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On Symmetric Information Divergence Measures of Csiszar's f - Divergence Class[1]

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Abstract

A non-parametric symmetric measure of divergence (in fact a series of divergence measures), which belongs to the family of Csiszar's *f* – divergence is proposed. Its properties are studied and bounds in terms of some well-known divergence measures are obtained. A parametric measure of information is also derived from the suggested non- parametric measure.

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1. INTRODUCTION

The Csiszar's *f* – divergence is a general class of divergence measures that includes several divergences used in measuring the distance or affinity between two probability distributions. This class is introduced by using a convex function *f*, defined on $(0, \infty)$. An important property of this divergence is that many known divergences can be obtained from this measure by appropriately defining the convex function *f*. These measures have been applied in a variety of fields such as anthropology, genetics, finance, economics, analysis of contingency tables, approximations of probability distributions, signal processing & pattern recognition. In this paper, we have introduced a new symmetric nonparametric information divergence measure, which belongs to the class of Csiszar's *f* – divergence [4, 5]. In section 2, we discuss the Csiszar's *f* – divergences and inequalities. New Symmetric divergence measure (in fact a series of divergence measures) is introduced in section 3. In section 4, we have derived some inequalities relating the new measure with other divergence measures. In section 5, we have derived information divergence inequalities providing bounds for the new measure in terms of some well-known divergence measures. In section 6, we have derived some new information divergence measures using the convexity property of divergence measures. In section 7, it is shown that the suggested measure can be applied to the parametric family of distributions. Section 8 concludes this paper.

2. CSISZAR'S F- DIVERGENCE AND INEQUALITIES

Let $\Omega = \{X_1, X_2 ----\}$ be a set with at least two elements and Π, the set of all probability distributions $P = \{ p(X): X \in \Omega\}$ on Ω. For a convex function $f: [0,\infty) \to R$, the f – divergence of the probability distributions *P* and *Q* by Csiszar, [4, 5] and Ali and Survey, [1] is defined as

$$C_f(P, Q) = \sum_{x \in \Omega} q(x)\, f\left[\frac{p(x)}{q(x)}\right] \tag{2.1}$$

Henceforth, for brevity we will denote C_f (P,Q), p(X), q(X) and $\sum_{x\in\Omega}$ by C (P,Q),p,q and $\sum$ respectively. Osterreicher [19] has discussed basic general properties of f-divergences including their axiomatic properties and some important classes. During the recent past there has been a considerable amount of work providing different kinds of bounds on the distance, information and divergence measures [27, 7, 8, and 9]. Taneja and Kumar [24] unified and generalized three theorems studied by Dragomir [7, 8, 9], which provide bounds on C (P, Q). The main result in [24] is the following theorem.

Theorem 1 Let $f: I \subset P_+ \to P$ be a mapping which is normalized i.e. $f(1) = 0$. Suppose that

(i) f is twice differentiable on (r, R), $0 \le r \le 1 \le R < \infty$ (f' and f'' denote the first and second order derivatives of f),

(ii) there exists real constants m, M such that m < M and $m \le x^{2-s} f''(x) \le M \;\forall\; x \in (r, R)$, $s \in R$

If P, Q $\in \Pi^2$ are discrete probability distributions with $0 < r \le \frac{p}{q} \le R < \infty$, then

$$m\,\Phi_s(P, Q) \le C(P, Q) \le M\Phi_s(P, Q). \tag{2.2}$$

and $m\,[\eta_s(P, Q) - \Phi_s(P, Q)] \le C_\rho(P, Q) - C(P, Q) \le M\,[\eta_s(P, Q) - \Phi_s(P, Q)]$ (2.3)
where

$$\Phi_s(P,Q) = \begin{cases} {}^2K_S(P,Q), & s \ne 0,1 \\ K(Q,P), & s = 0 \\ K(P,Q), & s = 1 \end{cases} \tag{2.4}$$

and $${}^2K_S(P, Q) = [s(s-1)]^{-1} \left[\sum p^s q^{1-s} - 1\right],\ s \ne 0, 1. \tag{2.5}$$

