1,4).

This can be easily proved for every coordinate function (8).) Equivalently, assuming (6), then f may be regarded as a function of variables (8_2),

$$f(G,\ldots,G^i_r,\ldots,\bar{x},\ldots,\bar{w}^j_s,\ldots)=\bar{\mathcal{F}}(\bar{x},\ldots,\bar{w}^j_s,\ldots)$$

and we denote $\bar{X}f=\bar{D}\bar{\mathcal{F}}$. If f does not depend on variables $\bar{x},\bar{w}^j_s$ (on x,w^i_r) then formally $Df=Xf$ ($\bar{D}f=\bar{X}f$, respectively) can be identified.

Let Ω ($\bar{\Omega}$) denotes the module of all finite sums $\sum a^i_r\omega^i_r$ ($\sum a^j_s\bar{\omega}^j_s$, respectively) of contact forms where the coefficients are arbitrary C^∞–smooth functions of variables (8). Then congruence (5) reads $\bar{\omega}^j_s\in\Omega$. Hence $\bar{\Omega}\subset\Omega$ in virtue of the prolongation formulae (4). Analogously $\Omega\subset\bar{\Omega}$ in accordance with (7) whence $\Omega=\bar{\Omega}$ is a consequence of the invertibility.

Assuming (1), then clearly $\mathrm{d}\bar{x}=\mathrm{d}F\cong XF\mathrm{d}x$ (mod Ω) as follows from (9) in the particular case $f=F$. Therefore $XF\neq 0$ if and only if $\mathrm{d}\bar{x}=\mathrm{d}F\notin\Omega$.

We turn to proper topic.

3. CONTACT TRANSFORMATIONS IN THE PLANE

The classical contact transformations arise if $m=S=R=1$ is supposed in Section 1. (In more detail: the assumption $m=1$ and the invertibility $\bar{\Omega}=\Omega$ together imply $S=R=1$ in equations (1, 6) with $r=0$ and moreover $S=R=0$ in equations (4, 6) with $s\geq 0, r>0$. We shall not need this fact, therefore a legthy direct proof may be omitted here.) Then we simplify the notation into

$$y_r=w^1_r,\bar{y}_s=\bar{w}^1_s,\eta_r=\omega^1_r,\bar{\eta}_r=\bar{\omega}^1_s,$$

by changing the letters and omitting the superscripts. Let us recall some (well–known) results with the use a somewhat strange method. First and foremost, our aim is to obtain the explicit formulae both for the contact transformation and for its inversion by a slight adaptation of classical reasonings.

Given a C^∞–smooth function f of four variables, we introduce the equations

$$f(x,y_0,\bar{x},\bar{y}_0)=0,\quad Df(x,y_0,y_1,\bar{x},\bar{y}_0)=0. \tag{10}$$

Assuming

$$\det\begin{pmatrix} f_{\bar{x}} & f_{\bar{y}_0} \\ (Df)_{\bar{x}} & (Df)_{\bar{y}_0}\end{pmatrix}\neq 0, \tag{11}$$

the implicit function theorem can be applied to (10) *and we obtain a* (*local*) *solution*

$$\bar{x}=F(x,y_0,y_1),\quad \bar{y}_0=F_0(x,y_0,y_1). \tag{12}$$

(More precisely: if the system (10) has at least one solution then the implicit function theorem can be applied and the definition domains are large enough. We shall omit analogous remark in all examples to follow, for brevity.)

Assuming moreover $\partial f/\partial y_0\neq 0$, *it follows that* $d\bar{x}=dF\notin\Omega$ *hence* $XF\neq 0$. (Hint. Much weaker condition is in fact sufficient. Calculate $\mathrm{d}\bar{x},\bar{\eta}_0$ by using the linear equations

$$0=\mathrm{d}f=Df\mathrm{d}x+f_{y_0}\eta_0+\bar{D}f\mathrm{d}\bar{x}+f_{\bar{y}_0}\bar{\eta}_0,$$

$$0=\mathrm{d}Df=D^2f\mathrm{d}x+(Df)_{y_0}\eta_0+(Df)_{y_1}\eta_1+\bar{D}Df\mathrm{d}\bar{x}+(Df)_{\bar{y}_0}\bar{\eta}_0$$

with the nonvanishing determinant

$$\Delta=\det\begin{pmatrix}\bar{D}f & f_{\bar{y}_0}\\ \bar{D}Df & (Df)_{\bar{y}_0}\end{pmatrix}=\det\begin{pmatrix} f_{\bar{x}} & f_{\bar{y}_0}\\ (Df)_{\bar{x}} & (Df)_{\bar{y}_0}\end{pmatrix}\neq 0.$$

We obtain

$$\mathrm{d}\bar{x}\cong-\frac{1}{\Delta}\det\begin{pmatrix} Df & f_{\bar{y}_0}\\ D^2f & (Df)_{\bar{y}_0}\end{pmatrix}\mathrm{d}x=-\frac{1}{\Delta}\det\begin{pmatrix}\cdots+y_1f_{y_0} & f_{\bar{y}_0}\\ \cdots+y_2f_{y_0} & (Df)_{\bar{y}_0}\end{pmatrix}\mathrm{d}x$$

modulo Ω. Therefore inequalities (11) and $\partial f/\partial y_0\neq 0$ ensure $\mathrm{d}\bar{x}\notin\Omega$. We shall omit the analogous calculation in all examples to follow, for brevity.) Then the prolongation

$$\bar{y}_s=F_s(x,y_0,\dots,y_{s+1}),\quad F_{s+1}=\frac{XF_s}{XF} \tag{13}$$

makes good sense. In reality *variable* y_{s+1} *does not occur here if* $s\geq 1$ (*so the hierarchy of finite order jet spaces is preserved and we indeed have the classical contact transformation*). This may be proved as follows.

