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On structural properties of random subgraphs of n-cube

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Abstract

We estimate the upper and the lower bound on the number of maximal subcubes contained in a random subgraph of n-cube Q_n. In addition, we study the sizes of maximal subcubes. Based on these results we estimate the lower bound on the number of maximal subcubes that cover random subgraph of Q_n.

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1. INTRODUCTION

Randomly induced subgraphs of cubes are related to minimization of Boolean functions in the class of disjunctive normal forms [Kostochka et al. 1992]. We study randomly induced subgraph of an n-cube Q_n. The model of random subgraphs is the following one: each edge is present in a subgraph with probability p $(0 < p < 1)$, independently on the presence of other edges. The properties of such random graphs were already studied [Toman and Stanek 2006], where we obtained the following results: a random graph does not contain subcubes of arbitrary order, we estimated the number of subcubes of given order, and for some orders we obtained an asymptotically tight estimates. Further, we estimated the number of subcubes of given order that contain some fixed vertex, and the relative number of vertices with this property. Using these parameters in analysis of greedy algorithm for vertex covering of Q_n by cubes, we were able to show that the number of subcubes covering all vertices of random graph is at most $2^n(1 - o(n))/\log_b n$, where $b = 1/p$.

We study the overall number of maximal subcubes of given order, as well as the number of all maximal subcubes (regardless of their order) contained in a random graph. We show that the majority of maximal subcubes of a random graph have order approximately λ (for some parameter λ depending on n). Moreover, we show that the number of maximal subcubes of order less than λ_1 or greater than λ_2 is significantly smaller than the overall number of maximal subcubes contained in a random graph. Using these parameters we prove the following lower bound of minimal vertex covering of a random graph by cubes: $2^n(1 - o(1))/(4 \log_b n)$. Precise formulations of studied problems and corresponding results are in Sections

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3 and 4.

The model of random graphs studied in this article was already explored by several works. Similar problems to those studied here, with respect to the edge covering of random graphs by cliques were studied in [Bollobás et al. 1993]. Connectedness of these graphs was analyzed in [Burtin 1977], components of random subgraphs were studied in [Erdös and Spencer 1979; Toman 1980; Ajtai et al. 1982], the radius of a random subgraph was estimated in [Toman 1982], perfect matchings were studied in [Bollobás 1990; Kostochka 1989], etc.

2. PRELIMINARIES

We will follow the notation used in [Toman and Stanek 2006]. In order to make the paper self-contained, we reintroduce the most important parts again.

Let G be a graph. The vertex set of G will be denoted by $V(G)$, and the edge set by $H(G)$.

Let Q_n be the graph consisting of 2^n vertices labelled by binary vectors of length n, and $n2^{n-1}$ edges joining vertices differing in exactly one coordinate. We denote by G^n the set of all subgraphs of Q_n with the complete set of vertices. Thus, every $G \in G^n$ has 2^n vertices. A random graph is a graph obtained from Q_n by independent removal of edges. The probability that the edge is not removed is denoted by p, where p is a constant $(0 < p < 1)$.

Random graphs create a probability space (G^n, P), where $P : G^n \to \langle 0, 1 \rangle$ is a probabilistic function defined as follows:

$$P(G) = p^{|H(G)|}(1-p)^{|H(Q_n)|-|H(G)|}.$$

In the model (G^n, P) we study properties of random graphs $G \in G^n$. Let M be some property. We shall say that G has a property M, if

$$\lim_{n\to\infty} P(G \text{ has property } M) = 1.$$

All characteristics of random graphs in this paper are real-valued random variables on a probability space (G^n, P), i.e. a random variable X is a function $X : (G^n, P) \to \mathbf{R}$. All random variables in this paper are non-negative integer random variables. The expectation of the random variable X will be denoted by $\mathrm{E}(X)$.

Let X be a non-negative random variable and let $t > 0$. Then we have (Markov's inequality):

$$\Pr[X \geq t \cdot \mathrm{E}(X)] \leq \frac{1}{t}. \tag{1}$$

We say that non-zero sequences $\{a_n\}_{n\geq 0}$, and $\{b_n\}_{n\geq 0}$ are asymptotically equal, notation $a_n \sim b_n$, if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. The symbol $\log x$ denotes the binary logarithm of x. We shall often use the logarithm to the base $\frac{1}{p}$. To simplify the notation, we put $b = \frac{1}{p}$ and write $\log_b x$ instead of $\log_{1/p} x$.

3. MAXIMAL SUBCUBES

In this section we study the number and distribution of maximal subcubes of given order in a random graph $G \in G^n$.

Definition 3.1. Let $G \in G^n$. We say that subcube K is a *maximal* subcube in G, if K is contained in G and there does not exist subcube $L \subseteq G$ such that K is contained in L, and L has higher order than K.

