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States on perfect pseudo-MV algebras

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Abstract

Pseudo-MV algebras were introduced by Georgescu and Iorgulescu as non-commutative extension of MV-algebras. States on pseudo-MV algebras were studied by Dvurečenskij. In this paper we will investigate the states on perfect pseudo-MV algebras. We will prove that any perfect pseudo-MV algebra admits a unique state and that there is a one to one correspondence between the local states on perfect pseudo-MV algebras and the states on ℓ-groups.

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1. INTRODUCTION

MV-algebras were introduced by C.C. Chang in [4] in 1958 as algebraic models for the Łukasiewicz infinite-valued logic. Due to D. Mundici [22], MV-algebras can be viewed as intervals of abelian ℓ-groups. More precisely, for any MV-algebra A there is an abelian ℓ-group $(G,+)$ with strong unit u such that A is isomorphic to $\Gamma(G,u) = [0,u]$ endowed with a canonical structure of MV-algebra: $x \oplus y = (x+y) \wedge u, \overline{x} = u - x, x \otimes y = (x+y-u) \vee 0$. Perfect MV-algebras were defined in [2].

A. Di Nola and A. Lettieri [20] proved that the category of perfect MV-algebras is equivalent to the category of abelian ℓ-groups. Namely, for each abelian ℓ-group $(G,+)$, considering the lexicographic product $\mathbb{Z} \times_{lex} G$ we can define the perfect MV-algebra $\Delta(G) = \Gamma(\mathbb{Z} \times_{lex} G, (1,0))$ with the operations: $(x,y) \oplus (u,v) = (1,0) \wedge (x+u, y+v)$, $(x,y)^- = (1,0) - (x,y) = (1-x,-y)$, $(x,y) \otimes (u,v) = (0,0) \vee (x+u-1, y+v)$. L.P. Belluce and A. Di Nola [1] established an isomorphism between an arbitrary MV-algebra and some subalgebra of a perfect MV-algebra.

In 1999 G. Georgescu and A. Iorgulescu defined pseudo-MV algebras as non-commutative extension of MV-algebras ([15]). A. Dvurečenskij proved in [7] that every pseudo-MV algebra is isomorphic with an interval in an ℓ-group and pseudo-MV algebras are categorical equivalent with the ℓ-groups with stong unit.

Local pseudo-MV algebras were investigated by I. Leuştean in [19]. She extended the Di Nola-Lettieri functors $\mathcal{D}$ and Δ, proving that the category of ℓ-groups is equivalent with a subcategory of perfect pseudo-MV algebras. She also proved that every pseudo-MV algebra is an interval algebra in some perfect pseudo-MV algebra, extending the Belluce-Di Nola results for the case of perfect MV-algebras.

The notion of a state is analogue to probability measure and it has a very important role in the theory of quantum structures ([11]). It was first introduced for MV-algebras by F. Kôpka, F. Chovanec ([18]) and by Mundici ([23]), while for the

case of BL-algebras the state was introduced by B. Riečan ([24]) and studied by Dvurečenskij and Hyčko ([10]). In case of non-commutative fuzzy structures, the states were introduced by A. Dvurečenskij ([8]) for pseudo-MV algebras, by G. Georgescu ([14]) for pseudo-BL algebras and by A. Dvurečenskij and J. Rachünek ([12], [13]) for bounded non-commutative Rℓ-monoids. In [3] the notion of state was extended to the case of non-commutative residuated lattices.
States on a particular class of MV-algebras, such as the perfect MV-algebras ([21]), proved to have special properties, so we will extend these results to perfect pseudo-MV algebras. We will prove that any perfect pseudo-MV algebra admits a unique state and that there is a one to one correspondence between the local states on the perfect pseudo-MV algebras and the states on ℓ-groups.

2. PRELIMINARIES ON PSEUDO-MV ALGEBRAS AND PERFECT PSEUDO-MV ALGEBRAS

In this section we will point out some basic definitions and results concerning pseudo-MV algebras. For unproved results or unexplained notions we refer to [15].