Employing (10) and (9), we have

$$0=\mathrm{d}f=f_{y_0}\eta_0+\bar{D}f\mathrm{d}\bar{x}+f_{\bar{y}_0}\bar{\eta}_0.$$

Assuming moreover (13), then congruence (5) ensures the inclusion $\bar{\eta}_0\in\Omega$, hence $\bar{D}f\mathrm{d}\bar{x}=\bar{D}f\mathrm{d}F\in\Omega$. However $\mathrm{d}F\notin\Omega$ implies the identity

$$0=\bar{D}f=f_{\bar{x}}+\bar{y}_1f_{\bar{y}_0} \tag{14}$$

which alternatively reads as better prolongation formula

$$\bar{y}_1=-\frac{\partial f/\partial\bar{x}}{\partial f/\partial\bar{y}_0}=F_1(x,y_0,y_1).$$

Variables (12) are substituted into the middle term here and $\partial f/\partial \bar{y}_0 \neq 0$ is supposed. From this particular result, the desired formulae $\bar{y}_s = F_s(x, y_0, \dots, y_s)$ easily follow for every $s \geq 1$ with the use of recurrences (4).

4. SUMMARY

PROPOSITION 4.1. *Assuming* (11) *and* $\partial f/\partial y_0 \cdot \partial f/\partial \bar{y}_0 \neq 0$, *equations* (10) *determine the transformation* (12, 13) *and then the prolongation formula* (14) *holds true. It follows that the role of* x, y_r *and* $\bar{x}, \bar{y}_s$ *can be interchanged. Assuming moreover*

$$\det \begin{pmatrix} f_x & f_{y_0} \\ (\bar{D}f)_x & (\bar{D}f)_{y_0} \end{pmatrix} \neq 0,$$

equations $f = 0, \bar{D}f = 0$ *provide the inverse transformation*

$$x = G(\bar{x}, \bar{y}_0, \bar{y}_1), \quad y_0 = G_0(\bar{x}, \bar{y}_0, \bar{y}_1)$$

and then (10_2) *may be interpreted as a mere prolongation formula.*

5. COMPLEMENTS

We state a few results without proofs in order to accentuate the distinction between operators D and X. Operators X are not frequently appearing in this article, however, they are useful in many respects.

Assuming interrelations (1, 4) among variables $x, y_r^i, \bar{x}, \bar{y}_s^j$, many identities can be derived by using the Lie derivative $\mathcal{L}_X$. For instance, (14) holds true whence also

$$\mathcal{L}_X \bar{D}f = X\bar{D}f = 0, \mathcal{L}_X^2 \bar{D}f = X^2\bar{D}f = 0, \dots$$

which provides the alternative prolongation recurrences

$$\begin{aligned} &0 = (D + XF \cdot \bar{D})\bar{D}f = D\bar{D}f + XF \cdot \bar{D}^2 f = \dots + XF \cdot \tfrac{\partial f}{\partial \bar{y}_0}\bar{y}_2, \\ &0 = (D + XF \cdot \bar{D})^2\bar{D}f = D^2\bar{D}f + \dots + XF \cdot \tfrac{\partial f}{\partial \bar{y}_0}\bar{y}_3, \\ &\dots \end{aligned}$$

in terms of functions F and f. Analogously, identities $\mathrm{d}f = 0, \mathcal{L}_X \mathrm{d}f = 0, \dots$ provide important recurrences for the forms $\bar{\omega}_s^j$ if the equations $\mathcal{L}_X \omega_r^i = \omega_{r+1}^i$, $\mathcal{L}_X \bar{\omega}_s^j = XF \cdot \bar{\omega}_{s+1}^j$ are applied.

6. DIGRESSION: THE COMMON APPROACH

For the convenience of reader, we compare the method of Section 3 with the common traditional approach in order to clarify some subtle distinctions. The distinctions are important, they enable us to discover the subsequent generalizations of the classical concepts.

In the *common approach*, a contact transformation

$$\bar{x} = F(x, y, y'), \ \bar{y} = G(x, y, y'), \ \bar{y}' = H(x, y, y') \quad (y' = \frac{\mathrm{d}y}{\mathrm{d}x}, \ \bar{y}' = \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}})$$

of curves in $\mathbb{R}^2$ is defined by the identity

$$\lambda(\mathrm{d}\bar{y} - \bar{y}'\mathrm{d}\bar{x}) = \lambda(\mathrm{d}G - H\mathrm{d}F) = \mu(\mathrm{d}y - y'\mathrm{d}x) \tag{15}$$

where $\lambda\mu \neq 0$. One can then obtain the unpleasant system

$$G_x - HF_x + y'(G_y - HF_y) = 0, \quad G_{y'} = HF_{y'}$$

for the functions F, G, H. Fortunately, the following easier method was invented. Assume a functional dependence $f(x, y, \bar{x}, \bar{y}) = f(x, y, F, G) = 0$ and identify the equation

$$\mathrm{d}f = f_x \mathrm{d}x + f_y \mathrm{d}y + f_{\bar{x}} \mathrm{d}\bar{x} + f_{\bar{y}} \mathrm{d}\bar{y} = 0$$

with identity (15). It follows that $\lambda = \partial f / \partial \bar{y}$, $\mu = -\partial f / \partial y$ whence (15) reads

$$f_{\bar{y}}(\mathrm{d}\bar{y} - \bar{y}'\mathrm{d}\bar{x}) + f_y(\mathrm{d}y - y'\mathrm{d}x) = 0$$

and moreover

$$f_x + y' f_y = 0, \quad f_{\bar{x}} + \bar{y}' f_{\bar{y}} = 0. \tag{16}$$

So we have obtained all crucial formulae of Section 3, in particular formulae (10, 14), however, in quite other arrangement.

In more detail, the distinctions are as follows. In the common method, the *invariance property* (15) of the Pfaffian equation $\mathrm{d}y - y'\mathrm{d}x = 0$ *is postulated* in order to obtain equations (16). In Section 3, *we have postulated* $f = Df = 0$ in order to directly obtain the prolongation $\bar{D}f = 0$. This is a slight but important rearrangement: in all examples of the generalized contact transformations to follow, there does not exist a sufficiently large supply of invariant Pfaffian equations since the hierarchy of the finite order jet spaces is destroyed. The common approach fails, however, the method of Section 3 based only on the inclusion $\bar{\Omega} \subset \Omega$ can be closely simulated.

7. THREE–DIMENSIONAL CASE

We will determine the *generalized contact transformations of curves in* $\mathbb{R}^3$, hence $m = 2$ is supposed in formulae of Section 1. Let us simplify the notation as

$$y_r = w_r^1, z_r = w_r^2, \eta_r = \omega_r^1, \zeta_r = \omega_r^2$$

and analogously with bars. Closely following the method of Section 3, completely new invertible *second–order contact transformations of curves* in $\mathbb{R}^3$ will be obtained.