Let $X_{n,k}$ be a random variable, where $X_{n,k}(G)$ denotes the number of maximal subcubes of order k in graph $G \in G^n$. Let s_n be a random variable, where $s_n(G)$ denotes the number of all maximal subcubes (subcubes of arbitrary orders) in $G \in G^n$. Trivially,

$$s_n(G) = \sum_{k=0}^{n} X_{n,k}(G) \tag{2}$$

We start by evaluating $\mathrm{E}(X_{n,k})$. First we prove the following technical lemma.

LEMMA 3.2. *Let K be a subcube of Q_n of order k, and let $G \in G^n$. Then*

$$\Pr[K \textit{ is maximal in } G \mid K \subseteq G] = \left(1 - p^{(k+2)2^{k-1}}\right)^{n-k}.$$

PROOF. For any given subcube K of order k there exist $n-k$ distinct subcubes L_i $(1 \leq i \leq n-k)$ of order $k+1$, such that $K \subseteq L_i$. Clearly, $K \subseteq G$ is maximal in G if and only if $L_i \nsubseteq G$ for any $i = 1, \ldots, n-k$. Let $R_i = L_i \setminus K$. Trivially, $R_i \cap R_j = \emptyset$, whenever $i \neq j$. Then K is maximal in G if and only if $R_i \nsubseteq G$ for any $i = 1, \ldots, n-k$. Let us compute the required probability:

$$\begin{aligned}\Pr[K \text{ is maximal in } G \mid K \subseteq G] &= \Pr[\forall i:\ R_i \nsubseteq G] = \prod_{i=1}^{n-k} \Pr[R_i \nsubseteq G] \\ &= \prod_{i=1}^{n-k} (1 - \Pr[R_i \subseteq G]) = \prod_{i=1}^{n-k} \left(1 - p^{|H(R_i)|}\right).\end{aligned}$$

Since $R_i = L_i \setminus K$ and $K \subseteq L_i$ we can easily evaluate $|H(R_i)|$:

$$|H(R_i)| = |H(L_i) - H(K)| = |H(L_i)| - |H(K)| = (k+1)2^k - k2^{k-1}.$$

Hence we get:

$$\Pr[K \text{ is maximal in } G \mid K \subseteq G] = \prod_{i=1}^{n-k} \left(1 - p^{(k+2)2^{k-1}}\right) = \left(1 - p^{(k+2)2^{k-1}}\right)^{n-k}.$$

□

We are ready to evaluate $\mathrm{E}(X_{n,k})$.

LEMMA 3.3. $\mathrm{E}(X_{n,k}) = \binom{n}{k} 2^{n-k} p^{k2^{k-1}} (1 - p^{(k+2)2^{k-1}})^{n-k}$.

PROOF. For every order k subcube K $(0 \leq k \leq n)$ in Q_n we define a random variable η_K (sometimes called an indicator):

$$\eta_K(G) = \begin{cases} 1, & \text{if } K \subseteq G \text{ and } K \text{ is maximal in } G, \\ 0, & \text{otherwise.} \end{cases}$$

The random variable $X_{n,k}$ can be expressed as $\sum_K \eta_K$, where the sum is computed over all subcubes of order k. Thus,

$$\mathrm{E}(X_{n,k}) = \mathrm{E}\left(\sum_K \eta_K\right) = \sum_K \mathrm{E}(\eta_K) = \sum_K \Pr[\eta_K = 1].$$

Let us compute the probability $\Pr[\eta_K = 1]$:

$$\begin{aligned}\Pr[\eta_K = 1] &= \Pr[K \text{ is maximal in } G \ \wedge \ K \subseteq G] \\ &= \Pr[K \text{ is maximal in } G \mid K \subseteq G] \cdot \Pr[K \subseteq G]\end{aligned}$$

It is easy to see that $\Pr[K \subseteq G] = p^{k2^{k-1}}$, and using Lemma 3.2 we get:

$$\Pr[\eta_K = 1] = \left(1 - p^{(k+2)2^{k-1}}\right)^{n-k} \cdot p^{k2^{k-1}}.$$

Substituting back, and using the fact that there are $\binom{n}{k}2^{n-k}$ subcubes of order k, we can finally express $\mathrm{E}(X_{n,k})$:

$$\mathrm{E}(X_{n,k}) = \sum_K \Pr[\eta_K = 1] = \binom{n}{k} 2^{n-k} p^{k2^{k-1}} \cdot \left(1 - p^{(k+2)2^{k-1}}\right)^{n-k}.$$

□

Let us introduce an important parameter λ. Later (in Lemma 3.4), we shall prove that $\mathrm{E}(X_{n,k})$ taken as a function of k attains maximum for $k = \lambda$ or $k = \lambda+1$. This results will be useful for further analysis of maximal subcubes in random graphs.