DEFINITION 2.1. *A pseudo-MV algebra is a structure* $(A, \oplus, ^-, ^\sim, 0, 1)$ *of type* $(2,1,1,0,0)$ *with an additional binary operation* $\odot$ *defined via* $y \odot x = (x^- \oplus y^-)^\sim$, *such that the following axioms hold for all* $x, y \in A$:
(A_1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A_2) $x \oplus 0 = 0 \oplus x = x$;
(A_3) $x \oplus 1 = 1 \oplus x = 1$;
(A_4) $1^\sim = 0; 1^- = 0$;
(A_5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A_6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;
(A_7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A_8) $(x^-)^\sim = x$.

We consider that the operation $\odot$ has priority to the operation $\oplus$.

EXAMPLE 2.2. *([15]) Consider an arbitrary* ℓ*-group* $(G, +, -, 0)$ *and let* $u \in G, u \geq 0$. *If we put by definition:*

$$x \oplus y = (x + y) \wedge u, \;\; x^- = u - x, \;\; x^\sim = -x + u,$$

then $(A = [0, u], \oplus, ^-, ^\sim, 0, u)$ *is a pseudo-MV algebra.*

EXAMPLE 2.3. *([6]) Let* $G = (\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}, +, (0,0,0), \leq)$ *be the* Scrimger 2-group. *The group operation* $+$ *is defined by:*

$$(k_1, m_1, n_1) + (k_2, m_2, n_2) := \begin{cases} (m_1 + k_2, m_2 + k_1, n_1 + n_2) \text{ if } n_2 \text{ is odd} \\ (k_1 + k_2, m_1 + m_2, n_1 + n_2) \text{ if } n_2 \text{ is even.} \end{cases}$$

The neutral element is $0 = (0,0,0)$ *and*

$$-(k, m, n) = \begin{cases} (-m, -k, -n)) \text{ if } n \text{ is odd} \\ (-k, -m, -n) \text{ if } n \text{ is even.} \end{cases}$$

The order relation on G *is* $(k_1, m_1, n_1) \leq (k_2, m_2, n_2)$ *iff*

$$(i)\ n_1 < n_2 \;\; or \;\; (ii)\;\; n_1 = n_2,\; k_1 \leq k_2,\; m_1 \leq m_2.$$

Then

$$(k_1,m_1,n_1)\vee(k_2,m_2,n_2)=\begin{cases}(k_1,k_2,n_1) & \text{if } n_1>n_2\\ (k_1\vee k_2,m_1\vee m_2,n_1\vee n_2) & \text{if} n_1=n_2\\ (k_2,m_2,n_2) & \text{if } n_1<n_2.\end{cases}$$

One can check that G is a non-abelian ℓ-group which is not linearly ordered and that $u=(1,1,1)$ is a strong unit of G. The positive cone of G is

$$G^+=\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}^+_{>0}\cup\mathbb{Z}^+\times\mathbb{Z}^+\times\{0\}.$$

The coresponding pseudo-MV algebra has the form

$$A=\Gamma(G,u)=\mathbb{Z}^+\times\mathbb{Z}^+\times\{0\}\times\mathbb{Z}_{\leq 1}\times\mathbb{Z}_{\leq 1}\times\{1\}$$

with the operations defined as follows:

$$\begin{aligned}&(k,m,0)^-=(1-k,1-m,1),\\ &(k,m,0)^\sim=(1-m,1-k,1),\\ &(k,m,1)^-=(1-m,1-k,0),\\ &(k,m,1)^\sim=(1-k,1-m,0),\\ &(k_1,m_1,0)\oplus(k_2,m_2,0)=(k_1+k_2,m_1+m_2,0),\\ &(k_1,m_1,0)\oplus(k_2,m_2,1)=((m_1+k_2)\wedge 1,(m_2+k_1)\wedge 1,1),\\ &(k_1,m_1,1)\oplus(k_2,m_2,0)=((k_1+k_2)\wedge 1,(m_1+m_2)\wedge 1,1),\\ &(k_1,m_1,1)\oplus(k_2,m_2,1)=(1,1,1).\end{aligned}$$

The properties of pseudo-MV algebras are deeply studied in [15].

In the sequel we will use the notations:
$(x^-)^-=x^{--}$; $(x^\sim)^\sim=x^{\sim\sim}$; $(x^-)^\sim=x^{-\sim}$; $(x^\sim)^-=x^{\sim -}$;
$0x=0$, $(n+1)x=nx\oplus x$ for $n\geq 1$; $x^0=1$, $x^{n+1}=x^n\odot x$ for $n\geq 1$.