Given a function f of six variables, we introduce the equations

$$\begin{aligned} f(x, y_0, z_0, \bar{x}, \bar{y}_0, \bar{z}_0) &= 0, \\ Df(x, y_0, z_0, y_1, z_1, \bar{x}, \bar{y}_0, \bar{z}_0) &= 0, \\ D^2 f(x, y_0, z_0, y_1, z_1, y_2, z_2, \bar{x}, \bar{y}_0, \bar{z}_0) &= 0. \end{aligned} \tag{17}$$

Assuming

$$\det \begin{pmatrix} f_{\bar{x}} & f_{\bar{y}_0} & f_{\bar{z}_0} \\ (Df)_{\bar{x}} & (Df)_{\bar{y}_0} & (Df)_{\bar{z}_0} \\ (D^2 f)_{\bar{x}} & (D^2 f)_{\bar{y}_0} & (D^2 f)_{\bar{z}_0} \end{pmatrix} \neq 0, \tag{18}$$

the implicit function proposition ensures the solution

$$\begin{aligned} \bar{x} &= F(x, y_0, z_0, y_1, z_1, y_2, z_2), \\ \bar{y}_0 &= F_0^1(x, y_0, z_0, y_1, z_1, y_2, z_2), \\ \bar{z}_0 &= F_0^2(x, y_0, z_0, y_1, z_1, y_2, z_2). \end{aligned} \tag{19}$$

Assuming moreover either $\partial f/\partial y_0 \neq 0$ or $\partial f/\partial z_0 \neq 0$, it may be proved that $\mathrm{d}\bar{x} = \mathrm{d}F \notin \Omega$ hence $XF \neq 0$. The prolongation

$$\bar{y}_s = F^1_s(x, y_0, z_0, \ldots, y_{s+2}, z_{s+2}), \bar{z}_s = F^2_s(x, y_0, z_0, \ldots, y_{s+2}, z_{s+2}) \qquad (20)$$

arising from formulae (4) makes good sense and we have $\bar{\Omega} \subset \Omega$. In reality the variables y_{s+2}, z_{s+2} do not occur here and the proof is as follows.

Assuming (17, 20), then the identities $0 = \mathrm{d}f, 0 = \mathrm{d}Df$ imply the inclusions $\bar{D}f\mathrm{d}\bar{x} \in \Omega, \bar{D}Df\mathrm{d}\bar{x} \in \Omega$ quite analogously as in Section 3. So we have

$$0 = \bar{D}f = \frac{\partial f}{\partial \bar{x}} + \bar{y}_1 \frac{\partial f}{\partial \bar{y}_0} + \bar{z}_1 \frac{\partial f}{\partial \bar{z}_0}, \quad 0 = \bar{D}Df = \frac{\partial Df}{\partial \bar{x}} + \bar{y}_1 \frac{\partial Df}{\partial \bar{y}_0} + \bar{z}_1 \frac{\partial Df}{\partial \bar{z}_0}$$

and if the inequality

$$\det \begin{pmatrix} f_{\bar{y}_0} & f_{\bar{z}_0} \\ (Df)_{\bar{y}_0} & (Df)_{\bar{z}_0} \end{pmatrix} \neq 0 \qquad (21)$$

holds true, then better prolongation formulae

$$\bar{y}_1 = F^1_1(x, y_0, z_0, y_1, z_1, y_2, z_2), \quad \bar{z}_1 = F^2_1(x, y_0, z_0, y_1, z_1, y_2, z_2)$$

follow. Moreover the identity $\bar{D}f = 0$ implies $\mathrm{d}\bar{D}f = 0$ whence $\bar{D}^2 f\mathrm{d}\bar{x} \in \Omega$ and therefore $\bar{D}^2 f = 0$ by analogous arguments as above.

It should be noted that variables y_{s+1}, z_{s+1} cannot be deleted from equations (20). It follows that the finite order jet spaces are destroyed, however, regardless of this poor prolongation result, *we have moreover proved the identities* $\bar{D}f = 0$, $\bar{D}^2 f = 0$ *as a consequence of* (17).

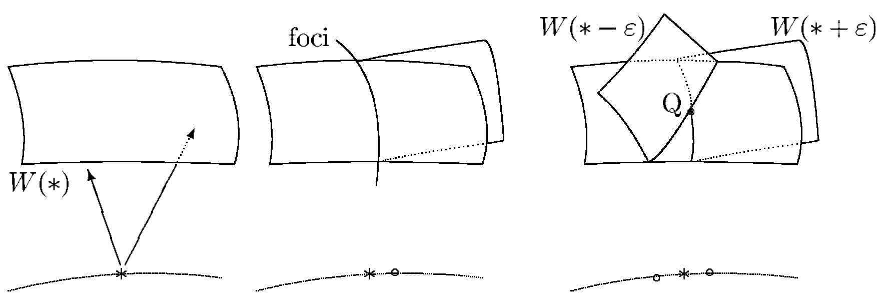
8. SUMMARY

PROPOSITION 8.1. *Assuming* (18) *and either* $\partial f/\partial y_0 \neq 0$ *or* $\partial f/\partial z_0 \neq 0$, *equations* (17) *determine the generalized contact transformation* (19, 20) *of curves in* $\mathbb{E}^3$ *satisfying moreover the identities* $\bar{D}f = \bar{D}^2 f = 0$. *Therefore the simple interchange of variables with bars and without bars provides the inverse transformation* (*defined by* $f = \bar{D}f = \bar{D}^2 f = 0$) *if either* $\partial f/\partial \bar{y}_0 \neq 0$ *or* $\partial f/\partial \bar{z}_0 \neq 0$ *and a certain Jacobian* (*a counterpart to* (18)) *is nonvanishing.*

A beautiful geometrical sense of the result is worth mentioning. Equation

$$f(*, \bar{x}, \bar{y}_0, \bar{z}_0) = 0$$

represents a wave $W(*)$ associated to the point $* = (x, y_0, z_0) \in \mathbb{R}^3$. Then the equation $Df = 0$ provides the "intersection" with the infinitesimally close wave $W(* + \varepsilon)$, the line of foci, and eventually the equation $D^2 f = 0$ determines the "intersection" of three infinitesimally close waves, the focus $Q = Q(*)$.



