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On the Solutions of the First Order Nonlinear Differential Equations

MAMRILLA DUŠAN and SEMAN JÁN and VAGASKÁ ALENA

Abstract

The systems of the quasilinear differential first order equations with the same element $f(t,x(t))$ on the main diagonal and with the antisymetric matrix have the property that $r'(t)=f(t,x(t))r(t)$, where $r(t)\geq 0$ is the polar function of the system. In special cases, when value $f(t,x(t))=f(r^2(t))$ for all $t\in J_0$, we can find the function $r(t)$, $t\in J_0$ and express the general solution explicitly without the entry of $\exp\left(\int_{t_0}^{t} f(s,x(s))ds\right)$.

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1 INTRODUCTION

This paper gives some asymptotical and oscillatory properties of the solutions to the system of the nonlinear differential equations:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' = \begin{pmatrix} f(t,x(t)) & 0 & 1 \\ 0 & f(t,x(t)) & 0 \\ -1 & 0 & f(t,x(t)) \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{1}$$

$$t>0,\ f(t,x)\in C_0(D\equiv J\times R^3, R).$$

We assume that each solution $x(t)=(x_1(t),x_2(t),x_3(t))$, $x_1(t_0)=x_1^0$, $x_2(t_0)=x_2^0$, $x_3(t_0)=x_3^0$, $t_0\in J$ exists on the interval J and we denote $h>t_0>0$ the right endpoint of the interval J and $J_0=[t_0,h)$.

We shall denote $g_1(t,x)=f(t,x)x_1+x_3$, $g_2(t,x)=f(t,x)x_2$, $g_3(t,x)=-x_1+f(t,x)x_3$. It is known that if $D_0\subset D$ is open nonempty set and derivatives $\dfrac{\partial g_i(t,x)}{\partial x_j}$ are continuous functions on D_0 for every $i,j\in\{1,2,3\}$ than each point $(t_0,x_1^0,x_2^0,x_3^0)\in D_0$ is passed by one and only one integral curve $x\in D$ of the system (1) ([4]).

DEFINITION 1. The solution $x(t)$ to the system (1) is called i*-trivial*, $i\in\{1,2,3\}$ is fixed, if $x_i(t)=0$ on the interval J_0. Otherwise $x(t)$ is i*-nontrivial* solution. If for at least one $i\in\{1,2,3\}$ the solution to the system (1) is i*-nontrivial*, shortly so solution $x(t)$ is said to be *nontrivial*.

It is obvious that system (1) has $1,2,3$*-trivial* solution, $1,3$*-trivial* and 2*-nontrivial* solution, $1,3$*-nontrivial* and 2*-trivial* solution, $1,2,3$*-nontrivial* solution.

DEFINITION 2. The solution $x(t)$ to the system (1) is called i*-positive (*i*-negative)*, $i\in\{1,2,3\}$ is fixed, if $x_i(t)$ is positive (negative) function on the interval J_0.

DEFINITION 3. The solution $x(t)$ to the system (1) is called i*-nondecreasing (*i*-nonincreasing)*, $i\in\{1,2,3\}$ is fixed, if $x_i(t)$ is nondecreasing (nonincreasing) function on the interval J_0.

It is obvious that if $f(t,x)x_2\geq 0$ $\left(f(t,x)x_2\leq 0\right)$ for any point $(t,x)\in D$ than arbitrary solution $x(t)$, $t\in J_0$ to the system (1) is 2*-nondecreasing* (2*-nonincreasing*).

DEFINITION 4. The solution $x(t)$ to the system (1) is called i*-bounded*, $i\in\{1,2,3\}$ is fixed, if $x_i(t)$ is the bounded function on the interval J_0. At other cases $x(t)$ is i*-unbounde* one which is called i*-from above (*i*-from below) unbounded*, $i\in\{1,2,3\}$ is fixed, if $x_i(t)$ is from above (from below) unbounded function on the interval J_0.