Proposition 2.4. *([15]) In a pseudo MV-algebra A the following are equivalent:*
(a) $x^-\oplus y=1$ *;*
(b) $y^\sim\odot x=0$ *;*
(c) $y=x\oplus x^\sim\odot y$ *;*
(d) $x=x\odot(x^-\oplus y)$ *;*
(e) there is $a\in A$ such that $y=x\oplus a$;
(f) $x\odot y^-=0$ *;*
(g) $y\oplus x^\sim=1$.

We define $x\leq y$ iff one of the above equivalent conditions holds and " $\leq$ " defines an order relation on A ([15]) . Moreover, A is a distributive lattice with the lattice operations defined as bellow:

$$x\vee y=x\oplus x^\sim\odot y=y\oplus y^\sim\odot x=x\odot y^-\oplus y=y\odot x^-\oplus x,$$
$$x\wedge y=x\odot(x^-\oplus y)=y\odot(y^-\oplus x)=(x\oplus y^\sim)\odot y=(x\oplus y^\sim)\odot y.$$

We can define two implications corresponding to the two negations:

$$x\to y=x^-\oplus y=(y^\sim\odot x)^- \text{ and } x\rightsquigarrow y=y\oplus x^\sim=(x\odot y^-)^\sim$$

for all $x,y\in A$.

On a pseudo-MV algebra A we can define two distance functions $d_1,d_2:A\times A\to A$:

$$d_1(x,y)=x\odot y^-\oplus y\odot x^-\ ,\ d_2(x,y)=x^\sim\odot y\oplus y^\sim\odot x.$$

Proposition 2.5. *([15]) In a pseudo-MV algebra A the following hold:*
(1) $d_1(x,y)=x\odot y^-\vee y\odot x^-,\quad d_2(x,y)=x^\sim\odot y\vee y^\sim\odot x$;
(2) $d_1(x,y)=0$ *iff* $x=y,\quad d_2(x,y)=0$ *iff* $x=y$;

(3) $d_1(x,0) = d_2(x,0) = x, \quad d_1(0,y) = d_2(0,y) = y$;
(4) $d_1(x,1) = x^-, \quad d_2(x,1) = x^\sim$;
(5) $d_1(1,y) = y^-, \quad d_2(1,y) = y^\sim$;
(6) $d_1(x,y) = d_1(y,x), \quad d_2(x,y) = d_2(y,x)$.

DEFINITION 2.6. *The order of an element x, denoted $ord(x)$, is the least $n \in \mathbb{N}$ such that $nx = 1$ if such n exists, and ∞ otherwise.*

PROPOSITION 2.7. *([15]) In any pseudo-MV algebra the following hold:*
(1) $ord(x^-) = ord(x^\sim)$;
(2) $ord(x) = ord(x^{--}) = ord(x^{\sim\sim})$.

If A is a pseudo-MV algebra, we denote:

$$D(A) = \{x \in A | ord(x) = \infty\}.$$

Obviously, $0 \in D(A)$.

LEMMA 2.8. *([19]) If A is a pseudo-MV algebra, then:*
$\{x \in A | x \geq y^- \text{ for some } y \in D(A)\} = \{x \in A | x \geq y^\sim \text{ for some } y \in D(A)\}$.

We denote:
$D(A)^* = \{x \in A | x \geq y^-$ for some $y \in D(A)\} = \{x \in A | x \geq y^\sim\}$ for some $y \in D(A)\}$.
Obviously, $1 \in D(A)^*$.

DEFINITION 2.9. *A nonempty subset I of a pseudo-MV algebra A is called a* ideal *if the following conditions are satisfied:*
$(I1)$ *if $x \in I$ and $y \in A$, $y \leq x$, then $y \in I$;*
$(I2)$ *if $x, y \in I$ then $x \oplus y \in I$.*
An ideal I is normal *if the following condition holds:*
$(I3)$ *for every $x, y \in A$, $y \odot x^- \in I$ iff $x^\sim \odot y \in I$.*
An ideal I of A is called proper *if $I \neq A$. An ideal I of A is called* maximal *if it is proper and for each ideal $J \neq I$, if $I \subseteq J$, then $J = A$.*
An ideal I of A is called principal *if there is an element $a \in A$ such that I is the ideal* generated *by $\{a\}$, i.e. $I = id(a) = \{x \in A | x \leq na$ for some $n \in \mathbb{N}\}$.*