Let λ be the smallest integer satisfying the following inequality (recall that $b = 1/p$):

$$\lambda + \log(\lambda + 2) \geq \log\log_b n. \tag{3}$$

Since λ is the smallest integer satisfying (3) we have

$$\lambda - 1 + \log(\lambda + 1) < \log\log_b n. \tag{4}$$

The inequalities (3) and (4) allow us to derive the following estimates:

$$\left(\frac{1}{p}\right)^{(\lambda+2)2^{\lambda}} \geq n \tag{5}$$

$$\left(\frac{1}{p}\right)^{(\lambda+1)2^{\lambda-1}} < n \tag{6}$$

The value of λ can be estimated as follows, for sufficiently large n:

$$\lambda \leq \lceil \log\log_b n - \log\log\log_b n + 1 \rceil, \tag{7}$$

$$\lambda > \log\log_b n - \log\log\log_b n, \tag{8}$$

$$\lambda \sim \log\log_b n = \log\log n - \log\log\frac{1}{p}. \tag{9}$$

LEMMA 3.4. *Let us take* $\mathrm{E}(X_{n,k})$ *as a function of* k*. Then, for sufficiently large* n*,* $\mathrm{E}(X_{n,k})$ *increases for* $k < \lambda$*, decreases for* $k > \lambda$*, and attains maximum for* $k = \lambda$ *or* $k = \lambda + 1$*.*

PROOF. Let us study the division $m_k = \mathrm{E}(X_{n,k+1})/\mathrm{E}(X_{n,k})$. Using Lemma 3.3 we get:

$$m_k = \frac{n-k}{2(k+1)} \frac{1}{(\frac{1}{p})^{(k+2)2^{k-1}} - 1} \left(\frac{1 - p^{(k+3)2^k}}{1 - p^{(k+2)2^{k-1}}} \right)^{n-k-1}$$

The following bounds are useful for estimating m_k:

$$1 \le \frac{1 - p^{(k+3)2^k}}{1 - p^{(k+2)2^{k-1}}} \le 1 + \frac{1}{(\frac{1}{p})^{(k+2)2^{k-1}} - 1} \tag{10}$$

For $k < \lambda$ we compute a lower bound of m_k:

$$m_k \ge \frac{n-k}{2(k+1)} \frac{1}{(\frac{1}{p})^{(k+2)2^{k-1}} - 1} \ge \frac{n-\lambda+1}{2\lambda} \frac{1}{(\frac{1}{p})^{(\lambda+1)2^{\lambda-2}} - 1}.$$

Using the bound (6) we get:

$$m_k \ge \frac{n-\lambda+1}{2\lambda} \frac{1}{\sqrt{n} - 1}.$$

Taking into account the estimate (9) we have (for $k < \lambda$): $\lim_{n\to\infty} m_k = \infty$. Hence, $m_k > 1$ for sufficiently large n.

Similarly, for $k > \lambda$ we compute an upper bound of m_k. Using bounds (10) and (5) we get:

$$\begin{aligned} m_k &\le \frac{n-k}{2(k+1)} \frac{1}{(\frac{1}{p})^{(k+2)2^{k-1}} - 1} \left(1 + \frac{1}{(\frac{1}{p})^{(k+2)2^{k-1}} - 1}\right)^{n-k-1} \\ &\le \frac{n-\lambda-1}{2(\lambda+2)} \frac{1}{(\frac{1}{p})^{(\lambda+3)2^{\lambda}} - 1} \left(1 + \frac{1}{(\frac{1}{p})^{(\lambda+3)2^{\lambda}} - 1}\right)^{n-\lambda-2} \\ &\le \frac{n-\lambda-1}{2(\lambda+2)} \frac{1}{n^{\frac{\lambda+3}{\lambda+2}} - 1} \left(1 + \frac{1}{n^{\frac{\lambda+3}{\lambda+2}} - 1}\right)^{n-\lambda-2} \\ &\le \frac{n-\lambda-1}{2(\lambda+2)} \frac{1}{n-1} \left(1 + \frac{1}{n-1}\right)^{n-\lambda-2}. \end{aligned}$$

Since $(1 + \frac{1}{n-1})^{n-\lambda-2} \le e$, the upper bound of m_k can be expressed as follows:

$$m_k \le \frac{n-\lambda-1}{2(\lambda+2)} \frac{e}{n-1}.$$

Taking into account the estimate (9) we have (for $k > \lambda$): $\lim_{n\to\infty} m_k = 0$. Hence, $m_k < 1$ for sufficiently large n.