It is obvious that if $y=(y_1,y_2,y_3)$ for every continuous function defined on the interval J_0:

a) $\sup_y \left(\int_{t_0}^{h} \left| f(t,y) y_2 \right| ds \right) < \infty$, than any solution $x(t)$, $t \in J_0$ to the system (1) is 2-*bounded*,

b) $\sup_y \left(\int_{t_0}^{h} f(t,y) y_2 dt \right) = -\infty$ $\left(\inf_y \left(\int_{t_0}^{h} f(t,y) y_2 dt \right) = \infty \right)$, than there exists a point $t^* \geq t_0$ and 2-*negative* (2-*positive*) solution $x(t)$, $t \in \left[t^*, h\right)$ to the system (1) such that it is 2-*from below* (2-*above*) unbounded.

DEFINITION 5. The solution $x(t)$ to the system (1) is called i-*oscillatory*, $i \in \{1,2,3\}$ is fixed, if $x_i(t)$ is the oscillatory function, i. e. if there exists the increasing sequence $\{ t_n \}_{n=1}^{\infty}$ such that $t_n \in J_0$, $t_n \to h$ and $x_i(t_n).x_i(t_{n+1}) < 0$ for each $n \in N$. The solution $x(t)$ is called i-*nonoscillatory* if there exists $h_1 < h$ such that $x_i(t)$ is not changing its sign on the interval $[h_1, h)$, resp. if it has maximally finite number of zero point on the interval $[t_0, h)$.

THEOREM 1. The general solution to the system (1) is generated by the trinity of the functions

$$x_1(t) = (C_2 \cos t - C_3 \sin t) \exp\left(\int_{t_0}^{t} f(s, x(s)) ds \right),\ x_2(t) = C_1 \exp\left(\int_{t_0}^{t} f(s, x(s)) ds \right),$$

$$x_3(t) = (-C_2 \sin t - C_3 \cos t) \exp\left(\int_{t_0}^{t} f(s, x(s)) ds \right),\text{ where } C_i (i = 1,2,3) \in R$$

are arbitrary constants.

PROOF. The characteristic quasipolynomial of the system (1) is

$$\det(A(t,x(t)) - \lambda(t,x(t))E) = (f(t,x(t)) - \lambda(t,x(t)))^3 + (f(t,x(t)) - \lambda(t,x(t))) = 0,$$

the solutions of which are the functions $\lambda_1(t,x(t)) = f(t,x(t))$ and $\lambda_{2,3}(t,x(t)) = f(t,x(t)) \pm i$. The fundamental system of the solutions to the system (1) is generated by the vector functions $X_1(t,x(t))$, $\operatorname{Re} X_2^c(t,x(t))$, $\operatorname{Im} X_2^c(t,x(t))$, where

$$X_1(t,x(t)) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \exp\left(\int_{t_0}^{t} f(s,x(s))ds \right),$$

$$X_2^c(t,x(t)) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right) \exp\left(\int_{t_0}^{t} f(s,x(s))ds \right).$$

This proves the theorem.

THEOREM 2. Let for all continuous functions y defined on the interval J_0 :

a) $\sup_y \left(\int_{t_0}^{h} |f(s,y)| ds \right) < \infty$, than each solution $x(t)$, $t \in J_0$ to the system (1) is $1,2,3$-*bounded*,

b) $\sup_y \left(\int_{t_0}^{h} f(s,y) ds \right) = -\infty$, than each solution $x(t)$, $t \in J_0$ to the system (1) is $1,2,3$-*bounded* and such that $x_1(t) \to 0$, $x_2(t) \to 0$, $x_3(t) \to 0$ for $t \to h$,

c) $\inf_y \left(\int_{t_0}^{h} f(s,y) ds \right) = \infty$, than each solution $x(t)$, $t \in J_0$ to the system (1) is such that it is i-*unbounded* at least for one $i \in \{1,2,3\}$.

PROOF. Theorem 1 implies that the general solution to the system (1) fulfils a condition

$x_1^2(t) + x_2^2(t) + x_3^2(t) = \left(C_1^2 + C_2^2 + C_3^2 \right) \exp\left(2 \int_{t_0}^{h} f(s,x(s))ds \right)$, and this implies the assertion of the theorem.