For any normal ideal I of A we can associate a congruence defined by:

$$x \cong_I y \quad \text{iff} \quad d_1(x,y) \in I \quad \text{iff} \quad d_2(x,y) \in I.$$

Denote by A/I the set of congruence classes and with x/I the congruence class of an element $x \in A$. Then, A/I becomes a pseudo-MV algebra with the operations induced by those of A. We remark that $x/I = 0/I$ iff $x \in I$.

DEFINITION 2.10. *A nonempty subset I of a pseudo-MV algebra A is called a* filter *if the following conditions are satisfied:*
$(F1)$ *if $x \in I$ and $y \in A$, $y \geq x$, then $y \in I$;*
$(F2)$ *if $x, y \in I$ then $x \odot y \in I$.*

DEFINITION 2.11. *A pseudo-MV algebra A is called* local *if for every $x, y \in A$ the following condition hold: $ord(x \oplus y) < \infty$ implies $ord(x) < \infty$ or $ord(y) < \infty$.*

REMARK 2.12. *If A is local then $ord(x) < \infty$ or $ord(x^-) < \infty$.*

PROPOSITION 2.13. *([19]) Let A be a pseudo-MV algebra. The following are equivalent:*

(a) A is local;
(b) $D(A)$ is an ideal of A;
(c) $D(A)$ is a proper ideal of A;
(d) $D(A)$ is the unique maximal ideal of A.

PROOF. (a) iff (c) iff (d) is proved in [19];
(c) implies (b) is obviously;
(b) implies (c): We have $ord(1) = 1$, hence $1 \notin D(A)$. $\square$

PROPOSITION 2.14. *([19]) If A is a local pseudo-MV algebra then:*
(1) *$D(A)^*$ is a filter of A;*
(2) *$D(A) \cap D(A)^* = \oslash$.*

DEFINITION 2.15. *([19]) A local pseudo-MV algebra A is called* perfect *if for any $x \in A$, $ord(x) < \infty$ implies $ord(x^-) = \infty$.*

PROPOSITION 2.16. *([19]) If A is a local pseudo-MV algebra then the following are equivalent:*
(a) A is perfect;
(b) $A = D(A) \cup D(A)^*$.

DEFINITION 2.17. *The intersection of all maximal ideals of a pseudo-MV algebra is denoted by $Rad(A)$ and it is called the* radical *of A.*

COROLLARY 2.18. (1) *If A is a local pseudo-MV algebra A, then $Rad(A) = D(A)$;*
(2) *A local pseudo-MV algebra A is perfect iff $A = Rad(A) \cup Rad(A)^*$.*

PROPOSITION 2.19. *([19]) If A is a perfect pseudo-MV algebra, then:*
(1) *$Rad(A)^* = \{y \in A \mid ord(y) < \infty\} = \{x^- \mid x \in Rad(A)\} = \{x^\sim \mid x \in Rad(A)\}$;*
(2) *$x \odot y = 0$ for all $x, y \in Rad(A)$;*
(3) *If $x \in Rad(A)$ and $y \in Rad(A)^*$ then $x \leq y$;*
(4) *$(Rad(A), \oplus, 0)$ is a cancellative monoid.*

PROPOSITION 2.20. *([19]) If A is a perfect pseudo-MV algebra, then $Rad(A)$ is a normal ideal of A.*

EXAMPLE 2.21. *([19]) Let $\mathbb{Z}$ the abelian ℓ-group of integers, G an ℓ-group, $g_0 \in G^+$ and $A = \Gamma(\mathbb{Z} \times G_{lex}, (1, g_0))$ where $\mathbb{Z} \times G_{lex}$ is the lexicographic product of $\mathbb{Z}$ and G.*
One can easily prove that
$$A = \{(0, g) \mid g \in G^+\} \cup \{(1, g) \mid g \leq g_0\} \quad \text{and} \quad D(A) = \{(0, g) \mid g \in G^+\}.$$
Obviously, $D(A)$ is an ideal of A, so A is a local pseudo-MV algebra.
Because $ord(0, g) = \infty$ for all $g \in G^+$ and $ord(1, g) = 2$ for all $g \leq g_0$, we conclude that A is a perfect pseudo-MV algebra.