We can summarize (from the properties of m_k): $\mathrm{E}(X_{n,k})$ increases for $k < \lambda$, decreases for $k > \lambda$, and attains maximum for $k = \lambda$ or $k = \lambda + 1$ □

The lemmas allow us to estimate $s_n(G)$ – the number of all maximal subcubes of random graph $G \in G^n$.

THEOREM 3.5. *The following inequality holds with probability tending to* 1 *as* $n \to \infty$*:*

$$s_n(G) \le n^{(1+\varepsilon_1(n)) \log\log_b n} \cdot 2^n,$$

where $G \in G^n$, *and* $\varepsilon_1(n) = O\left(\frac{1}{\log\log_b n}\right)$.

PROOF. The upper bound on $s_n(G)$ can be estimated through Markov's inequality (1) with parameter $\varepsilon = \frac{n^2}{n+1}$:

$$s_n(G) \le \frac{n^2}{n+1}\ \mathrm{E}(s_n).$$

Using 2 we have:

$$\mathrm{E}(s_n) \le \sum_{k=0}^{n} \max_k \mathrm{E}(X_{n,k}) = (n+1)\max_k \mathrm{E}(X_{n,k}).$$

According to Lemma 3.4, $\mathrm{E}(X_{n,k})$ attains maximum for $k \in \{\lambda, \lambda+1\}$. Thus,

$$s_n(G) \le n^2 \cdot \max_{k\in\{\lambda,\lambda+1\}} \mathrm{E}(X_{n,k}).$$

Using Lemma 3.3 we get:

$$\begin{aligned} s_n(G) &\le n^2 \cdot \max_{k\in\{\lambda,\lambda+1\}} \binom{n}{k} 2^{n-k} p^{k2^{k-1}} (1 - p^{(k+2)2^{k-1}})^{n-k} \\ &\le n^2 \binom{n}{\lambda+1} 2^{n-\lambda} p^{\lambda 2^{\lambda-1}}. \end{aligned}$$

Since $\binom{n}{\lambda+1} < n^{\lambda+1}$, the bound can be simplified:

$$s_n(G) \le n^{\lambda+3} p^{\lambda 2^{\lambda-1}} \cdot 2^{n-\lambda}.$$

Inequality (5) offers further simplification:

$$s_n(G) \le n^{\lambda+3} n^{-\frac{\lambda}{2(\lambda+2)}} \cdot 2^{n-\lambda} \le n^{\lambda+3} 2^{n-\lambda}.$$

Hence,

$$s_n(G) \le n^{(1+\varepsilon_1'(n))\lambda} \cdot 2^n,$$

where $\varepsilon_1'(n) = \frac{3\log n - \lambda}{\lambda \log n}$. Using the estimate (9), $\varepsilon_1'(n) = O\left(\frac{1}{\log\log_b n}\right) = \varepsilon_1(n)$, and $s_n(G) \le n^{(1+\varepsilon_1(n))\log\log_b n} \cdot 2^n$. □

THEOREM 3.6. *The following inequality holds with probability tending to* 1 *as* $n \to \infty$*:*

$$s_n(G) \ge n^{(1-\varepsilon_2(n))\log\log_b n} \cdot 2^n,$$

where $G \in G^n$, *and* $\varepsilon_2(n) = O\left(\frac{1}{\log\log_b n}\right)$.

PROOF. Let μ be the smallest integer satisfying inequality

$$\mu - 1 + \log(\mu+1) \ge \log\log_b 2^n.$$

According to [Toman and Stanek 2006], the random graph does not contain subcubes of order greater than μ. For the rest of the proof we can work with random graphs $G \in G^n$ that do not contain subcubes of order greater that μ. Hence, every maximal subcube in G contains at most $\binom{\mu}{\lambda}2^{\mu-\lambda}$ subcubes of order λ. Moreover, every subcube is contained in at least one maximal subcube. Therefore

$$s_n(G) \geq \frac{\binom{n}{\lambda}\left(2^{n-\lambda}p^{\lambda 2^{\lambda-1}} - \varphi(n)\sqrt{2^{n-\lambda}p^{\lambda 2^{\lambda-1}}}\right)}{\binom{\mu}{\lambda}2^{\mu-\lambda}}, \qquad (11)$$

where the numerator (let us denote it by $i_{n,\lambda}$) is the lower bound on the number of order λ subcubes contained in a random graph G [Toman and Stanek 2006], and $\varphi(n)$ is an arbitrary increasing function.

