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On Criteria of Nonoscillation of n-th Order Nonlinear Differential Equations

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Abstract

The paper deals with the criteria of nonoscillation for n-th order nonlinear differential equations $L_n y + P(t)L_{n-2}y = f\Big(t, y, L_1 y, L_2 y, ..., L_{n-3}y\Big)$.

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1. INTRODUCTION

Let us consider the following differential equation of the n-th order with classical derivatives

$$y^{(n)} = f\Big(t, y, y', y'', ..., y^{(n-1)}\Big),$$

such that a function $f\big(t, u_1, u_2, u_3, ..., u_n\big)$ is continuous and positive on a set $S = [a, \infty) \times T$, $T \subset E_1^n$. If a function $y(t)$ is an arbitrary solution of the preceding differential equation on the interval $[a, \infty)$, then, owing to the assumptions, $y^{(n)}(t) > 0$ on $[a, \infty)$. From this it implies that $y(t)$ is a nonoscillatory function (see Definition 1), i.e. the considered differential equation is nonoscillatory (see Definition 2). Such assertions which assure that considered differential equations are nonoscillatory is called the *criteria of nonoscillation.* They were derived in many papers for linear (see, for instance, [1]) as well as nonlinear differential equations. A main aim of the article is to establish the criteria of nonoscillation for certain class of nonlinear differential equations of the n-th order.

Now, let us take into account the following nonlinear differential equation of the n-th order with quasiderivatives

$$L_n y + P(t)L_{n-2}y = f\big(t, y, L_1 y, L_2 y, ..., L_{n-3}y\big), \qquad (1)$$

where

$$\begin{aligned} L_0 y(t) &= y(t), \\ L_k y(t) &= p_k(t)\big(L_{k-1}y(t)\big)', \\ L_n y(t) &= \big(L_{n-1}y(t)\big)', \end{aligned}$$

$k = 1, 2, ..., n-1$, $p_i(t)$, $i = 1, 2, ..., n-1$, $P(t)$ resp. $f\big(t, u_1, u_2, ..., u_{n-2}\big)$ are real-valued continuous functions defined on an half-closed real interval $I_a = [a, \infty)$ resp. on a set $A = I_a \times B$, $B \subset E_1^{n-2}$. We suppose $p_i(t) > 0$, $t \in I_a$, $i = 1, 2, ..., n-1$, too. Terms $L_k y(t)$, $k = 0, 1, ..., n$ are k-th *quasiderivatives* of a function $y(t)$. Let a symbol $C^{(n)}$ denote the set of all functions $y(t)$ having all continuous quasiderivatives $L_k y(t)$, $t \in I_a$, $k = 0, 1, ..., n$. It is assumed throughout that

$$\begin{array}{l}\text{for all functions } y(t) \in C^{(n)} \text{ it holds, that the function} \\ f\big(t, y(t), L_1 y(t), L_2 y(t), ..., L_{n-3} y(t)\big) \text{ is nonoscillatory.}\end{array} \tag{2}$$

We note that a concept of the nonoscillatory function is stated in Definition 1. In the whole paper we shall use a notation $I_b = [b, \infty)$ for any $b \geq a$. The main result of the article is Theorem 1; it yields the criterion of nonoscillation of the equation (1). The special case of the preceding assertion for $P(t) \leq 0$, $t \in I_a$, is a content of Theorem 2. The preceding results are illustrated on three examples.

2. DEFINITIONS AND PRELIMINARY RESULTS

Definition 1. A real function $y(t)$ defined on I_a is said to be *oscillatory* if its set of zeros is not bounded from above. Otherwise, it is called *nonoscillatory.*

Definition 2. A differential equation is said to be *oscillatory* if it admits an oscillatory solution on some interval I_a. Otherwise, it is called *nonoscillatory.*