3. STATES ON PSEUDO-MV ALGEBRAS

In this section we will present some notions and results regarding the states on a pseudo-MV algebra.

In [9] there is defined a partial binary operation $+$ on a pseudo-MV algebra A in the following way: $x+y$ is defined iff $x \leq y^-$ and in this case $x+y = x \oplus y$. Obviously, $x+y$ is defined iff $x \leq y^-$ iff $y \leq x^\sim$ iff $y \odot x = 0$.

DEFINITION 3.1. *([8]) A* state *on a pseudo-MV algebra A is a function $s : A \to [0,1]$ such that:*
(s_1) $s(1) = 1$;
(s_2) $s(x+y) = s(x) + s(y)$ *whenever $x+y$ is defined in A.*

EXAMPLE 3.2. *([8]) Consider the pseudo-MV algebra $A = \Gamma(G,u)$ from Example 2.3 and define $s(k,m,0) = 0$ and $s(k,m,1) = 1$. Then s is the unique state on A.*

PROPOSITION 3.3. *([8]) Let s be a state on a pseudo-MV algebra A. Then, for all $x, y \in A$ the following hold:*
(1) $s(0) = 0$;
(2) *if $x \leq y$, then $s(x) \leq s(y)$;*
(3) *if $x \leq y$, then $s(y) - s(x) = s(x^\sim \odot y) = s(y \odot x^-)$;*
(4) $s(x^-) = s(x^\sim) = 1 - s(x)$;
(5) $s(x^{--}) = s(x^{\sim\sim}) = s(x)$;
(6) $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$;
(7) $s(x \oplus y) + s(y \odot x) = s(x) + s(y)$.

PROPOSITION 3.4. *Let s be a state on a pseudo-MV algebra A. Then, for all $x, y \in A$ the following hold:*
(1) $s(x) + s(x \to y) = s(y) + s(y \to x)$;
(2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$;
(3) $s(d_1(x,y)) = s(d_2(x,y)) = s(x \vee y) - s(x \wedge y)$.

PROOF. Since the pseudo-MV algebras are particular cases of pseudo-BL algebras, the above assertions follow from a similar result proved in [14] for pseudu-BL algebras. $\square$

PROPOSITION 3.5. *([8]) If s is a state on a pseudo-MV algebra A, then $Ker(s) = \{x \in A \mid s(x) = 0\}$ is a proper normal ideal of A.*

PROPOSITION 3.6. *([8]) Let A be a pseudo-MV algebra and s a state on A. For all $x, y \in A$, the following are equivalent:*
(a) $x/Ker(s) = y/Ker(s)$;
(b) $s(x \wedge y) = s(x \vee y)$;
(c) $s(x) = s(y) = s(x \wedge y)$.

PROPOSITION 3.7. *([8]) Let A be a pseudo-MV algebra and s a state on A. Then:*
(1) *the function $\hat{s} : A/Ker(s) \to [0,1]$ defined by $\hat{s}(x/Ker(s)) = s(x)$ is a state on $A/Ker(s)$;*
(2) $s(x \oplus y) = s(y \oplus x)$;
(3) $s(x \odot y) = s(y \odot x)$.

Let us denote by $[0,1]$ the standard MV-algebra of the real unit interval $[0,1]$ with the operations: $x \oplus y = \min\{x+y, 1\}$, $x \odot y = \max\{x+y-1, 0\}$, $x^- = 1 - x$.

DEFINITION 3.8. *([8]) Let A be a pseudo-MV algebra. A* state-morphism *on A is a function $s : A \to [0,1]$ such that the following conditions hold for all $x, y \in A$:*
(sm_1) $s(x \oplus y) = s(x) \oplus s(y)$;
(sm_2) $s(x^-) = s(x^\sim) = 1 - s(x)$;
(sm_3) $s(1) = 1$.

Obviously, any state-morphism is a state, but in general, the converse does not hold.