Let $\varphi(n) = n$. Using inequality $\binom{n}{\lambda} > \left(\frac{n}{\lambda}\right)^{\lambda}$, the numerator in (11) can be lower-bounded:

$$i_{n,\lambda} > n^{\lambda}2^{n-\lambda}p^{\lambda 2^{\lambda-1}}\lambda^{-\lambda}\left(1 - \sqrt{n^2 2^{\lambda-n}p^{-\lambda 2^{\lambda-1}}}\right)$$

Similarly, using the inequality $\binom{\mu}{\lambda} < \mu^{\lambda}$, the denominator in (11) can be upper-bounded:

$$\binom{\mu}{\lambda}2^{\mu-\lambda} < \mu^{\lambda}2^{\mu-\lambda}.$$

Putting these bounds together we get:

$$s_n(G) \geq n^{\lambda}2^{n-\mu}p^{\lambda 2^{\lambda-1}}\lambda^{-\lambda}\mu^{-\lambda}\left(1 - \sqrt{n^2 2^{\lambda-n}p^{-\lambda 2^{\lambda-1}}}\right).$$

The bound (6) in the form $p^{\lambda 2^{\lambda-1}} > n^{-1}$ simplifies the bound further:

$$s_n(G) \geq n^{\lambda-1}2^{n-\mu}\lambda^{-\lambda}\mu^{-\lambda}\left(1 - \sqrt{n^3 2^{\lambda-n}}\right).$$

Hence,

$$s_n(G) \geq n^{(1-\varepsilon_2'(n))\lambda}2^n,$$

where $\varepsilon_2'(n) = \frac{\log n + \mu + \lambda\log\lambda + \lambda\log\mu - \log(1-\sqrt{n^3 2^{\lambda-n}})}{\lambda\log n}$. Using the estimate (9), $\varepsilon_2'(n) = O\left(\frac{1}{\log\log_b n}\right) = \varepsilon_2(n)$, and $s_n(G) \geq n^{(1-\varepsilon_2(n))\log\log_b n} \cdot 2^n$. □

The following theorem is the main result of this section, it summarizes the results of previous two theorems.

Theorem 3.7 (Counting maximal subcubes). *The following inequality holds with probability tending to* 1 *as* $n \to \infty$*:*

$$n^{(1-\varepsilon_2(n))\log\log_b n} \cdot 2^n \leq s_n(G) \leq n^{(1+\varepsilon_1(n))\log\log_b n} \cdot 2^n$$

where $G \in G^n$, *and* $\varepsilon_1(n) = O\left(\frac{1}{\log\log_b n}\right) = \varepsilon_2(n)$.

Proof. The theorem trivially follows from Theorem 3.5 and Theorem 3.6. □

4. DISTRIBUTION OF MAXIMAL SUBCUBES SIZES

According [Toman and Stanek 2006], the random graph does not contain subcubes of order greater than $\mu = \log n - \log\log\frac{1}{p}$ (for sufficiently large n). We studied the number of maximal subcubes of order k in graph $G \in G^n$ in Section 3. The corresponding random variable was denoted as $X_{n,k}$. From Lemma 3.4 one can deduce that the majority of maximal subcubes in G have order close to λ. We will estimate parameter λ_1 (λ_2), such that the number of maximal subcubes of order smaller than λ_1 (larger than λ_2), is substantially smaller than the overall number of maximal subcubes in G. In order to study this problem formally, let us introduce the following notation:

$$X^+_{n,k}(G) = \sum_{j>k} X_{n,k}(G),$$
$$X^-_{n,k}(G) = \sum_{j<k} X_{n,k}(G).$$

Further, let $V^+_{n,k}$ ($V^-_{n,k}$) be a random variable, where $V^+_{n,k}(G)$ ($V^-_{n,k}(G)$) denotes the number of vertices in G covered by subcubes of orders larger (smaller) than k. Let $H^+_{n,k}$ ($H^-_{n,k}$) be a random variable, where $H^+_{n,k}(G)$ ($H^-_{n,k}(G)$) denotes the number of edges in G covered by subcubes of orders larger (smaller) than k.

Let h_n be a random variable, where $h_n(G)$ denotes the number of edges in $G \in G^n$. Theorem 3.7 gives an estimate for the number of maximal subcubes in G (i.e. the quantity $s_n(G)$). A result from [Toman and Stanek 2006] gives an estimate for $h_n(G)$. Together with the fact that G has 2^n vertices we can reformulate our problem. We try to find λ_1 and λ_2 such that:

$$X^-_{n,\lambda_1}(G) = o(s_n(G)) = X^+_{n,\lambda_2}(G),$$
$$V^-_{n,\lambda_1}(G) = o(2^n) = V^+_{n,\lambda_2}(G),$$
$$H^-_{n,\lambda_1}(G) = o(h_n(G)) = H^+_{n,\lambda_2}(G).$$

In the following, due to Lemma 4.1, we can restrict our analysis to those graphs $G \in G^n$, that satisfy:

(1) G does not contain subcubes of order larger than μ;

(2) $X_{n,k}(G) < n\,\mathrm{E}(X_{n,k})$, for $k = 0, \dots, n$.