$$K(P, Q) = \sum p \ln(p/q) \tag{2.6}$$

$$C_\rho(P, Q) = C_{f'}\left(\frac{P^2}{Q}, P\right) - C_{f'}(P, Q) = \sum (p - q) f'\left(\frac{p}{q}\right) \tag{2.7}$$

and

$$\eta_s(P,Q) = C_{\Phi's}\left(\frac{P^2}{Q}, P\right) - C_{\Phi's}(P,Q) = \begin{cases} (s-1)^{-1} \sum (p - q)\left(\frac{p}{q}\right)^{s-1}, & s \ne 1 \\ \sum (p - q)\ln\left(\frac{p}{q}\right), & s = 1 \end{cases} \tag{2.8}$$

The following information inequalities which are interesting from the *information-theoretic* point of view, are obtained from Theorem 1 and discussed in [24]

(i) The case s = 2 provides the information bounds in terms of the chi-square divergence $\chi^2(P, Q)$.

$$\frac{m}{2}\chi^2(P, Q) \le C(P, Q) \le \frac{M}{2}\chi^2(P, Q). \tag{2.9}$$

and

$$\frac{m}{2}\chi^2(P, Q) \le C_\rho(P, Q) - C(P, Q) \le \frac{M}{2}\chi^2(P, Q). \tag{2.10}$$

where

$$\chi^2 (P, Q) = \sum \frac{(p - q)^2}{q} \tag{2.11}$$

(ii) The case s = 1 provides information bounds in terms of the Kullback-Leibler divergence, K (P, Q),

$$mK(P,Q) \le C(P,Q) \le MK(P,Q) \tag{2.12}$$

and $$m\,K(Q, P) \le C_\rho(P, Q) - C(P,Q) \le M\,K(Q, P) \tag{2.13}$$

(iii) The case s = $\frac{1}{2}$ provides the information bounds in terms of the Hellinger's discrimination,

H (P, Q)[14],

$$4mH(P,Q) \le C(P,Q) \le 4MH(P,Q) \tag{2.14}$$

and $$4m\,[\eta_{1/2}(P, Q) - H(P,Q)] \le C_\rho(P,Q) - C(P,Q) \le 4M\,[\eta_{1/2}(P, Q) - H(P, Q)] \tag{2.15}$$

where

$$H(P,Q) = \sum \frac{(\sqrt{p} - \sqrt{q})^2}{2} \tag{2.16}$$

For s = 0, the information bounds in terms of the Kullback-Leibler and χ^2 – divergences.

$$mK(P,Q) \le C(P,Q) \le MK(P,Q) \tag{2.17}$$

and

$$m\,[\chi^2(Q, P) - K(Q, P)] \le C_\rho(P, Q) - C(P, Q) \le M\,[\chi^2(Q, P) - K(Q, P)] \tag{2.18}$$

3. NEW INFORMATION DIVERGENCE MEASURE

We consider the function $f : (0, \infty) \rightarrow R$ given by

$$f_k(u) = \frac{(u - 1)^{k+1}}{u^{k/2}}, \qquad k = 1, 3, 5, 7- - - - \tag{3.1}$$

Since

$$f_k'(u) = \frac{(u-1)^k (uk + 2u + k)}{2u^{\frac{k}{2}+1}}, \tag{3.2}$$

and $$f_k''(u) = \frac{(u-1)^{k-1}\,(u^2k^2 + 2u^2k + 2uk^2 + k^2 + 2k)}{4u^{\frac{k}{2}+2}} \tag{3.3}$$

which is obviously positive for k = 1, 3, 5, 7- - - -. Hence $f(u)$ is convex for all u > 0. Thus we have the following series of convex functions

$$\frac{(u-1)^2}{u^{1/2}},\ \frac{(u-1)^4}{u^{3/2}},\ \frac{(u-1)^6}{u^{5/2}},\ \frac{(u-1)^8}{u^{7/2}}\ldots\ldots\ldots\ldots .$$