PROPOSITION 3.9. *([8]) A state s on a pseudo-MV algebra A is a state-morphism iff*

$$s(x \wedge y) = min\{s(x), s(y)\} \text{ for all } x, y \in A.$$

PROPOSITION 3.10. *([8]) A state s on a pseudo-MV algebra A is a state-morphism iff $Ker(s)$ is a maximal ideal of A.*

PROPOSITION 3.11. *([8]) If I is a normal and maximal ideal of a pseudo-MV algebra A, then there is a unique state-morphism on A such that $Ker(s) = I$.*

PROPOSITION 3.12. *([8]) Let s_1 and s_2 be two state-morphisms on a pseudo-MV algebra A such that $Ker(s_1) = Ker(s_2)$. Then $s_1 = s_2$.*

4. STATES ON PERFECT PSEUDO-MV ALGEBRAS

In this section we will prove first that any perfect pseudo-MV algebra admits a unique state. We will define the local states on a perfect pseudo-MV algebra and we will investigate their relation with the states defined on the corresponding abelian ℓ-group.

THEOREM 4.1. *Any perfect pseudo-MV algebra A admits a unique state.*

PROOF. If A is a perfect pseudo-MV algebra, then by Proposition 2.13, Corollary 2.18 and Proposition 2.20, $Rad(A)$ is the unique normal and maximal ideal of A. (With the other words, the unique maximal ideal of a perfect pseudo-MV algebra is normal). According to Proposition 3.11, there exists a unique state-morphism s on A such that $Ker(s) = Rad(A)$. By Proposition 3.10, the above state-morphism s is a state. □

PROPOSITION 4.2. *Let A be a perfect pseudo-MV algebra and s the unique state on A. Then $A = Ker(s) \cup Ker(s)^*$.*

PROOF. Since A is perfect it follows that $A = Rad(A) \cup Rad(A)^*$ and $Rad(A) = Ker(s)$. Hence, $A = Ker(s) \cup Ker(s)^*$. □

COROLLARY 4.3. *If s is the unique state on a perfect pseudo-MV algebra A, then $s(A) = \{0, 1\}$.*

PROOF. Consider $x \in A$, so $x \in Ker(s)$ or $x \in Ker(s)^*$. It follows that $x \in Ker(s)$ or $x^- \in Ker(s)$, so $s(x) = 0$ or $s(x^-) = 0$. Hence, $s(x) = 0$ or $s(x) = 1$. Thus, $s(A) = \{0, 1\}$. □

DEFINITION 4.4. *If A is a perfect pseudo-MV algebra, then a* local state *on A is a function $s : Rad(A) \to \mathbb{R}_+$ satisfying the conditions:*
(ls_1) $s(0) = 0$;
(ls_2) $s(x \oplus y) = s(x) + s(y)$ *for all* $x, y \in Rad(A)$.
If $a \in A$ such that $Rad(A) = Id(a)$ then a local state s on A is normalized *if $s(a) = 1$. A local state s is* faithful *if $s(x) \neq 0$ for all $x \in Rad(A)$, $x \neq 0$.*

PROPOSITION 4.5. *Let A be a perfect pseudo-MV algebra and s a local state on A. Then, for all $x, y \in Rad(A)$ the following hold:*
(1) *if $x \leq y$ then $s(y) - s(x) = s(y \odot x^-) = s(x^\sim \odot y)$;*
(2) $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$;
(3) $s(x \oplus y) + s(y \odot x) = s(x) + s(y)$.

PROOF. (1) From $y \in Rad(A)$, $y \odot x^- \leq y$ and $x^\sim \odot y \leq y$, we get $y \odot x^- \in Rad(A)$ and $x^\sim \odot y \in Rad(A)$. From $x \leq y$ and (A_6) we get $y = x \oplus x^\sim \odot y = y \odot x^- \oplus x$.
Hence, $s(y) = s(x) + s(x^\sim \odot y) = s(y \odot x^-) + s(x)$.
Thus, $s(y) - s(x) = s(x^\sim \odot y) = s(y \odot x^-)$.
(2) Since $x \wedge y \leq y$, it follows that $s(y) - s(x \wedge y) = s(y \odot (x \wedge y)^-) = s(y \odot x^-)$.
Similarly, from $x \leq x \vee y$ we get $s(x \vee y) - s(x) = s((x \vee y) \odot x^-) = s(y \odot x^-)$.
It follows that $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$;
(3) According to [15] Prop. 1.25 we have $x = (x \oplus y) \odot y^- \oplus y \odot x$.
Hence, $s(x) = s((x \oplus y) \odot y^- \oplus y \odot x) = s((x \oplus y) \odot y^-) + s(y \odot x)$.
Because $y \leq x \oplus y$, we get $s(x \oplus y) - s(y) = s((x \oplus y) \odot y^-) = s(x) - s(y \odot x)$.
Thus, $s(x \oplus y) + s(y \odot x) = s(x) + s(y)$. $\square$