LEMMA 4.1. *The following properties hold with probability tending to* 1 *as* $n \to \infty$*:*

(1) G does not contain subcubes of order larger than μ;

(2) $X_{n,k}(G) < n\,\mathrm{E}(X_{n,k})$, for $k = 0, \dots, n$.

PROOF. Let us define

$$M_1 = \{G \in G^n;\ G \text{ does not contain subcubes of order larger than } \mu\},$$
$$M_2 = \bigcap_{k \le \mu} M_{n,k},$$

where $M_{n,k}$ denotes an event that $X_{n,k}(G) < n\,\mathrm{E}(X_{n,k})$ for given k. According [Toman and Stanek 2006], $\lim_{n\to\infty} \Pr[M_1] = 1$, i.e. $\lim_{n\to\infty} \Pr[G^n \setminus M_1] = 0$. Let us compute $\Pr[G^n \setminus M_2]$:

$$\begin{aligned}\Pr[G^n \setminus M_2] &= \Pr\left[\bigcup_{k\le\mu} G^n \setminus M_{n,k}\right]\\ &\le \sum_{k=0}^{n} \Pr[G^n \setminus M_{n,k}]\\ &= \sum_{k=0}^{n} \Pr[X_{n,k} \ge n\,\mathrm{E}(X_{n,k})].\end{aligned}$$

Applying Markov's inequality we get:

$$\Pr[G^n \setminus M_2] \le \sum_{k=0}^{\mu} \frac{1}{n} = \frac{\mu+1}{n}.$$

Considering the value of μ it can be easily seen that

$$\lim_{n\to\infty} \Pr[G^n \setminus M_2] \le \lim_{n\to\infty} \frac{\mu+1}{n} = 0.$$

To finish the proof we have to show that $\lim_{n\to\infty} \Pr[M_1 \cap m_2] = 1$:

$$\begin{aligned}\lim_{n\to\infty} \Pr[M_1 \cap M_2] &= 1 - \lim_{n\to\infty} \Pr[G^n \setminus (M_1 \cap M_2)]\\ &\ge 1 - \lim_{n\to\infty} \Pr[G^n \setminus M_1] - \lim_{n\to\infty} \Pr[G^n \setminus M_2] = 1.\end{aligned}$$

□

Lemma 4.1 allows us to establish an upper bound for random variables $X^+_{n,k}(G)$, $X^-_{n,k}(G)$, $V^+_{n,k}(G)$, $V^-_{n,k}(G)$, $H^+_{n,k}(G)$, $H^-_{n,k}(G)$.

LEMMA 4.2. *The following inequalities hold with probability tending to* 1 *as* $n \to \infty$*:*

(1) $X^+_{n,k}(G) < \sum_{k<j\le\mu} n\,\mathrm{E}(X_{n,j})$
(2) $X^-_{n,k}(G) < \sum_{j<k} n\,\mathrm{E}(X_{n,j})$
(3) $V^+_{n,k}(G) < \sum_{k<j\le\mu} n\,\mathrm{E}(X_{n,j}) \cdot 2^j$
(4) $V^-_{n,k}(G) < \sum_{j<k} n\,\mathrm{E}(X_{n,j}) \cdot 2^j$
(5) $H^+_{n,k}(G) < \sum_{k<j\le\mu} n\,\mathrm{E}(X_{n,j}) \cdot j \cdot 2^{j-1}$
(6) $H^-_{n,k}(G) < \sum_{j<k} n\,\mathrm{E}(X_{n,j}) \cdot j \cdot 2^{j-1}$

PROOF. Using Lemma 4.1 we get:

$$\begin{aligned}X^+_{n,k}(G) &= \sum_{k<j} X_{n,j}(G) = \sum_{k<j\le\mu} X_{n,j}(G) < \sum_{k<j\le\mu} n\,\mathrm{E}(X_{n,j}),\\ X^-_{n,k}(G) &= \sum_{j<k} X_{n,j}(G) < \sum_{j<k} n\,\mathrm{E}(X_{n,j}).\end{aligned}$$

Similarly, using Lemma 4.1 and the trivial fact that a subcube of order j covers at most 2^j vertices we have:

$$V^+_{n,k}(G) \le \sum_{k<j} X_{n,j}(G)\cdot 2^j = \sum_{k<j\le\mu} X_{n,j}(G)\cdot 2^j < \sum_{k<j\le\mu} n\,\mathrm{E}(X_{n,j})\cdot 2^j,$$
$$V^-_{n,k}(G) \le \sum_{j<k} X_{n,j}(G)\cdot 2^j < \sum_{j<k} n\,\mathrm{E}(X_{n,j})\cdot 2^j.$$

Analogously, using Lemma 4.1 and the fact that a subcube of order j covers at most $j\cdot 2^{j-1}$ vertices we have:

$$H^+_{n,k}(G) \le \sum_{k<j} X_{n,j}(G)\cdot j\cdot 2^{j-1} < \sum_{k<j\le\mu} n\,\mathrm{E}(X_{n,j})\cdot j\cdot 2^{j-1},$$
$$H^-_{n,k}(G) \le \sum_{j<k} X_{n,j}(G)\cdot j\cdot 2^{j-1} < \sum_{j<k} n\,\mathrm{E}(X_{n,j})\cdot j\cdot 2^{j-1}.$$

□

Previous results allow us to estimate λ_1.

