Journal of Applied Mathematics, Statistics and Informatics (JAMSI), **3** (2007), No. 2

On the Connectedness of Jet Groups and Groups of Automorphisms of Weil Algebras[1]

MIROSLAV KUREŠ

Abstract

The connectedness of jet groups for Zariski topology and usual real topology is completely solved. Jet groups are also described as certain groups of automorphisms of Weil algebras, too. Further, it is demonstrated that both connectedness and disconnectedness can occur for groups of automorphisms of general Weil algebras.

Mathematics Subject Classification 2000: 22E15, 58A20, 20G20
General Terms: Algebraic group, Lie group, Weil algebra, group of automorphisms

1. STARTING POINTS

Let K be a field. We recall (see [2]) that by an affine variety we mean the zero set in K^n of finite collection of polynomials. A variety endowed with the structure of a group is called the *algebraic group*. We study the case $K = \mathbb{R}$ in this paper. A (real) differentiable manifold with the structure of a group is called the *Lie group*. We investigate the problem of connectedness of some groups: first, we recall the classical case of general linear groups, second, we study jet groups, investigated only as Lie groups up to now. We also compare a classical view to jet groups with their realizing as groups of automorphisms of the Weil algebra $\mathbb{D}_n^r$ and bring essential examples for the study of the connectedness of Weil algebras.

Usual topology on $\mathbb{R}^n$ is induced by the Euclidean metric with open sets obtained as unions of open balls. *Zariski topology* is defined by decreeing that the closed sets are to be precisely affine varieties. If we fasten on these particular topologies, we use prefixes $\mathcal{U}-$ and $\mathcal{Z}-$, respectively (e.g. $\mathcal{U}-$connectedness, etc.). We remark that the Zariski topology is coarser than the usual topology.

The set $\mathrm{GL}(n,\mathbb{R})$ of invertible matrices of the order n over $\mathbb{R}$ is $\mathcal{Z}$-closed. It can be proved by using of the embedding of $\mathrm{GL}(n,\mathbb{R})$ into $\mathbb{R}^{n^2+1}$, in which $A \in \mathrm{GL}(n,\mathbb{R})$ is mapped onto $(A, \frac{1}{\det A})$. Then $\mathrm{GL}(n,\mathbb{R})$ is represented by zeros of the polynomial

$$\det(x_1,\dots,x_{n^2})x_{n^2+1} - 1$$

in $\mathbb{R}^{n^2+1}$ and that is why it is $\mathcal{Z}$-closed. Simultaneously, the set $\mathrm{GL}(n,\mathbb{R})$ is $\mathcal{Z}$-open, as its complement represents zeros of $\det(x_1,\dots,x_{n^2})$. In $\mathcal{U}$-topology, the set $\mathrm{GL}(n,\mathbb{R})$ is $\mathcal{U}$-open. The group structure of $\mathrm{GL}(n,\mathbb{R})$ is given by the matrix multiplication and $\mathrm{GL}(n,\mathbb{R})$ is viewed as one of the most important examples of

[1]The author was supported by GA ČR, grant No. 201/05/0523.

algebraic groups and Lie groups, too. Further examples can be construct from the fact that $\mathcal{Z}-$closed ($\mathcal{U}-$closed) subgroups of an algebraic (a Lie) group is again algebraic (Lie) groups.

Topological space X is *connected*, if it is not possible to express it as a disjoint union of two or more open sets. If every neighbourhood of a point contains a connected open neighbourhood, then X is called *locally connected*. If for any two points x_0, x_1 in X there exists a path from x_0 to x_1 lying in X, then X is called *pathly connected*.

The connectedness coincides with the path connectedness for differentiable manifolds endowed with the $\mathcal{U}-$topology: it follows from the known property that the connectedness and the local connectedness yields the path connectedness. Thus, we can use the path connectedness for a demonstration of the connectedness in the $\mathcal{U}-$topology. For this, we introduce so-called elementary paths.