If the point $*$ moves along a curve, then the multiple focus $Q(*)$ runs over the transformed curve and the "reverse wave" with the variables interchanged conversely determines the point $*$ in terms of $Q(*)$.

Analogous interpretations of all contact–like transformations to follow are possible, as well.

9. TWO FURTHER GENERALIZATIONS

(ι) *More variables.* Given a C^∞–smooth function f of $2(m+1)$ variables ($m \geq 3$), we introduce the equations

$$\begin{aligned} f(x, w_0^1, \ldots, w_0^m, \bar{x}, \bar{w}_0^1, \ldots, \bar{w}_0^m) &= 0, \\ Df(x, w_0^1, \ldots, w_0^m, w_1^1, \ldots, w_1^m, \bar{x}, \bar{w}_0^1, \ldots, \bar{w}_0^m) &= 0, \\ &\ldots \\ D^m f(x, w_0^1, \ldots, w_0^m, \ldots, w_m^1, \ldots, w_m^m, \bar{x}, \bar{w}_0^1, \ldots, \bar{w}_0^m) &= 0. \end{aligned}$$

They implicitly determine the (local) transformation

$$\bar{x} = F(x, w_0^1, \ldots, w_m^m), \quad \bar{w}_0^j = F_0^j(x, w_0^1, \ldots, w_m^m) \quad (j = 1, \ldots, m)$$

if the Jacobian (not written here) is nonvanishing. Assuming moreover either function $\partial f/\partial w_0^i$ $(i = 1, \ldots, m)$ nonvanishing, then $XF \neq 0$ and we may introduce the prolongation (4). However, also the identities

$$\bar{D}f = \bar{D}Df = \ldots, \bar{D}D^{m-1}f = \bar{D}^2 f = \bar{D}^2 Df = \cdots = \bar{D}^m f = 0$$

are valid after this prolongation. It follows that *the inverse transformation is determined by the implicit system $f = \bar{D}f = \ldots = \bar{D}^m f = 0$ if the relevant Jacobian is nonvanishing.*

($\iota\iota$) *Lower dimensional wave.* Given functions f and g, each dependent on eight variables, we introduce the equations

$$\begin{aligned} f(x, w_0^1, w_0^2, w_0^3, \bar{x}, \bar{w}_0^1, \bar{w}_0^2, \bar{w}_0^3) &= 0, \quad g(x, w_0^1, w_0^2, w_0^3, \bar{x}, \bar{w}_0^1, \bar{w}_0^2, \bar{w}_0^3) = 0, \\ Df(x, w_0^1, \ldots, w_1^3, \bar{x}, \bar{w}_0^1, \bar{w}_0^2, \bar{w}_0^3) &= 0, \quad Dg(x, w_0^1, \ldots, w_1^3, \bar{x}, \bar{w}_0^1, \bar{w}_0^2, \bar{w}_0^3) = 0. \end{aligned}$$

If the implicit function proposition can be applied, we obtain the (local) solution

$$\bar{x} = F(x, w_0^1, \ldots, w_1^3), \quad \bar{w}_0^j = F_0^j(x, w_0^1, \ldots, w_1^3) \quad (j = 1, 2, 3).$$

Assuming moreover $XF \neq 0$, prolongation (4) makes good sense whence $\bar{\Omega} \subset \Omega$. (One can check that the condition

$$\operatorname{rank} \begin{pmatrix} \partial f/\partial w_0^1 & \partial f/\partial w_0^2 & \partial f/\partial w_0^3 \\ \partial g/\partial w_0^1 & \partial g/\partial w_0^2 & \partial g/\partial w_0^3 \end{pmatrix} = 2$$

is sufficient.) Then, employing (9) applied to the identities $\mathrm{d}f = 0$, $\mathrm{d}g = 0$, it follows that the equations $\bar{D}f = 0$, $\bar{D}g = 0$ are satisfied, too. *So we obtain the inverse transformation*

$$x = G(\bar{x}, \bar{w}_0^1, \ldots, \bar{w}_1^3), \quad w_0^i = G_0^i(\bar{x}, \bar{w}_0^1, \ldots, \bar{w}_1^3) \quad (i = 1, 2, 3)$$

implicitly defined by the equations $f = 0, g = 0, \bar{D}f = 0, \bar{D}g = 0$ *if the relevant Jacobian is nonvanishing.*

Unlike all previous examples, *two* equations $\bar{D}f = 0, \bar{D}g = 0$ *do not* completely determine the *three* prolongation functions F_1^1, F_1^2, F_1^3 here but it does not matter as the invertibility is concerned.

10. TWO INDEPENDENT VARIABLES

The above reasonings can be carried over the multidimensional case of several independent variables, however, then the technical machinery obscures the simple idea. Therefore, for the convenience of exposition, we restrict ourselves to the quite instructive case of two independent variables.

We are interested in the invertible transformations, determined by the equations

$$\begin{aligned} \bar{x}^k &= F^k(x^1, x^2, \ldots, w^i_{rs}, \ldots) \qquad (k = 1, 2), \\ \bar{w}^j_{00} &= F^j_{00}(x^1, x^2, \ldots, w^i_{rs}, \ldots) \ (j = 1, \ldots, m), \end{aligned} \tag{22}$$

between two families

$$w^i = w^i(x^1, x^2) \ \ (i = 1, \ldots, m), \quad \bar{w}^j = \bar{w}^j(\bar{x}^1, \bar{x}^2) \ \ (j = 1, \ldots, m)$$

of C^∞–smooth (germs of) two–dimensional surfaces. In order to delete some indices, we use alternative notation $x = x^1, y = x^2$ of independent variables and of derivatives $w^i_{rs} = \partial^{r+s} w^i / \partial x^r \partial y^s$ here (and analogously for the variables with bars). Denoting moreover

$$X = X_1 = \frac{\partial}{\partial x} + \sum w^i_{r+1,s} \frac{\partial}{\partial w^i_{rs}}, \quad Y = X_2 = \frac{\partial}{\partial y} + \sum w^i_{r,s+1} \frac{\partial}{\partial w^i_{rs}}$$

and assuming

$$\det(X_k F^l) = XF^1 \cdot YF^2 - XF^2 \cdot YF^1 \neq 0, \tag{23}$$

the prolongation

$$\bar{w}^j_{uv} = F^j_{uv}(x, y, \ldots, w^i_{rs}, \ldots) \quad (j = 1, \ldots, m; \ u, v = 0, 1, \ldots) \tag{24}$$

is determined by the implicit recurrences

$$XF^j_{uv} = F^j_{u+1,v} XF^1 + F^j_{u,v+1} XF^2, \quad YF^j_{uv} = F^j_{u+1,v} YF^1 + F^j_{u,v+1} YF^2. \tag{25}$$