THEOREM 4.3. *Let* $\lambda_1 = \log\log_b n - \log\log\log_b n$. *The following properties hold for every* $G \in G^n$ *with probability tending to* 1 *as* $n \to \infty$*:*

(*1*) $V^-_{n,\lambda_1}(G) = o(2^n)$

(*2*) $H^-_{n,\lambda_1}(G) = o(h_n(G))$

(*3*) $X^-_{n,\lambda_1}(G) = o(s_n(G))$

PROOF. First, we show that the theorem is valid for $\lambda_1 = \lambda$. Then we complete the proof by considering the inequality (8).

Using Lemma 4.2 we have:

$$V^-_{n,\lambda}(G) < n \sum_{j\le\lambda-1} \mathrm{E}(X_{n,j})\cdot 2^j$$
$$H^-_{n,\lambda}(G) < n \sum_{j\le\lambda-1} \mathrm{E}(X_{n,j})\cdot j\cdot 2^{j-1}$$

According Lemma 3.4, the value of $\mathrm{E}(X_{n,k})$ increases for $k \le \lambda - 1$. The fact is used for these estimates:

$$V^-_{n,\lambda}(G) < n\,\mathrm{E}(X_{n,\lambda-1}) \sum_{j\le\lambda-1} 2^j < n\,\mathrm{E}(X_{n,\lambda-1})\cdot 2^\lambda \tag{12}$$
$$H^-_{n,\lambda}(G) < n\,\mathrm{E}(X_{n,\lambda-1})\cdot\lambda \sum_{j\le\lambda-1} 2^{j-1} < n\,\mathrm{E}(X_{n,\lambda-1})\cdot\lambda\cdot 2^{\lambda-1} \tag{13}$$

We express $\mathrm{E}(X_{n,\lambda-1})$ by applying Lemma 3.3, and estimate by using inequalities $\binom{n}{\lambda-1} < n^{\lambda-1}$, (5), and (6):

$$\mathrm{E}(X_{n,\lambda-1}) = \binom{n}{\lambda-1} 2^{n-\lambda+1} p^{(\lambda-1)2^{\lambda-2}} (1 - p^{(\lambda+1)2^{\lambda-2}})^{n-\lambda+1}$$
$$< n^{\lambda-1}\, 2^{n-\lambda+1} \left(1 - \frac{1}{\sqrt{n}}\right)^{n-\lambda+1} \tag{14}$$

In order to prove the first statement, we use formulas (12) and (14):

$$V^-_{n,\lambda_1}(G) < 2^{n+1}n^\lambda\left(1-\frac{1}{\sqrt{n}}\right)^{n-\lambda+1}.$$

Applying the estimate (8) we get

$$V^-_{n,\lambda_1}(G) = 2^n \cdot o(1) = o(2^n).$$

In order to prove the first statement, we use formulas (13) and (14):

$$H^-_{n,\lambda_1}(G) < 2^n\lambda n^\lambda\left(1-\frac{1}{\sqrt{n}}\right)^{n-\lambda+1}.$$

Applying the estimate (9) we get

$$H^-_{n,\lambda_1}(G) = n \cdot 2^{n-1} \cdot o(1) = o(n \cdot 2^{n-1}).$$

Since $h_n(G) \sim n2^{n-1}p$, see [Toman and Stanek 2006], we have

$$H^-_{n,\lambda_1}(G) = o(h_n(G)).$$

The last property can be derived from the asymptotic equality $V^-_{n,\lambda_1}(G) = o(2^n)$. We immediately get $X^-_{n,\lambda_1}(G) = o(2^n)$. Considering the estimate for $s_n(G)$ from Theorem 3.7 we have

$$X^-_{n,\lambda_1}(G) = o(s_n(G)).$$

□

Similarly to Theorem 4.3 we can prove an estimate for λ_2.