2. ELEMENTARY PATHS IN $\mathrm{GL}(N,\mathbb{R})$

We fasten onto the $\mathcal{U}-$topology case in this section. By an *elementary transvection* $V_{ij}(r)$ $1 \le i \ne j \le n$, $r \in \mathbb{R}$, we mean a matrix $[a_{\nu\mu}]$ from $\mathrm{GL}(n,\mathbb{R})$ with

$$a_{\nu\mu} = \begin{cases} 1 & \text{for } \nu = \mu \\ r & \text{for } \nu = i, \mu = j \\ 0 & \text{otherwise} \end{cases}$$

and by an *elementary dilation* $S_i(r)$, $1 \le i \le n$, $r \in \mathbb{R}$, $r \ne 0$, we mean a matrix $[a_{\nu\mu}]$ from $\mathrm{GL}(n,\mathbb{R})$ with

$$a_{\nu\mu} = \begin{cases} 1 & \text{for } \nu = \mu \ne i \\ r & \text{for } \nu = \mu = i \\ 0 & \text{otherwise.} \end{cases}$$

We have obtain all elements of $\mathrm{GL}(n,\mathbb{R})$ as finite products of elementary transvections and elementary dilations (in general, it is not true for a ring in place of $\mathbb{R}$, see [5]). We define the *elementary transvection path* $v_{ij}(r)\colon [0,1] \to V_{ij}(r)$ by

$$v_{ij}(r)(t) = V_{ij}(rt)$$

and, for $r > 0$, the *elementary dilation path* $s_i(r)\colon [0,1] \to S_i(r)$ by

$$s_i(r)(t) = S_i(r^t).$$

Let every individual ϕ, ψ be an elementary transvection path or an elementary dilaton path and Φ, Ψ be their corresponding elementary transvections or dilations (dilations only with $r > 0$). We define the composition of paths by $\phi\psi\colon [0,1] \to \Phi\Psi$ and by the usual rule

$$\phi\psi(t) = \begin{cases} \phi(2t) & \text{for } t \in [0, \frac{1}{2}] \\ \psi(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Let $A \in \mathrm{GL}(n,\mathbb{R})$. The path in $\mathrm{GL}(n,\mathbb{R})$ will be called the *elementary path with the origin in A* if it is a multiple of A by a composition of a finite number of a elementary transvection paths and elementary dilation paths from the left.

3. THE GENERAL LINEAR GROUP

First, we recall known results about the general linear group $\mathrm{GL}(n, \mathbb{R})$.

PROPOSITION 1.

*(**A**) $\mathrm{GL}(n, \mathbb{R})$ is $\mathcal{Z}-$connected.*

*(**B**) $\mathrm{GL}(n, \mathbb{R})$ is $\mathcal{U}-$disconnected and has two $\mathcal{U}-$connected components.*

PROOF.

(**A**) Open sets are complement to closed; it means they are non-zeros of finite collection of polynomials. A typical non-empty open set can be written as the union of so-called principal open sets — sets of non-zeros of individual polynomials. The group $\mathrm{GL}(n, \mathbb{R})$ is such a principal open set in $\mathcal{Z}-$ topology, because it is defined by the nonvanishing of the individual polynomial: the determinant of the n-th order matrix. Thus, it is not possible to express it as a disjoint union of two open sets.

(**B**) There is no path between $A, B \in \mathrm{GL}(n, \mathbb{R})$ having determinants of opposite signs, because the determinant is a continuous function and there must be a point for which the determinant is equal to zero because of Bolzano theorem. Evidently, such a point does not lie in $\mathrm{GL}(n, \mathbb{R})$. On the other hand, elementary paths with the origin in A reach all points having a determinant of the same sign as A. The $\mathcal{U}-$connected component of identity in $\mathrm{GL}(n, \mathbb{R})$ is a normal subgroup in $\mathrm{GL}(n, \mathbb{R})$ having two cosets.