They are equivalent to the inclusion

$$\bar{\omega}^j_{uv} = \mathrm{d}F^j_{uv} - F^j_{u+1,v} \mathrm{d}F^1 - F^j_{u,v+1} \mathrm{d}F^2 \in \Omega, \tag{26}$$

where Ω is the module generated by the contact forms

$$\omega^i_{rs} = \mathrm{d}w^i_{rs} - w^i_{r+1,s} \mathrm{d}x - w^i_{r,s+1} \mathrm{d}y.$$

Moreover inequality (23) means that differentials $\mathrm{d}x = \mathrm{d}F^1, \mathrm{d}y = \mathrm{d}F^2$ are linearly independent modulo Ω. The inversion of equations (22, 24) is given by certain formulae

$$x^k = G^k(\bar{x}^1, \bar{x}^2, \ldots, \bar{w}^j_{uv}, \ldots), \quad w^i_{rs} = G^i_{rs}(\bar{x}^1, \bar{x}^2, \ldots, \bar{w}^j_{uv}, \ldots) \tag{27}$$

where the obvious recurrences are automatically satisfied.

11. TWO EXAMPLES OF NEW INVERTIBLE TRANSFORMATIONS

The classical contact transformations appear if m=1 and functions F^1, F^2, F^1_{00}, F^1_{10}, F^1_{01} are independent of variables w^1_{rs} $(r+s>1)$. We shall not discuss this well–known case here and pass to the *new higher–order generalizations.*

(ι) *Several dependent variables.* Given a C^∞–smooth function f of twelve variables, let us introduce the equations

$$\begin{gathered} f(x,y,w^1_{00},\ldots,w^4_{00},\bar{x},\bar{y},\bar{w}^1_{00},\ldots,\bar{w}^4_{00})=0,\\ D_1f=0,\ D_2f=0,\ D_1^2f=0,\ D_1D_2f=0,\ D_2^2f=0. \end{gathered} \tag{28}$$

Here D_1,D_2 are operators *formally* defined as X_1,X_2 above, hence

$$\mathrm{d}f=D_1f\mathrm{d}x+D_2f\mathrm{d}y+\sum\frac{\partial f}{\partial w^i_{rs}}\omega^i_{rs}+\bar{D}_1f\mathrm{d}\bar{x}+\bar{D}_2f\mathrm{d}\bar{y}+\sum\frac{\partial f}{\partial\bar{w}^j_{uv}}\bar{\omega}^j_{uv}, \tag{29}$$

analogously to (9), with obvious operators $\bar{D}_1,\bar{D}_2$. (Recall Section 2: operators $D_k,\bar{D}_k$ are applied if the total family of dashed and undashed letters denotes *independent variables.* Otherwise, if certain interrelations as (22, 23, 26) are assumed, we use operators X_k and $\bar{X}_k$.) Assuming

$$\det\begin{pmatrix} f_{\bar{x}} & f_{\bar{y}} & f_{\bar{w}^1_{00}} & \cdots & f_{\bar{w}^4_{00}}\\ (D_1f)_{\bar{x}} & (D_1f)_{\bar{y}} & (D_1f)_{\bar{w}^1_{00}} & \cdots & (D_1f)_{\bar{w}^4_{00}}\\ (D_2f)_{\bar{x}} & (D_2f)_{\bar{y}} & (D_2f)_{\bar{w}^1_{00}} & \cdots & (D_2f)_{\bar{w}^4_{00}}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ (D_2^2f)_{\bar{x}} & (D_2^2f)_{\bar{y}} & (D_2^2f)_{\bar{w}^1_{00}} & \cdots & (D_2^2f)_{\bar{w}^4_{00}} \end{pmatrix}\neq 0, \tag{30}$$

implicit function proposition can be applied and we obtain a (local) solution

$$\begin{aligned} \bar{x}^k&=F^k(x,y,w^1_{00},\ldots,w^1_{00},\ldots,w^1_{02},\ldots,w^4_{02}) && (k=1,2),\\ \bar{w}^j_{00}&=F^j_{00}(x,y,w^1_{00},\ldots,w^1_{00},\ldots,w^1_{02},\ldots,w^4_{02}) && (j=1,\ldots,4). \end{aligned} \tag{31}$$

Assuming moreover either of the values $\partial f/\partial w^i_{00}$ $(i=1,\ldots,4)$ nonvanishing, then differentials $\mathrm{d}\bar{x}=\mathrm{d}F^1$ and $\mathrm{d}\bar{y}=\mathrm{d}F^2$ are linearly independent modulo Ω, therefore (23) holds true and we may introduce the prolongation (24).

Let us turn to the inversion. Assuming (31), then the identities (28) are satisfied. In particular identities $f=D_1f=D_2f=0$ imply

$$\sum\frac{\partial f}{\partial w^i_{00}}\omega^i_{00}+\bar{D}_1f\mathrm{d}\bar{x}+\bar{D}_2f\mathrm{d}\bar{y}+\sum\frac{\partial f}{\partial\bar{w}^j_{00}}\bar{\omega}^j_{00}=0$$

if (29) is employed. However $\bar{\omega}^i_{00}\in\Omega$ and it follows that $\bar{D}_1f=\bar{D}_2f=0$ identically since $\mathrm{d}\bar{x},\mathrm{d}\bar{y}$ are linearly independent modulo Ω.