Further we know that if $f_1(u), f_2(u), f_3(u), f_4(u)\ldots\ldots\ldots$ are convex functions then the function

$$c_1 f_1(u) + c_2 f_2(u) + c_3 f_3(u) + c_4 f_4(u) + \ldots\ldots\ldots\ldots$$

is also convex where $c_1, c_2, c_3, c_4\ldots\ldots\ldots$ are positive constants such that at least one c_i is not equal to zero. Now taking

$$c_1 = 1,\ c_2 = 1,\ c_3 = \frac{1}{2!},\ c_4 = \frac{1}{3!}\ldots\ldots \text{ and}$$

$$f_1(u) = \frac{(u-1)^2}{u^{1/2}}, f_2(u) = \frac{(u-1)^4}{u^{3/2}}, f_3(u) = \frac{(u-1)^6}{u^{5/2}}, f_4(u) = \frac{(u-1)^8}{u^{7/2}}\ldots\ldots$$

we have the following convex function

$$\frac{(u-1)^2}{u^{1/2}} + \frac{(u-1)^4}{u^{3/2}} + \frac{(u-1)^6}{2!u^{5/2}} + \frac{(u-1)^8}{3!u^{7/2}} + \ldots\ldots\ldots\ldots$$

$$= \frac{(u-1)^2}{u^{1/2}}\left\{1 + \frac{(u-1)^2}{u} + \frac{(u-1)^4}{2!u^2} + \frac{(u-1)^6}{3!u^3}\ldots\ldots\right\}$$

$$= \frac{(u-1)^2}{u^{1/2}}\exp\left\{\frac{(u-1)^2}{u}\right\}$$

where exp {.} denotes the exponential function and the following divergence measure of Csiszar's f – divergence class

$$\sum \frac{(p - q)^2}{\sqrt{pq}} \exp\left\{ \frac{(p - q)^2}{pq} \right\} \tag{3.4}$$

Similarly if we take

$$c_1 = 1,\ c_2 = 1,\ c_3 = \frac{1}{2!},\ c_4 = \frac{1}{3!} \text{.............. and}$$

$$f_1(u) = \frac{(u-1)^4}{u^{3/2}},\ f_2(u) = \frac{(u-1)^6}{u^{5/2}},\ f_3(u) = \frac{(u-1)^8}{u^{7/2}},\ f_4(u) = \frac{(u-1)^{10}}{u^{9/2}} \text{........}$$

then we obtain the following divergence measure of Csiszar's f – divergence class

$$\sum \frac{(p - q)^4}{(pq)^{3/2}} \exp\left\{ \frac{(p - q)^2}{pq} \right\} \tag{3.5}$$

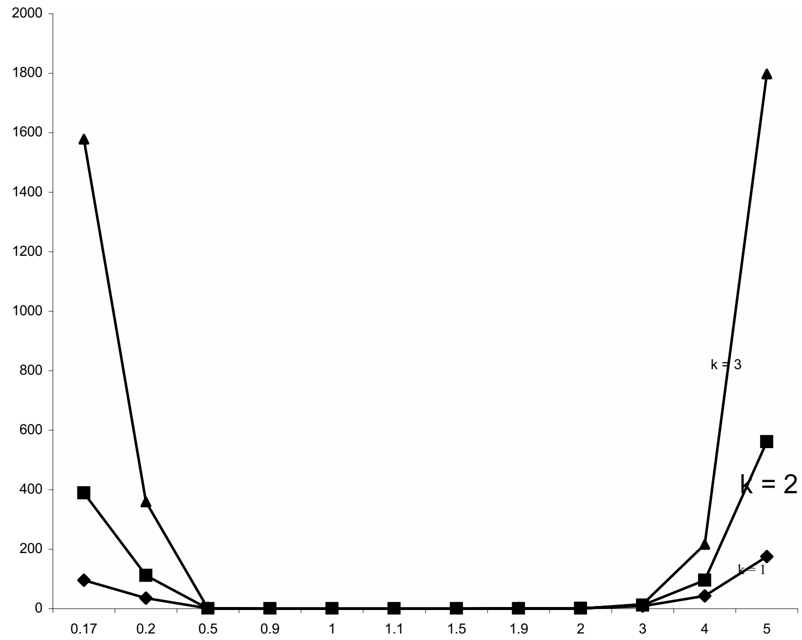
Similarly an appropriate selection of constants and convex functions will result in the following series
of convex functions

$$f_k^*(u) = \frac{(u - 1)^{k+1}}{(u)^{k/2}} \exp\left\{ \frac{(u - 1)^2}{u} \right\} \quad , k = 1, 3, 5, 7\text{- - - -} \tag{3.6}$$

and the following series of divergence measures of Csiszar's f – divergence class

$$J_k^*(P,Q) = \sum \frac{(p - q)^{k+1}}{(pq)^{k/2}} \exp\left\{ \frac{(p - q)^2}{pq} \right\}, k = 1, 3, 5, 7\text{- - - -} \tag{3.7}$$