We assume that for each nontrivial solution $x(t)$, $t \in J_0$ to the system (1) there exists the trinity of the functions $r(t) > 0$, $u(t)$, $v(t) \in C_1(J_0, R)$ such that the coordinates $x_i(t)$, $t \in J_0$, $i = 1,2,3$ fulfil ([1]):

$$x_1(t) = r(t)\cos u(t),$$
$$x_2(t) = r(t)\sin u(t)\cos v(t), \qquad (*)$$
$$x_3(t) = r(t)\sin u(t)\sin v(t),$$
$$r'(t) = x_1'(t)\cos u(t) + x_2'(t)\sin u(t)\cos v(t) + x_3'(t)\sin u(t)\sin v(t),$$
$$r(t)u'(t) = -x_1'(t)\sin u(t) + x_2'(t)\cos u(t)\cos v(t) + x_3'(t)\cos u(t)\sin v(t),$$
$$r(t)\sin u(t)v'(t) = -x_2'(t)\sin v(t) + x_3'(t)\cos v(t).$$

The function $r(t)$ is called the polar, $u(t)$ the first angle function and $v(t)$ the second angle function. From this after equivalent arrangement for nontrivial solutions to the system (1) we get:

$$r'(t) = f(t,x(t))r(t),\ u'(t) = -\sin v(t),\ \sin u(t)v'(t) = -\cos u(t)\cos v(t). \qquad (2)$$

DEFINITION 6. The solution $x(t)$ to the system (1) is called *strongly singular* with respect to the substitution of variables (*) if $r(t)\sin u(t) = 0$ for all $t \in J_0$.

It is obvious that system (1) has not 1-*nontrivial* and $2,3$-*trivial* solution, hence there are no strongly singular solution.

Let us consider the positive solutions to the quasilinear differential equation:

$$r'(t) = f(t,r(t),u(t),v(t))r(t)\ ,\ f(t,r,u,v) \in C_0\ \left(D \equiv J \times R^3, R\right) \qquad (3)$$

We shall assume that each solution $r(t)$, $r(t_0) = r_0$, $t_0 \in J$ exists on the interval J and we shall denote $h > t_0 > 0$ the right endpoint of the interval J and $J_0 = [t_0, h)$. It is known that if $D_0 \subset D$ is open nonempty set and the derivative $\frac{\partial(f(t,r,u,v))r}{\partial r}$ is the continuous function on D than each point $(t_0, r_0, u_0, v_0) \in D$ is passed by one and only one integral curve $r \in D$ of the equation (3) ([4]).

It is clear that the general solution $r(t)$ to the equation (3) is the polar function of the system (1) which fulfils

$r(t) = r_0 \exp\left(\int_{t_0}^{t} f(t, r(s), u(s), v(s))ds\right)$, where $r_0 = r(t_0)$ is a positive constant. This implies that if $f(t,r,u,v) \geq 0$ $\left(f(t,r,u,v) \leq 0\right)$ on D for any point (t,r,u,v) than arbitrary positive solution $r(t)$, $t \in J_0$ to the equation (2) is nondecreasing (nonincreasing).

One can easily see that the following theorem holds.

THEOREM 3. Let for all continuous functions $y = (r, u, v)$ defined on the interval J_0:

a) $\inf_y \left(\int_{t_0}^{h} f(t, y) r dt \right) = \infty$, than any positive solution $r(t)$, $t \in J_0$ to the equation (3) is unbounded from above,

b) $\sup_y \left(\int_{t}^{h} |f(t, y)| dt \right) < \infty$, than any positive solution $r(t)$, $t \in J_0$ to the equation (3) is bounded,

c) $\sup_y \left(\int_{t_0}^{h} f(t, y) dt \right) = -\infty$, than any positive solution $r(t)$, $t \in J_0$ to the equation (3) is bounded and such that $r(t) \to 0$ for $t \to h$.