THEOREM 4.4. *Let $\lambda_2 = \log\log_b n + 2$. The following properties hold for every $G \in G^n$ with probability tending to 1 as $n \to \infty$:*

(1) $V^+_{n,\lambda_2}(G) = o(2^n)$

(2) $H^+_{n,\lambda_2}(G) = o(h_n(G))$

(3) $X^+_{n,\lambda_2}(G) = o(s_n(G))$

PROOF. In order to estimate $V^+_{n,\lambda_2}(G)$, $H^+_{n,\lambda_2}(G)$, and $X^+_{n,\lambda_2}(G)$ we use Lemma 3.3 and inequalities from Lemma 4.2. For $V^+_{n,\lambda_2}(G)$ we have:

$$\begin{aligned} V^+_{n,\lambda_2}(G) &< \sum_{\lambda_2<j\le\mu} n\,\mathrm{E}(X_{n,j}) \cdot 2^j \\ &< \sum_{\lambda_2<j\le\mu} n\binom{n}{j}2^{n-j}p^{j2^{j-1}}(1-p^{(j+2)2^{j-1}})^{n-j} \cdot 2^j \end{aligned}$$

We apply the inequality $\binom{n}{j} < n^j$. Further simplification yields:

$$V^+_{n,\lambda_2}(G) < 2^n \sum_{\lambda_2<j\le\mu} n^{j+1}p^{j2^{j-1}}.$$

The sequence $\{n^{j+1}p^{j2^{j-1}}\}_{j>\lambda_2}$ decreases (in fact, it decreases for $j \geq \lambda+1$). Hence

$$\sum_{\lambda_2<j\leq\mu} n^{j+1}p^{j2^{j-1}} < \mu n^{\lambda_2+1}p^{\lambda_2 2^{\lambda_2-1}} = \mu n^{\lambda_2+1}n^{-2\lambda_2} = o(1), \tag{15}$$

where the last step employs the estimate of μ from [Toman and Stanek 2006]. Putting all together we get:

$$V^+_{n,\lambda_2}(G) = 2^n \cdot o(1) = o(2^n).$$

The second case is similar, for $H^+_{n,\lambda_2}(G)$ we have:

$$\begin{aligned} H^+_{n,\lambda_2}(G) &< \sum_{\lambda_2<j\leq\mu} n\,\mathrm{E}(X_{n,j}) \cdot j \cdot 2^{j-1} \\ &< \sum_{\lambda_2<j\leq\mu} n\binom{n}{j}2^{n-j}p^{j2^{j-1}}(1-p^{(j+2)2^{j-1}})^{n-j} \cdot j \cdot 2^j \\ &< n2^{n-1}\sum_{\lambda_2<j\leq\mu} n^{j+1}p^{j2^{j-1}}. \end{aligned}$$

Applying (15) gives:

$$H^+_{n,\lambda_2}(G) = n2^{n-1}o(1) = o(n2^{n-1}).$$

Since $h_n(G) \sim n2^{n-1}$ (see [Toman and Stanek 2006]), this finishes the proof for $H^+_{n,\lambda_2}(G)$.

The last statement can be derived from the estimate $V^+_{n,\lambda_2}(G) = o(2^n)$. Immediately we have $X^+_{n,\lambda_2}(G) = o(2^n)$, and by Theorem 3.7 we get $X^+_{n,\lambda_2}(G) = o(s_n(G))$. □

Definition 4.5. A set $\mathcal{B} = \{G_1, \ldots, G_m\}$ of subcubes of a graph G is called a vertex covering of graph G (by cubes), if $\bigcup_{i=1}^m V(G_i) = V(G)$. The value m is the size of the vertex covering. The smallest possible m is the size of a minimal vertex covering.

The size of a minimal vertex covering can be viewed as a random variable in the probability space (G^n, P). Thus, let p_n be a random variable, where $p_n(G)$ denotes the size of a minimal vertex covering of $G \in G^n$. As a final result we prove a lower bound for $p_v(G)$.

THEOREM 4.6. *The following inequality holds with probability tending to* 1 *as* $n \to \infty$*:*

$$p_v(G) \geq \frac{2^n(1-o(1))}{4\log_b n}.$$

PROOF. According to Theorem 4.4, the following property holds for $\lambda_2 = \log\log_b n$-2 and for every $G \in G^n$ with probability tending to 1 as $n \to \infty$:

$$V^+_{n,\lambda_2}(G) = o(2^n).$$

Hence, we can restrict our analysis to graphs satisfying this property. The vertices of such graph G are covered by subcubes of order at most λ_2. There are $|V(G)| - |V^+_{n,\lambda_2}(G)|$ vertices not covered by subcubes of order less than λ_2. Hence, the lower bound for $p_v(G)$ can by obtained by calculating the number of subcubes of order λ_2 needed to cover these vertices:

$$p_v(G) \geq \frac{|V(G)| - |V^+_{n,\lambda_2}(G)|}{2^{\lambda_2}} > \frac{2^n - o(2^n)}{4\log_b n} = \frac{2^n(1 - o(1)))}{4\log_b n}.$$

This ends proof of theorem. □

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