The same reasonings can be successively applied to the functions D_1f, D_2f, $\bar{D}_1f$, $\bar{D}_2f$ instead of f. It follows that

$$\bar{D}_kD_1f=\bar{D}_kD_2f=\bar{D}_k\bar{D}_1f=\bar{D}_k\bar{D}_2f=0\quad(k=1,2)$$

identically. Therefore *the inversion appears by the interchange of variables and it is determined by the implicit equations*

$$f=\bar{D}_1f=\bar{D}_2f=\bar{D}_1{}^2f=\bar{D}_1\bar{D}_2f=\bar{D}_2{}^2f=0$$

if the relevant Jacobian is nonvanishing.

($\iota\iota$) *Lower dimensional wave.* Given three C^∞–smooth functions f, g, h of ten variables $(\cdot) = (x, y, w^1_{00}, w^2_{00}, w^3_{00}, \bar{x}, \bar{y}, \bar{w}^1_{00}, \bar{w}^2_{00}, \bar{w}^3_{00})$, we introduce the equations

$$\begin{aligned} &f(\cdot) = g(\cdot) = h(\cdot) = 0, \quad D_1 f \cdot D_2 g = D_1 g \cdot D_2 f, \\ &D_1 f \cdot D_2 h = D_1 h \cdot D_2 f, \; D_1 g \cdot D_2 h = D_1 h \cdot D_2 g. \end{aligned} \tag{32}$$

(Assuming moreover $D_1 f \neq 0$ or $D_2 f \neq 0$, the last equation may be omitted.) The implicit function proposition ensures a (local) solution

$$\begin{aligned} \bar{x}^k &= F^k(x, y, w^1_{00}, \dots, w^3_{01}) \qquad (k = 1, 2), \\ \bar{w}^j_{00} &= F^j_{00}(x, y, w^1_{00}, \dots, w^3_{01}) \qquad (j = 1, 2, 3) \end{aligned} \tag{33}$$

if the Jacobian (not written here) is nonvanishing. One can also check certain (rather clumsy) conditions which ensure that differentials $\mathrm{d}\bar{x} = \mathrm{d}F^1$ and $\mathrm{d}\bar{y} = \mathrm{d}F^2$ are linearly independent. Then (23) holds true and the prolongation (24) makes good sense.

Let us turn to the inversion problem. Consider identity (29) together with the analogous development

$$\mathrm{d}g = D_1 g \mathrm{d}x + D_2 g \mathrm{d}y + \sum \frac{\partial g}{\partial w^i_{rs}} \omega^i_{rs} + \bar{D}_1 g \mathrm{d}\bar{x} + \bar{D}_2 g \mathrm{d}\bar{y} + \sum \frac{\partial g}{\partial \bar{w}^i_{rs}} \bar{\omega}^i_{rs} \tag{34}$$

of function g. Assuming moreover (33), then (32) holds true, in particular

$$0 = \mathrm{d}f = \mathrm{d}g, \quad D_1 f \cdot D_2 g = D_1 g \cdot D_2 f.$$

It follows that we have a certain nontrivial relations

$$\lambda D_1 f = \mu D_1 g, \quad \lambda D_2 f = \mu D_2 g$$

and therefore

$$0 = \lambda \mathrm{d}f + \mu \mathrm{d}g \cong \lambda(\bar{D}_1 f \mathrm{d}\bar{x} + \bar{D}_2 f \mathrm{d}\bar{y}) + \mu(\bar{D}_1 g \mathrm{d}\bar{x} + \bar{D}_2 g \mathrm{d}\bar{y})$$

modulo the contact forms. Since the differentials $\mathrm{d}\bar{x}, \mathrm{d}\bar{y}$ are assumed linearly independent and (23) is satisfied, clearly

$$\lambda \bar{D}_1 f = \mu \bar{D}_1 g, \quad \lambda \bar{D}_2 f = \mu \bar{D}_2 g$$

whence $\bar{D}_1 f \cdot \bar{D}_2 g = \bar{D}_1 g \cdot \bar{D}_2 f$. Remaining identities (32) for the operators $\bar{D}_i$ can be derived, as well.

Altogether taken, *inversion* (27) *of the transformation* (33) *is determined by the implicit equations*

$$\begin{aligned} &f(\cdot) = g(\cdot) = h(\cdot) = 0, \quad \bar{D}_1 f \cdot \bar{D}_2 g = \bar{D}_1 g \cdot \bar{D}_2 f, \\ &\bar{D}_1 f \cdot \bar{D}_2 h = \bar{D}_1 h \cdot \bar{D}_2 f, \; \bar{D}_1 g \cdot \bar{D}_2 h = \bar{D}_1 h \cdot \bar{D}_2 g \end{aligned} \tag{35}$$

if the relevant Jacobian is nonvanishing.

12. CONCLUDING COMMENTS

Immense literature on classical contact structures on manifolds need not be referred to. Let us just remind the primary expositions [2; 4; 7] with a nostalgia. On this occasion, we mention other generalized transformations which can be reasonably applied only to *very special classes* of differential equations. The ancient *Laplace substitution* $\bar{w} = w_x + bw$ in the theory of hyperbolic equations $w_{xy} + aw_x + bw_y + cw = M$ [5] serve as a typical example and a prototype of *differential substitutions* $\bar{w} = g(x, w, w_x, \dots, w_{x \dots x})$ in the theory of evolutional and nonlinear hyperbolic equations [10; 11; 12] with one unknown function w of two independent variables. Also the *Lie–Bäcklund correspondences* [1] and *Darboux transformations* [8] with deep applications on solitons [9] are worth mentioning. Although the independent variables are always preserved in these examples, a general theory including all such transformations is still lacking and the invertibility in the common sense fails since such transformations are not applied to the total jet spaces.

We intentionally use only the most elementary tools of algorithmical nature in our article since they provide transparent and explicit results. Alternative coordinate–free exposition would be rather lengthy. (See, e.g., [6] pages 146 -156 devoted to analogous theory of canonical transformations.) Our generalized contact transformations can be easily inverted. This is not the case for the general mappings (1, 4), however, an *universal algorithm for explicit determination* of all such *invertible transformations* is already available [3].