COROLLARY 4.6. *If s is a local state on a perfect pseudo-MV algebra A and $x, y \in Rad(A)$ such that $x \leq y$, then $s(x) \leq s(y)$.*

PROPOSITION 4.7. *Let A be a perfect pseudo-MV algebra and a function $s : Rad(A) \to \mathbb{R}_+$. Then, for all $x, y \in Rad(A)$ the following are equivalent:*
(a) $s(x \oplus y) = s(x) + s(y)$;
(b) $s(d_1(x, y)) = s(d_2(x, y)) = s(x \vee y) - s(x \wedge y)$.

PROOF. $(a) \Rightarrow (b)$: Since $x \wedge y \leq x \vee y$, applying Prop. 4.5 (1) we have:

$$s(x \vee y) - s(x \wedge y) = s((x \vee y) \odot (x \wedge y)^-) = s(d_1(x, y)),$$
$$s(x \vee y) - s(x \wedge y) = s((x \wedge y)^\sim \odot (x \vee y)) = s(d_2(x, y)).$$

$(b) \Rightarrow (a)$: If $a \leq b$, then $a \odot b^- = 0$ and $b^\sim \odot a = 0$. Hence:

$$d_1(a, b) = (a \odot b^-) \vee (b \odot a^-) = b \odot a^-,$$
$$d_2(a, b) = (a^\sim \odot b) \vee (b^\sim \odot a) = a^\sim \odot b.$$

It follows that:

$$s(d_1(a, b)) = s(a \vee b) - s(a \wedge b), \text{ so } s(b \odot a^-) = s(b) - s(a),$$
$$s(d_2(a, b)) = s(a \vee b) - s(a \wedge b), \text{ so } s(a^\sim \odot b) = s(b) - s(a).$$

For $a = x$, $b = y \oplus x$ we get $s((y \oplus x) \odot x^-) = s(y \oplus x) - s(x)$.
Hence, $s(x^- \wedge y) = s(y \oplus x) - s(x)$. Since $x, y \in Rad(A)$ we have $y \odot x = 0$, so $y \leq x^-$.
Thus, $x^- \wedge y = y$ and $s(y) = s(y \oplus x) - s(x)$.
For $a = x$, $b = x \oplus y$ we get $s(x^\sim \odot (x \oplus y)) = s(x \oplus y) - s(x)$.
Hence, $s(x^\sim \wedge y) = s(x \oplus y) - s(x)$. Since $x, y \in Rad(A)$ we have $x \odot y = 0$, so $y \leq x^\sim$.

Thus, $x^{\sim} \wedge y = y$ and $s(y) = s(x \oplus y) - s(x)$.
We conclude that $s(x \oplus y) = s(y \oplus x) = s(x) + s(y)$. $\square$

DEFINITION 4.8. *([19]) A perfect pseudo-MV algebra A is called* strong perfect *iff*

$$x^{-} = x^{\sim} \text{ for all } x \in A.$$

REMARK 4.9. *([19]) One can easily check that the above condition is equivalent with each of the following conditions:*
(a) $x^{-} = x^{\sim}$ *for all* $x \in A$;
(b) $x = x^{--} = x^{\sim\sim}$ *for all* $x \in Rad(A)$;
(c) $x = x^{--} = x^{\sim\sim}$ *for all* $x \in A$.

LEMMA 4.10. *([19]) Let A be a strong perfect pseudo-MV algebra, $G = \mathcal{D}(A)$ and*
$f : Rad(A) \to G^{+}$ an isomorphism of latticial monoids. If $a \in Rad(A)$, then the following are equivalent:
(a) $\{a\}$ generates $Rad(A)$;
(b) $f(a)$ is a strong unit of G.