Fig 3.1 Graph of the convex function $f^*_k(u)$



X Axis – u

Y Axis- $f^*_k(u)$

It is clear from the above graph that the convex function $f^*_k(u)$ gives a steeper slope with increase in value of k. Further $f^*_k(1) = 0$ so that $J^*_k(P,P) = 0$ and the convexity of the function $f^*_k(u)$ ensures that the measure (3.7) is nonnegative. Thus we can say that the measure (3.7) is nonnegative and convex in the pair of probability distributions (P, Q) Є Ω. Since

$$E_1^*(P,Q) = \sum \frac{(p-q)^2}{(pq)^{1/2}} \exp\{\frac{(p-q)^2}{pq}\}$$

$$= \sum \frac{1}{2}\{\frac{(p-q)^2(p+q)}{pq}\}\{\frac{2}{p+q}\}\{\sqrt{pq}\}\exp\frac{1}{2}\{\frac{(p-q)^2(p+q)}{pq}.\frac{2}{p+q}\}$$

$$E_2^*(P,Q) = \sum \frac{(p-q)^4}{(pq)^{3/2}} \exp\{\frac{(p-q)^2}{pq}\}$$

$$= \sum \frac{1}{4}\{\frac{(p-q)^2(p+q)}{pq}\}^2\{\frac{2}{p+q}\}^2\{\sqrt{pq}\}\exp\frac{1}{2}\{\frac{(p-q)^2(p+q)}{pq}.\frac{2}{p+q}\}$$

$$E_3^*(P,Q) = \sum \frac{(p-q)^6}{(pq)^{5/2}} \exp\{\frac{(p-q)^2}{pq}\}$$

$$= \sum \frac{1}{8}\{\frac{(p-q)^2(p+q)}{pq}\}^3\{\frac{2}{p+q}\}^3\{\sqrt{pq}\}\exp\frac{1}{2}\{\frac{(p-q)^2(p+q)}{pq}.\frac{2}{p+q}\}$$

and so on. Therefore we can say that the measure $J_k^*(P,Q)$ is made up of symmetric Chi-square, arithmetic and geometric mean divergence measures.

4. RELATIONSHIP WITH OTHER DIVERGENCE MEASURES

The following measures of divergence are well known in the literature on Information theory & Statistics.

- **Triangular Discrimination [Δ (P,Q)]**

$$\Delta(P,Q) = 2\,[1 - W(P,Q)] = \sum \frac{(p-q)^2}{p+q}$$

where $W(P,Q) = \sum \frac{2pq}{p+q}$ is the well-known Harmonic mean divergence.

- **Symmetric Chi-square Divergence [Ψ (P,Q)]**

$$\Psi(P,Q) = \chi^2(P,Q) + \chi^2(Q,P) = \sum \frac{(p-q)^2(p+q)}{pq} \tag{4.2}$$

- **J – Divergence [J (P,Q)]**

$$J(P,Q) = \sum (p - q) \ln \frac{p}{q} \tag{4.3}$$

- **Jensen – Shannon Divergence [I (P,Q)]**

$$I(P,Q) = \frac{1}{2}\left[\sum p \ln \frac{2p}{p+q} + \sum q \ln \frac{2q}{p+q}\right] \tag{4.4}$$