We deal with quite simple examples in order to demonstrate our approach clearly. It is a matter of a routine to analyze more general waves

$$f^k(x, w_0^1, \dots, w_0^m, \bar{x}, \bar{w}_0^1, \dots, \bar{w}_0^{\bar{m}}) = 0 \quad (k = 1, \dots, K;\ 1 \le K \le m-1)$$

of dimension $m - K$ and even the multi–dimensional case. Assuming $m = \bar{m}$, one can obtain analogous results as above. Assuming $m \neq \bar{m}$, then our method provides explicitly solvable differential equations, examples to the *Monge problem* [13].

For instance, the wave $f(x, y_0, \bar{x}, \bar{y}_0, \bar{z}_0) = 0$ leads to the implicit system $f = Df = D^2 f = 0$ with a certain solution

$$\bar{x} = F(x, y_0, y_1, y_2),\ \bar{y}_0 = F_0^1(x, y_0, y_1, y_2),\ \bar{z}_0 = F_0^2(x, y_0, y_1, y_2). \tag{36}$$

Then the inversion

$$x = G(\bar{x}, \bar{y}_0, \bar{y}_1, \bar{z}_0, \bar{z}_1),\ y_0 = G_0^1(\bar{x}, \bar{y}_0, \bar{y}_1, \bar{z}_0, \bar{z}_1)$$

is determined by the implicit system $f = \bar{D} f = 0$ with the compatibility condition

$$\bar{D}^2 f(x, y_0, \bar{x}, \bar{y}_0, \bar{y}_1, \bar{y}_2, \bar{z}_0, \bar{z}_1, \bar{z}_2) = 0 \tag{37}$$

where G, G_0^1 should be inserted for x, y_0. It follows that formulae (36) provide explicit solution for the underdetermined differential equation (37) with two unknown functions $\bar{y}_0$ and $\bar{z}_0$. Analogously the multi–dimensional case provides some explicitly solvable examples of partial differential equations. The classical theory of complete integral [4] is trivially involved as a very particular subcase.

It is to be noted that Lie together with Bäcklund were also interested in the existence of the higher–order contact transformations. They did not succeed, consult the remarkable history and extensive comments to the relevant Lie–Bäcklund nonexistence proposition in [1].

13. APPENDIX

For the convenience of reader, let us mention the *Lie–Bäcklund proposition* since it is not easily available in literature. In order to simplify the exposition, we shall deal only with the case of one independent variable. Then the multidimensional case causes only technical difficulties with clumsy multiindices and formaly extensive systems of equations.

So, continuing the idea of Section 1, let us consider transformation (1, 4) of curves (2) of the following special kind:

$$\begin{aligned}\bar{x} &= F(x, w_0^1, \dots, w_0^m, \dots, w_S^1, \dots, w_S^m),\\ \bar{w}_s^j &= F_s^j(x, w_0^1, \dots, w_0^m, \dots, w_S^1, \dots, w_S^m),\ (j = 1, \dots, m;\ s = 0, \dots, S)\end{aligned} \tag{38}$$

(of a certain order S, $S \geq 0$) and therefore

$$\bar{w}_{S+s}^j = F_{S+s}^j(x, w_0^1, \dots, w_S^m, \dots, w_{S+s}^1, \dots, w_{S+s}^m) \tag{39}$$

($j = 1, \dots, m$; $s = 0, 1, \dots$) for the higher order derivatives, by prolongation. In other words, *we suppose that all* ($S + s$)*–order jet spaces are transformed into themselves.*

The *Lie–Bäcklund proposition* reads: *If* (38) *is an invertible system then we either deal with the prolonged point transformation* (*therefore* $S = 0$ *may be supposed*) *or* $m = 1$ *and we have the classical prolonged contact transformation.*

One can easily see that if the system (38) is invertible by certain formulae

$$\begin{aligned}x &= G(\bar{x}, \bar{w}_0^1, \dots, \bar{w}_0^m, \dots, \bar{w}_S^1, \dots, \bar{w}_S^m),\\ w_s^j &= G_s^j(\bar{x}, \bar{w}_0^1, \dots, \bar{w}_0^m, \dots, \bar{w}_S^1, \dots, \bar{w}_S^m),\ (j = 1, \dots, m;\ s \leq S)\end{aligned} \tag{40}$$

then the infinite system (38, 39) can be inverted by a mere prolongation of formulae (40). Conversely, if the infinite system (38, 39) is invertible, then the finite part (38) is separately invertible, too. We therefore deal with the same invertibility as in the sense of Section 1.

We shall systematically use the bars in order to denote the pull–back (of functions and differential forms) with respect to transformation (38, 39).

For every $l = 0, 1, \dots$, let $\Omega_l \subset \Omega$ be the submodule of all contact forms

$$\omega = \sum a_r^i \omega_r^i = \sum a_r^i (\mathrm{d}w_r^i - w_{r+1}^i \mathrm{d}x) \quad (i = 1, \dots, m;\ r \leq l)$$

with the generators ω_r^i ($r \leq l$) of the order at most l. Analogously, let $\bar{\Omega}_l \subset \Omega$ be the submodule of all forms

$$\sum b_r^i \bar{\omega}_r^i = \sum b_r^i (\mathrm{d}F_r^i - F_{r+1}^i \mathrm{d}F) \quad (i = 1, \dots, m;\ r \leq l).$$

Then (38) clearly implies $\bar{\Omega}_S \subset \Omega_S$. Assume $S > 1$, $\omega \in \Omega_{S-1}$. Then $\mathrm{d}\omega \cong 0$ (mod Ω_S) hence $\mathrm{d}\bar{\omega} \cong 0$ (mod $\bar{\Omega}_S$) and consequently $\mathrm{d}\bar{\omega} \cong 0$ (mod Ω_S). However, this implies $\bar{\omega} \in \Omega_{S-1}$ (easy direct proof) and the inclusion $\bar{\Omega}_{S-1} \subset \Omega_{S-1}$ follows. Continuing, we obtain even the whole series of the inclusions $\bar{\Omega}_{S-2} \subset \Omega_{S-2}, \dots, \bar{\Omega}_0 \subset \Omega_0$ for the transformation (38).