□

4. THE JET GROUP

Let

$$G_n^r = \mathrm{inv} J_0^r(\mathbb{R}^n, \mathbb{R}^n)_0$$

be the *r-th order jet group*, i.e. the group of all invertible jets of the order r from $\mathbb{R}^n$ into $\mathbb{R}^n$ with the source and the target in 0 with the jet composition as a group operation. We remark that

$$G_n^1 = \mathrm{GL}(n, \mathbb{R}).$$

Hence we generalize the previous case. The pleasant introduction to jet groups is in [3], where it is showed that G_n^r is a Lie group.

It follows from the construction of jets that $n\kappa$ real numbers (jet coordinates), where $\kappa = \binom{r+n}{n} - 1$ (we present this formula also as a correction of the formula written by author in [4], where the subtracted 1 is missing by mistake), express an element of G_n^r. The invertibility of jets means the invertibility of elements of underlying group $G_n^1 = \mathrm{GL}(n, \mathbb{R})$ formed by 1-jet coordinates. G_n^r is an $\mathcal{U}-$open subset of the real space of dimension $n\kappa$, which we can extend to the real space of dimension κ^2: in this space we add some new fixed coordinates by such a way that matrices of the order κ over $\mathbb{R}$ remain invertible. Such an embedding will be called *admissible*. Thus, G_n^r can be viewed as $\mathcal{U}-$open subset of $\mathrm{GL}(\kappa^2, \mathbb{R})$. We follow these facts up with the easy assertion.

Lemma 1. *The group G_n^r is an algebraic group.*

Proof. We must prove that G_n^r is $\mathcal{Z}-$closed. We can use its embedding into $\mathrm{GL}(\kappa^2,\mathbb{R})$ and denote by G_n^1 the underlying group formed by 1-jet coordinates and by $A_1 \in G_n^1$ the underlying 1-jet to a jet $A \in G_n^r$. Now, we can use further embedding of $\mathrm{GL}(\kappa^2,\mathbb{R})$ into $\mathbb{R}^{\kappa^2+2}$, in which $A \in \mathrm{GL}(\kappa^2,\mathbb{R})$ is mapped onto $(A, \frac{1}{\det A}, \frac{1}{\det A_1})$. Then G_n^r is represented by zeros of the polynomials

$$\det(x_1,\dots,x_{\kappa^2})x_{\kappa^2+1} - 1$$

and $\det(x_1,\dots,x_{n^2})x_{\kappa^2+2} - 1$ in $\mathbb{R}^{\kappa^2+2}$ and that is why it is $\mathcal{Z}$-closed. □

Consequently, we acquire finding about the connectedness.

Proposition 2.

(**A**) *G_n^r is $\mathcal{Z}-$connected.*
(**B**) *G_n^r is $\mathcal{U}-$disconnected and has two $\mathcal{U}-$connected components.*

Proof.

(**A**) The group G_n^r is a principal open set in $\mathbb{R}^{n\kappa}$, because it is given by the nonvanishing of the individual polynomial: the determinant of the n-th order matrix.
(**B**) There is no path between $A, B \in \mathrm{GL}(n,\mathbb{R})$ having determinants of the group G_n^1 formed by 1-jet coordinates of opposite signs. On the other hand, elementary paths with the origin in A reach all points having this determinant of the same sign as A. The $\mathcal{U}-$ connected component of identity in G_n^r is a normal subgroup in G_n^r having two cosets.

□

5. THE GROUP OF AUTOMORPHISMS OF WEIL ALGEBRA $\mathbb{D}_N^R$

The *Weil algebra* A is a local commutative $\mathbb{R}$-algebra with identity, the nilpotent ideal $\mathfrak{n}_A$ of which has a finite dimension as a vector space and $A/\mathfrak{n}_A = \mathbb{R}$. One can assume that A is expressed as a factor algebra of the algebra $\mathbb{D}_n^r$, where

$$\mathbb{D}_n^r \coloneqq \mathbb{R}[x_1,\dots,x_n]/\mathfrak{m}^{r+1},$$

$\mathbb{R}[x_1,\dots,x_n]$ being the algebra of real polynomials in several indeterminates and $\mathfrak{m}$ the maximal ideal of $\mathbb{R}[x_1,\dots,x_n]$. The group G_n^r can be viewed just as the group of algebra automorphisms of the algebra $\mathbb{D}_n^r$.