- **Arithmetic – Geometric Mean Divergence [T (P,Q)]**

$$T(P,Q) = \sum \frac{p+q}{2} \ln \frac{p+q}{2\sqrt{pq}} \tag{4.5}$$

- **Symmetric Chi-square, Arithmetic and Geometric Mean Divergence [Ψ M (P,Q)]**

$$\Psi M(P,Q) = \sum \frac{(p^2 - q^2)^2}{2(pq)^{3/2}} \tag{4.6}$$

We call the measures given in (4.1), (4.2), (4.3), (4.4), (4.5) and (4.6) as symmetric divergence measures since they are symmetric w.r.t probability distributions P and Q. The measure (4.2) is due to Draqomir et.al. [12], and recently has been studied by Taneja [25]. The measure (4.3) is due to Jeffreys [15] and later Kullback – Leibler [17] studied it extensively. Sometimes it is called Jeffreys–Kullback–Leibler's J- divergence. The measure (4.4) is due to Sibson [22] and later Burbea and Rao [2, 3] studied it extensively. Now a days it is famous as Jenson- Shannon divergence. The measure (4.5) is due to Taneja [23] and is known by arithmetic–geometric divergence or by arithmetic–geometric mean divergence measure. The measure (4.6) is due to P. Kumar and A. Johnson [18]. The following inequalities among the symmetric divergence measure hold.

$$\frac{1}{4}\Delta(P,Q) \leq I(P,Q) \leq H(P,Q) \leq \frac{1}{8} J(P,Q) \leq T(P,Q) \leq \frac{1}{16}\Psi(P,Q) \tag{4.7}$$

[for H (P, Q) see (2.16)]

In this section we will derive inequalities relating $J_k^*(P,Q)$ (for the case k = 1) with the above divergence measures. To start with, we will derive inequalities for the divergence measures given by

$$E_1^*(P,Q)=\sum\frac{(p-q)^2}{\sqrt{pq}} \quad \text{and} \quad E_2^*(P,Q)=\sum\frac{(p-q)^4}{(pq)^{3/2}}$$

and then we will use these inequalities for relating $J_1^*(P,Q)$ with other divergence measures.

We start with the following inequalities

$$\ln x\le\frac{x-1}{\sqrt{x}}\le x-1 \quad \textit{for} \quad x>1$$

$$\frac{x-1}{\sqrt{x}}\le\ln x\le x-1 \quad \textit{for} \quad 0<x<1$$

Equalities occur for x=1.

We are interested in the inequality for $x>1$. Replacing x by $\frac{(p+q)^2}{4pq}$ we obtain

$$\ln\frac{(p+q)^2}{4pq}\le\frac{(p-q)^2}{2(p+q)\sqrt{pq}}\le\frac{(p-q)^2}{4pq}$$

Now multiplying the above equation by $\frac{(p+q)}{4}$ and summing over all x, we obtain

$$\sum\frac{p+q}{2}\ln\frac{p+q}{2\sqrt{pq}}\le\sum\frac{(p-q)^2}{8\sqrt{pq}}\le\sum\frac{(p-q)^2(p+q)}{16pq}$$

or

$$T(P,Q)\le\frac{1}{8}E_1^*(P,Q)\le\frac{1}{16}\Psi(P,Q) \qquad (4.8)$$

where T (P,Q) , Ψ (P, Q) and $E_k^*(P,Q)$ are given by (4.5),(4.2) and (3.2) respectively.

or

$$\frac{1}{4}\ \Delta\,(P, Q) \leq I\,(P, Q) \leq H\,(P, Q) \leq \frac{1}{8} J\,(P, Q) \leq T\,(P, Q) \leq \frac{1}{8} E_1^*(P,Q) \leq \frac{1}{16}\,\Psi\,(P,|\,Q)$$

[from (4.7)] (4.9)

Again consider the inequality

$$\exp\{x\} \ \geq\ x + 1\ \forall\ x\ \varepsilon\ R$$

Replacing x by $\frac{(u\ -\ 1)^2}{u}$, we obtain

$$\exp\{\frac{(u\ -\ 1)^2}{u}\}\ \geq\ \frac{u^2 - u + 1}{u}$$

or

$$\frac{(u\ -\ 1)^2}{u^{1/2}} \exp\{\frac{(u\ -\ 1)^2}{u}\}\ \geq\ \frac{(u\ -\ 1)^2(u^2 - u + 1)}{u^{3/2}} \tag{4.10}$$

$$= \frac{(u\ -\ 1)^2}{u^{1/2}} + \frac{(u\ -\ 1)^4}{u^{3/2}}$$

Therefore we have

$$\frac{(u\ -\ 1)^2}{u^{1/2}} \exp\{\frac{(u\ -\ 1)^2}{u}\}\ \geq \frac{(u\ -\ 1)^2}{u^{1/2}} + \frac{(u\ -\ 1)^4}{u^{3/2}}$$