THEOREM 4. Any 2-*trivial* (2-*nontrivial*) solution $x(t)$ to the system (1) can be expressed in the form

$$x_1(t) = r_0 \cos(u_0 + t_0 - t) \exp\left(\int_{t_0}^{t} f(s, x(s)) ds \right),$$

$$x_2(t) = 0 \quad \left(x_2(t) = x_2(t_0) \exp\left(\int_{t_0}^{t} f(s, x(s)) ds \right) \right) \tag{4}$$

where $0 \neq x_2(t_0) \in R$ is a constant,

$$x_3(t) = r_0 \sin(u_0 + t_0 - t) \exp\left(\int_{t_0}^{t} f(s, x(s)) ds \right),$$

and any $1,3$-*nontrivial* solution is $1,3$-*oscillatory*.

PROOF. If in (2) we shall put $v(t) = \frac{\pi}{2} + 2k\pi$, $k \in Z$ for 2-*trivial* solutions than the angle functions of the system (1) fulfil $v'(t) = 0$, $u'(t) = -1$, $u(t) = u_0 + t_0 - t$, where $u(t_0) = u_0$ is a constant. By the analogical way it holds for $v(t) = -\frac{\pi}{2} + 2k\pi$, $k \in Z$. This completes the proof of the theorem.

EXAMPLE. Let us consider the solutions $x(t)$ to the system of three quasilinear first order differential equations (1), with $f(t, x(t)) = -\sqrt{x_1^2(t) + x_2^2(t) + x_3^2(t)}$, $t > 0$. This system has $1,2,3$ -*nontrivial* solution on the interval $J = \left(\frac{r_0 t_0 - 1}{r_0}, \infty \right)$.

The polar function $r(t) > 0$ of the arbitrary $1,2,3$ -*nontrivial* solution $x(t)$ to this system according to (2) fullfils $r'(t) = -r^2(t)$, $r(t) = \frac{r_0}{1 - r_0 t_0 + r_0 t}$, where $r(t_0) = r_0 > 0$ is a constant, $t > \frac{r_0 t_0 - 1}{r_0}$, $t_0 \geq \frac{1}{r_0}$ and $f(t, x(t)) = -r(t)$. So $x_2(t) = \frac{x_2(t_0)}{1 - r_0 t_0 + r_0 t}$, $t \in J$, where $x_2(t_0) \in R$ is a constant, $x_2(t_0) \neq 0$.

For the first angle function $u(t)$, $t \in J$ and the second angle function $v(t)$, $t \in J$ of the arbitrary 2 -*trivial* solution $x(t)$ to this system it is valid $u'(t) = -1$, $u(t) = u_0 + t_0 - t$, where $u(t_0) = u_0$ is a constant and $v(t) = \frac{\pi}{2} + 2k\pi$, $k \in Z$.

So any $1,2,3$ -*nontrivial* solution $x(t)$ to the considered system fullfils

$$(x_1(t), x_2(t), x_3(t)) = \frac{r_0}{1 - r_0 t_0 + r_0 t} \left(\cos(u_0 + t_0 - t), \frac{x_2(t_0)}{r_0}, \sin(u_0 + t_0 - t) \right),$$

$t \in J$, $x_2(t_0) \neq 0$. The given solution is $1,3$ -*oscillatory* and $1,2,3$ -*bounded* and such that $x_i(t) \to 0$ for every $i \in \{1,2,3\}$ if $t \to \infty$. Let us add that the assumptions b) of the theorem 2 and c) of the theorem 3 can be expressed as

$$\sup_y \left(\int_{t_0}^{h} f(s, y) ds \right) = -\int_{t_0}^{\infty} \frac{r_0}{1 - r_0 t_0 + r_0 t} dt = -\infty .$$

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Mamrilla Dušan
Department of Quantitative Methods and Managerial Informatics,
Faculty of Management University of Prešov in Prešov,
Ul. 17. novembra 1, 080 01 Prešov, Slovak Republik
e-mail: mamrilla@unipo.sk

Seman Ján
Department of Quantitative Methods and Managerial Informatics,
Faculty of Management University of Prešov in Prešov,
Ul. 17. novembra 1, 080 01 Prešov, Slovak Republik
e-mail: seman@unipo.sk

Vagaská Alena
Department of Mathematics, Informatics and Kybernetics,
Faculty of Manufacturing Technologies of the Technical University of Košice with a seat in Prešov,
Bayerova 1, 080 01 Prešov,
e-mail: vagaska.alena@fvt.sk

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