We can write elements of G_n^r as automorphisms of $\mathbb{D}_n^r$ by this way:

$$\begin{aligned} 1 &\mapsto 1 \\ x_1 &\mapsto C_{1,1}x_1 + \dots C_{1,n}x_n + Q_1(x_1,\dots,x_n) \\ &\dots \\ x_n &\mapsto C_{n,1}x_1 + \dots C_{n,n}x_n + Q_n(x_1,\dots,x_n) \end{aligned}$$

The real matrix $\mathbf{C} = \begin{pmatrix} C_{1,1} & \dots & C_{1,n} \\ \dots & \dots & \dots \\ C_{n,1} & \dots & C_{n,n} \end{pmatrix}$ is invertible, therefore belonging to $\mathrm{GL}(n,\mathbb{R})$; $Q_1(x_1,\dots,x_n)$, …, $Q_n(x_1,\dots,x_n)$ are polynomials without absolute and linear terms. It is clear that

$$\begin{aligned} (x_1)^2 &\mapsto (C_{1,1}x_1 + \dots C_{1,n}x_n + Q_1(x_1,\dots,x_n))^2 \\ & \quad \text{(modulo } (r+1)\text{-th powers)} \\ \text{etc.,} & \end{aligned}$$

because we have algebra automorphisms of $\mathbb{D}^r_n$. In a different guise, we see that $n\kappa$ coefficients of an automorphism are completed by the determinated coefficients to the total number κ^2 and a matrix of the order κ remains invertible, i.e. we have an admissible embedding.

6. THE $\mathcal{U}$-CONNECTEDNESS IN THE CASE OF GENERAL WEIL ALGEBRA

An arbitrary Weil algebra A is given by a factorization of $\mathbb{D}^r_n$ by a finitely generated ideal. Therefore the above-mention form of automorphisms is enriched by a finite number of relations between coefficients in a general case. The number n is $\dim(\mathfrak{n}_A/\mathfrak{n}_A^2)$; this number is called the *width* of a Weil algebra A. Mentioned relations are nothing that algebraic equations, so it is clear that $\mathrm{Aut}A$ is a Lie group and an algebraic group.

PROPOSITION 3. *Both cases ($\mathcal{U}-$disconnectedness and $\mathcal{U}-$connectedness) can occur for Weil algebras with width greater or equal 2.*

PROOF. For the proof, it suffices to show examples with the width 2. The first example is

$$A = \mathbb{D}^5_2/\langle xy^2 + x^5, x^2y + y^5\rangle.$$

The basis is $\mathcal{B}(A) = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^4, y^4\}$.
$\mathrm{Aut}A$ is $\mathcal{U}-$disconnected and has eight connected components:

1st component

$$\begin{aligned} x &\mapsto -x + C_{1,3}x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,10}x^4 + C_{1,11}y^4 \\ y &\mapsto -y + C_{2,4}xy + C_{2,5}y^2 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 + C_{2,11}y^4; \end{aligned}$$

2nd component

$$\begin{aligned} x &\mapsto x + C_{1,3}x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,10}x^4 + C_{1,11}y^4 \\ y &\mapsto -y + C_{2,4}xy + C_{2,5}y^2 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 + C_{2,11}y^4; \end{aligned}$$

3rd component

$$\begin{aligned} x &\mapsto -x + C_{1,3}x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,10}x^4 + C_{1,11}y^4 \\ y &\mapsto y + C_{2,4}xy + C_{2,5}y^2 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 + C_{2,11}y^4; \end{aligned}$$