Now replacing u by $\frac{p}{q}$, multiplying by q and finally summing over all x in the above inequality, we obtain

$$J_1^*(P,Q) \geq\ E_1^*(P,Q) + E_2^*(P,Q) \tag{4.11}$$

Now (4.9) and (4.11) gives

$$\frac{1}{4}\ \Delta(P,Q) \leq I(P,Q) \leq H(P,Q) \leq \frac{1}{8} J(P,Q) \leq T(P,Q) \leq \frac{1}{8} E_1^*(P,Q) \leq \frac{1}{8} J_1^*(P,Q) - \frac{1}{8} E_2^*(P,Q) \tag{4.12}$$

Again from (4.10), we have

$$\frac{(u-1)^2}{u^{1/2}} \exp\left\{\frac{(u-1)^2}{u}\right\} \geq \frac{(u-1)^2(u^2-u+1)}{u^{3/2}}$$

$$= \frac{(u^2-1)^2}{u^{3/2}} - \frac{4\,(u-1)^2}{u^{1/2}}$$

Therefore we have

$$\frac{(u-1)^2}{u^{1/2}} \exp\left\{\frac{(u-1)^2}{u}\right\} \geq \frac{(u^2-1)^2}{u^{3/2}} - \frac{4\,(u-1)^2}{u^{1/2}}$$

Now replacing u by $\frac{p}{q}$, multiplying by q and finally summing over all x in the above inequality, we obtain

$$E_1^*(P,Q) \geq \frac{1}{2}\ \Psi M(P,Q) - \frac{1}{4} J_1^*(P,Q) \tag{4.13}$$

The above inequality relates $\Psi M(P,Q)$ with $J_1^*(P,Q)$. We can similarly relate $J_k^*(P,Q)$ for k = 2, 3, 4, 5… with other divergence measures.

5. INFORMATION BOUNDS

We can easily derive information divergence inequalities providing bounds for $J_k^*(P,Q)$ in terms of some well known divergence measures using the inequalities mentioned in section 2 for different values of k. The readers can refer (18) for examples. Here we are omitting the details.

6. SOME NEW INFORMATION DIVERGENCE MEASURES

In this section, we will derive some new information divergence measures using the convexity property of divergence measures. We proceed as follows.

Since the sum of two convex functions is again a convex function, therefore we have the following convex functions

$$\frac{(u-1)^2}{u^{1/2}} + \frac{(u-1)^4}{u^{3/2}} = \frac{(u-1)^2(u^2-u+1)}{u^{3/2}},$$

$$\frac{(u-1)^4}{u^{3/2}} + \frac{(u-1)^6}{u^{5/2}} = \frac{(u-1)^4(u^2-u+1)}{u^{5/2}},$$

$$\frac{(u-1)^6}{u^{5/2}} + \frac{(u-1)^8}{u^{7/2}} = \frac{(u-1)^6(u^2-u+1)}{u^{7/2}}$$

……………………………

and as a result the following divergence measures of Csiszar's *f* – divergence class

$$\sum \frac{(p-q)^2\,(p^2-pq+q^2)}{(pq)^{\frac{3}{2}}},$$

$$\sum \frac{(p-q)^4\,(p^2-pq+q^2)}{(pq)^{\frac{5}{2}}},$$

$$\sum \frac{(p-q)^6\,(p^2-pq+q^2)}{(pq)^{\frac{7}{2}}}$$

……………………………………………………………………………………………

Similarly we can generate various other series of divergence measures using the properties of convex functions. Further results about these divergence measures will be discussed elsewhere.