By extending the Di Nola-Lettieri functors $\mathcal{D}$ and Δ, I. Leuştean proved in [19] that the category of ℓ-groups is equivalent with the category of strong perfect pseudo-MV algebras. Let A be a strong perfect pseudo-MV algebra. We only remind the steps of construction for the extended functors $\mathcal{D}$ and Δ in the case of non-commutative structures:
(S_1) There exists a ℓ-group $\mathcal{D}(A)$ associated with A such that $Rad(A)$ and $\mathcal{D}(A)^{+}$ are isomorphic latticial monoids and $\mathcal{D}$ is a functor from the category of strong perfect pseudo-MV algebras to the category of ℓ-groups. More precisely, if $a, b \in Rad(A)$ let b_a be the unique element in $Rad(A)$ such that $a + b_a = b + a$. We define the equivalence " $\equiv$ " on $Rad(A) \times Rad(A)$: $(a, b) \equiv (c, d)$ iff $a + d_b = c + b$. Then $\mathcal{D}(A) = Rad(A) \times Rad(A)/_{\equiv}$ is a ℓ-group with the operations: $[a, b] + [c, d] = [a + c_b, d + b]$, $-[a, b] = [b, a]$ where $[a, b]$ is the equivalence class of the element (a, b). $Rad(A)$ can be identified with the set $\{[x, 0] \mid x \in Rad(A)\}$;
(S_2) If G is a ℓ-group, then $\Delta(G) = \Gamma(\mathbb{Z} \times G_{lex}, (1, 0))$ is a strong perfect pseudo-MV algebra and Δ is a functor from the category of ℓ-groups to the category of strong perfect pseudo-MV algebras; ;
(S_3) A and $\Delta(\mathcal{D}(A))$ are isomorphic pseudo-MV algebras. If G is an ℓ-group, then G and $\mathcal{D}(\Delta(A))$ are isomorphic ℓ-groups;
(S_4) The functors $\mathcal{D}$ and Δ establish a categorical equivalence between the category of strong perfect pseudo-MV algebras and the category of ℓ-groups;
(S_5) The restrictions of functors $\mathcal{D}$ and Δ establish a categorical equivalence between the category of strong perfect pseudo-MV algebras with $Rad(A)$ principal ideal and the category of unital ℓ-groups;
(S_6) The category of pseudo-MV algebras is equivalent with the category of strong perfect pseudo-MV algebras with $Rad(A)$ principal ideal. Indeed, if A is a pseudo-MV algebra, then according to [7] there exists a unital ℓ-group (G, u) such that A is isomorphic with $\Gamma(G, u)$. The strong perfect pseudo-MV algebra associated to A will be $\bar{A} = \Delta(G)$. Conversely, if A is a strong MV-algebra such that $Rad(A)$ is a

principal ideal then consider $(\mathcal{D}(A), u)$ the unital ℓ-group associated with A. The pseudo-MV algebra corresponding to A is $\Gamma(\mathcal{D}(A), u)$.

THEOREM 4.11. *If A is a strong perfect pseudo-MV algebra and G an ℓ-group such that $G = \mathcal{D}(A)$, then there is a one to one correspondence between the local states on A and the states on G. Under this correspondence, if $Rad(A) = Id(a)$ and $u = [a, 0]$ is a strong unit of G, the normalized local states on A are mapped onto normalized states on G.*

PROOF. We recall that a state on a unital ℓ-group (G, u) with strong unit u is a function $s : G \to \mathcal{R}$ such that: (i) $s(g_1 + g_2) = s(g_1) + s(g_2)$ for all $g_1, g_2 \in G$, (ii) $s(g) \geq 0$ for all $g \in G$. A state s on (G, u) is *normalized* if $s(u) = 1$.
If s is a state on G, we define a local state s_A on A by $s_A(x) = s([x, 0])$ for all $x \in Rad(A)$.
Conversely, if s is a local state on A, we define a state s_G on G by $s_G([x, 0]) = s(x)$ for each $[x, 0] \in G^+$. $\square$

REMARK 4.12. *It was proved in [16] that even in Abelian ℓ-group there exists a nonunital Dedeking ℓ-group G such that a unique real valued positive mapping is the zero function. Consequently, there is a perfect MV algebra such that a unique local state is the zero function, namely in $\Delta(G)$.*

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