Passing to the main part of the proof, the last inclusion means that $\bar{\omega}_0^j \in \Omega_0$ for all forms $\omega_0^i \in \Omega_0$. Explicitly,

$$\bar{\omega}_0^j = \mathrm{d}F_0^j - F_1^j \mathrm{d}F = \sum a_i^j \omega_0^i \quad (a_i^j = \frac{\partial F_0^j}{\partial w_0^i} - F_1^j \frac{\partial F}{\partial w_0^i}), \tag{41}$$

$$\frac{\partial F_0^j}{\partial w_r^i} - F_1^j \frac{\partial F}{\partial w_r^i} = 0 \quad (r > 0). \tag{42}$$

Omitting for a moment the identities (42), we will need the congruence

$$\mathrm{d}\bar{\omega}_0^j \cong \mathrm{d}x \wedge \sum a_i^j \omega_1^i \quad (\mathrm{mod}\ \Omega_0) \tag{43}$$

which easily follows from (41). On the other hand, formula (41) also implies the congruence

$$\bar{\omega}_1^j \cong \frac{1}{XF} \sum a_i^j \omega_1^i \quad (\mathrm{mod}\ \Omega_0).$$

(Hint. Employing $X \rfloor \omega_r^i = 0$, $\bar{\Omega} \subset \Omega$, $\mathrm{d}\bar{x} \cong XF\mathrm{d}x$ (mod Ω), we have

$$X \rfloor \mathrm{d}\bar{\omega}_0^j = X \rfloor (\mathrm{d}\bar{x} \wedge \bar{\omega}_1^j) = XF \cdot \bar{\omega}_1^j$$

but alternatively $X \rfloor \mathrm{d}\bar{\omega}_0^j = X \rfloor \mathrm{d} \sum a_i^j \omega_0^i$ whence even the equality

$$XF \cdot \bar{\omega}_1^j = \sum X a_i^j \cdot \omega_0^i + \sum a_i^j \omega_1^i$$

can be obtained.) So we have another formula

$$\mathrm{d}\bar{\omega}_0^j = \mathrm{d}\bar{x} \wedge \bar{\omega}_1^j \cong (\mathrm{d}x + \sum b_r^k \omega_r^k) \wedge \sum a_i^j \omega_1^i \quad (\mathrm{mod}\ \Omega_0) \tag{44}$$

for the exterior differential where the expansion

$$\mathrm{d}\bar{x} = XF \cdot (\mathrm{d}x + \sum b_r^k \omega_r^k) \qquad (b_r^k = \frac{1}{XF} \cdot \frac{\partial F}{\partial w_r^k})$$

was inserted.

Assume $m = 1$ *and* $a_1^1 \neq 0$. Then (43, 44) imply $b_r^1 = 0$ $(r > 1)$, hence $\partial F / \partial w_r^1 = 0$ $(r > 1)$. Moreover $\bar{\Omega}_1 \subset \Omega_1$ and therefore

$$\mathrm{d}F_s^1 = \mathrm{d}\bar{w}_s^1 = \bar{\omega}_s^1 + \bar{w}_{s+1}^1 \mathrm{d}F \cong \bar{\omega}_s^1 + \bar{w}_{s+1}^1 XF\mathrm{d}x \cong 0 \quad (\mathrm{mod}\ \mathrm{d}x, \Omega_1)$$

for $s = 0, 1$. Altogether taken, functions F, F_0^1, F_1^1 do not depend on variables w_r^1 $(r > 1)$ and *we have the classical contact transformation.*

Assume $m > 1$ *and*

$$\mathrm{rank}\,(a_j^i) \geq 2. \tag{45}$$

Then (43, 44) imply

$$b_r^k = 0 \ (r > 1), \quad \sum b_1^k \omega_1^k \wedge \sum a_i^j \omega_1^i = 0.$$

The last equality means that the forms $\sum b_1^k \omega_1^k$, $\sum a_i^j \omega_1^i$ are proportional for every $i = 1, \ldots, m$, therefore $b_1^k = 0$ identically by virtue of (45). Applying moreover the inclusion $\bar{\Omega}_0 \subset \Omega_0$, we obtain the congruences

$$\mathrm{d}F, \mathrm{d}F_0^j, \mathrm{d}F_1^j \cong 0 \quad (\mathrm{mod}\ \mathrm{d}x, \Omega_0)$$

for every $j = 1, \ldots, m$ and *we have the (prolonged) point transformation.*

The invertibility assumption was not yet mentioned.

In order to obtain the above Lie–Bäcklund result, we still have to prove that *the invertibility of system* (38) *implies* $a_1^1 \neq 0$ *for the case* $m = 1$ *and the condition* (45) *for the case* $m > 1$.

Assuming the invertibility of (38), then the Jacobian is nonvanishing:

$$\det \begin{pmatrix} \partial F/\partial x & \partial F/\partial w_0^1 & \dots & \partial F/\partial w_0^m & \dots & \partial F/\partial w_r^i & \dots \\ \partial F_0^1/\partial x & \partial F_0^1/\partial w_0^1 & \dots & \partial F_0^1/\partial w_0^m & \dots & \partial F_0^1/\partial w_r^i & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \partial F_0^m/\partial x & \partial F_0^m/\partial w_0^1 & \dots & \partial F_0^m/\partial w_0^m & \dots & \partial F_0^m/\partial w_r^i & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \partial F_s^j/\partial x & \partial F_s^j/\partial w_0^1 & \dots & \partial F_s^j/\partial w_0^m & \dots & \partial F_s^j/\partial w_r^i & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \neq 0.$$

The entries $\partial F/\partial x, \partial F_s^j/\partial x$ of the first column can be replaced by the functions XF, XF_s^j, respectively. Then identities (41, 42) with $F_1^j = XF_0^j/XF$ may be applied to the first $m+1$ rows and we obtain the condition

$$\det \begin{pmatrix} XF & \partial F/\partial w_0^1 & \dots & \partial F/\partial w_0^m & \dots & \partial F/\partial w_r^i & \dots \\ 0 & a_1^1 & \dots & a_m^1 & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & a_1^m & \dots & a_m^m & \dots & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ XF_s^j & \partial F_s^j/\partial w_0^1 & \dots & \partial F_s^j/\partial w_0^m & \dots & \partial F_s^j/\partial w_r^i & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \neq 0.$$

It follows that $\det(a_j^i) \neq 0$ which concludes the proof.

In fact we have obtained better version of the Lie–Bäcklund result since the condition (45) is much weaker that the invertibility assumption if $m > 2$.

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