7. PARAMETRIC MEASURE OF INFORMATION

The parametric measures of information are applicable to regular families of probability distributions i.e. to the families for which the regularity conditions are assumed to be satisfied. Let for $\theta = (\theta_1, \theta_2, ...\theta_k)$, the Fisher [16] information matrix be

$$I_x(\theta) = E_\theta \left[\frac{\partial}{\partial\theta} \log f(x,\theta) \right]^2 \text{ ,if } \theta \text{ is unvariate}$$

$$= \left\| \; E_\theta \left[\frac{\partial}{\partial\theta_i} \log f(X,\theta) \frac{\partial}{\partial\theta_j} \log f(X,\theta) \right] \; \right\| ,$$

if θ is k – variate (7.1)

Where $\| \quad \|$ denotes a k x k matrix. The regularity conditions are

R1) $f(x,\theta) > 0$ for $x \in \Omega$ and $\theta \in \Theta$

R2) $\frac{\partial}{\partial\theta_i} f(X,\theta)$ exists for all $x \in \Omega$ and $\theta \in \Theta$ and all i =1, 2 - - - k.

R3) $\frac{d}{d\theta_i} \int_A f(x,\theta)\, d\mu = \int_A \frac{d}{d\theta_i} f(x,\theta)\, d\mu$ for any

$A \in$ A [measurable space (X, A) in respect of a finite or a finite measure μ), for all $\theta \in \Theta$ and for all i.

Ferentimos and Papiopanmou [13] suggested the following method to construct the parametric measure from the non-parametric measure.

Let $k(\theta)$ be a one to one transformation of the parameter space Θ onto itself with $k(\theta) \neq \theta$.

The quantity

$$I_x[\theta, k(\theta)] = I_x[f(x,\theta), f(x, k(\theta)] \tag{7.2}$$

can be considered as a parametric measure of information based on $k(\theta)$.

This method is employed to construct the modified Csiszar's measure of information about univariate θ contained in X and based on $k(\theta)$.

$$I_x^C[\theta, k(\theta)] = \int f(x,\theta)\phi\left(\frac{f[x,k(\theta)]}{f(x,\theta)}\right) d\mu \tag{7.3}$$

Now we have the following proposition for providing the parametric measure of information from

$J_k^*(P,Q)$.

Proposition 1 – Consider the convex function

$$\phi(u) = \frac{(u-1)^{k+1}}{(u)^{k/2}} \exp\left\{ \frac{(u-1)^2}{u} \right\}, \ k = 1, 3, 5, 7- - - - \tag{7.4}$$

and the corresponding non parametric divergence measure

$$J_k^{*c}(P,Q) = \sum \frac{(p-q)^{k+1}}{(pq)^{k/2}} \exp\left\{ \frac{(p-q)^2}{pq} \right\} \tag{7.5}$$

Then the parametric measure $J_k^{*c}(P,Q)$ is the same as $J_k^*(P,Q)$.

Proof – For discrete random variables X the expression (7.3) can be written as.

$$I_X^C[\theta, k(\theta)] = \sum_{x\in\Omega} p(x)\phi\left[\frac{q(x)}{p(x)}\right] \tag{7.6}$$

From (7.4)

we have $$\phi\left(\frac{q(x)}{p(x)}\right) = \sum \frac{(p-q)^{k+1}}{p^{k/2+1}q^{k/2}} \exp\left\{ \frac{(p-q)^2}{pq} \right\} \tag{7.7}$$

where we denote p(x) and q(x) by p and q respectively. This $J_k^{*c}(P,Q)$ follows from (7.6) and (7.7) as.

$$J_k^{*c}(P,Q) = I_X^C[\theta, k(\theta)] = \sum \frac{(p-q)^{k+1}}{(pq)^{k/2}} \exp\left\{ \frac{(p-q)^2}{pq} \right\} \tag{7.8}$$

and hence the proposition.

Note that the parametric measure $J_k^{*^c}(P,Q)$ is the same as the non-parametric measure $J_k^*(P,Q)$. Further since the properties of $J_k^*(P,Q)$ do not require any regularity conditions, $J_k^*(P,Q)$ is applicable to the broad family of probability distributions including the non-regular ones.

8. CONCLUDING REMARKS

We have introduced a new symmetric divergence measure (in fact a series of divergence measures) by considering a convex function and have investigated its properties. Work on one parametric generalization of this measure is in progress and will be reported elsewhere.

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