4th component

$$\begin{aligned} x &\mapsto x + C_{1,3}x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,10}x^4 + C_{1,11}y^4 \\ y &\mapsto y + C_{2,4}xy + C_{2,5}y^2 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,9}y^3 + C_{2,10}x^4 + C_{2,11}y^4; \end{aligned}$$

5th component

$$\begin{aligned} x &\mapsto -y + C_{1,4}xy + C_{1,5}y^2 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 + C_{1,11}y^4 \\ y &\mapsto -x + C_{2,3}x^2 + C_{2,4}xy + C_{2,6}x^3 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,10}x^4 + C_{2,11}y^4; \end{aligned}$$

6th component

$$\begin{aligned}
x &\mapsto y + C_{1,4}xy + C_{1,5}y^2 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 + C_{1,11}y^4\\
y &\mapsto -x + C_{2,3}x^2 + C_{2,4}xy + C_{2,6}x^3 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,10}x^4 + C_{2,11}y^4;
\end{aligned}$$

7th component

$$\begin{aligned}
x &\mapsto -y + C_{1,4}xy + C_{1,5}y^2 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 + C_{1,11}y^4\\
y &\mapsto x + C_{2,3}x^2 + C_{2,4}xy + C_{2,6}x^3 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,10}x^4 + C_{2,11}y^4;
\end{aligned}$$

8th component

$$\begin{aligned}
x &\mapsto y + C_{1,4}xy + C_{1,5}y^2 + C_{1,7}x^2y + C_{1,8}xy^2 + C_{1,9}y^3 + C_{1,10}x^4 + C_{1,11}y^4\\
y &\mapsto x + C_{2,3}x^2 + C_{2,4}xy + C_{2,6}x^3 + C_{2,7}x^2y + C_{2,8}xy^2 + C_{2,10}x^4 + C_{2,11}y^4;
\end{aligned}$$

$C_{i,j} \in \mathbb{R}$.

The second example is

$$A = \mathbb{D}_2^6/\langle x^3 + y^4, x^4 + y^5\rangle.$$

The basis is $\mathcal{B}(A) = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, x^2y^2, xy^3, x^2y^3\}$.
Aut$A$ is $\mathcal{U}-$connected:

singleton component

$$\begin{aligned}
x &\mapsto x + \frac{1}{3}(-3C_{1,4} + 4(C_{2,4} + C_{2,5}))x^2 + C_{1,4}xy + C_{1,6}x^3 + C_{1,7}x^2y + C_{1,8}xy^2\\
&\quad -\frac{4C_{2,3}}{3}y^3 + C_{1,10}x^3y + C_{1,11}x^2y^2 + C_{1,12}xy^3 + C_{1,13}x^2y^3\\
y &\mapsto y + C_{2,3}x^2 + C_{2,4}xy + C_{2,5}y^2 + C_{2,6}x^3 + C_{2,7}x^2y + C_{2,8}xy^2\\
&\quad +C_{2,9}y^3 + C_{2,10}x^3y + C_{2,11}x^2y^2 + C_{2,12}xy^3 + C_{2,13}x^2y^3
\end{aligned}$$

□

Acknowledgement. The author thanks to organizers of the conference Trends in Applied Mathematics and Statistics, Budmerice 2007, for the perfect organization, hospitality and interest.

REFERENCES

[1] Brown, W.C., Matrices over Commutative Rings, Marcel Dekker 1993

[2] Humphreys, J.E., Linear Algebraic Groups, Springer Verlag 1998

[3] Kolář, I., Michor, P.W., and Slovák, J., Natural Operations in Differential Geometry, Springer Verlag 1993

[4] Kureš, M., Local approach of higher order contact elements, Reports on Mathematical Physics 58 (2006), No.2, 393–409

[5] Kureš, M., Skula, L., Smith normal form of a matrix of generalized polynomials with rational exponents, preprint

Miroslav Kureš,
Institute of Mathematics,
Brno University of Technology,
Technická 2, 61669 Brno, Czech Republic,
e-mail: kures@fme.vutbr.cz