Lemma 1. Let $A(t, s)$ be a nonnegative and continuous function for $a \leq t_0 \leq s \leq t$. If $g(t)$, $\phi(t)$ are continuous on I_{t_0} and

$$\phi(t) \leq g(t) + \int_{t_0}^{t} A(t, s)\phi(s)\, ds \text{ for } t \in I_{t_0},$$

then every solution $x(t)$ of an integral equation

$$x(t) = g(t) + \int_{t_0}^{t} A(t, s)x(s)\, ds \text{ on } I_{t_0}$$

satisfies an inequality

$$x(t) \geq \phi(t) \text{ in } I_{t_0}.$$

Proof See [2, Lemma 2], pages 168–169. □

3. AUXILIARY RESULT

Let us consider a linear differential equation of the n-th order with the quasiderivatives of the form

$$L_n y + P(t) L_{n-2} y = 0. \tag{3}$$

Lemma 2. Let $P(t) \leq 0$ on I_a. If $y(t)$ is any solution of (3) on I_a satisfying conditions $L_k y(a) > 0$, $k = n-2, n-1$, then $L_k y(t) > 0$ on I_a, $k = n-2, n-1$.

Proof An integration of (3) on $[a, t]$, $a < t$ it yields

$$L_{n-1} y(t) \;=\; L_{n-1} y(a) + \int_a^t L_n y(u)\, du$$

$$
\begin{aligned}
&= L_{n-1}y(a) + \int_a^t -P(u)\Big(L_{n-2}y(a) \\
&\quad + \int_a^u \frac{L_{n-1}y(s)}{p_{n-1}(s)}\,ds\Big)\,du \\
&= L_{n-1}y(a) + L_{n-2}y(a)\int_a^t -P(u)\,du \\
&\quad + \int_a^t -P(u)\left(\int_a^u \frac{L_{n-1}y(s)}{p_{n-1}(s)}\,ds\right)du.
\end{aligned}
$$

A change of an order of the integration in the last two-dimensional integral it gives

$$
L_{n-1}y(t) = g(t) + \int_a^t A(t,s)L_{n-1}y(s)\,ds,
$$

where

$$
g(t) = L_{n-1}y(a) + L_{n-2}y(a)\int_a^t -P(u)\,du,
$$

$$
A(t,s) = \int_s^t \frac{-P(u)}{p_{n-1}(s)}\,du.
$$

However, $g(t) > 0$ (and continuous) on I_a; similarly, $A(t,s) \geq 0$ (and continuous) for $a \leq s \leq t$. If we put $t_0 = a$, $x(t) = L_{n-1}y(t)$, $\phi(t) = g(t)$ both on I_a in Lemma 1, then we obtain $L_{n-1}y(t) \geq g(t) \geq L_{n-1}y(a) > 0$ on I_a. From this we conclude

$$
\begin{aligned}
L_{n-2}y(t) &= L_{n-2}y(a) + \int_a^t \frac{L_{n-1}y(s)}{p_{n-1}(s)}\,ds \\
&> 0 \text{ on } I_a.
\end{aligned}
$$

The lemma is proved. □

4. MAIN RESULTS

In this section we establish the mentioned criteria of nonoscillation for the nonlinear differential equation (1).

THEOREM 1. Let (2) hold. If the equation (3) admits on I_a such a solution $y(t)$ that $L_{n-2}y(t) \neq 0$ on I_a, then the equation (1) is nonoscillatory.

Proof Let $y(t)$ be a solution of the equation (3) on I_a satisfying the assumptions of the theorem. Let $z(t)$ be a solution of (1) on I_a. Let us define a function $W(t)$ in this way

$$
W(t) = \begin{vmatrix} L_{n-2}y(t) & L_{n-2}z(t) \\ p_{n-1}(t)L'_{n-2}y(t) & p_{n-1}(t)L'_{n-2}z(t) \end{vmatrix}.
$$

Then for every $t \in I_a$ it holds that (a symbol $W'(t)$ denotes the derivative of the determinant $W(t)$)

$$
W'(t) = \begin{vmatrix} L_{n-2}y(t) & L_{n-2}z(t) \\ \big(p_{n-1}(t)L'_{n-2}y(t)\big)' & \big(p_{n-1}(t)L'_{n-2}z(t)\big)' \end{vmatrix}
$$

$$\begin{aligned}
&= \begin{vmatrix} L_{n-2}y(t) & L_{n-2}z(t) \\ L_n y(t) & L_n z(t) \end{vmatrix} \\
&= L_{n-2}y(t)L_n z(t) - L_{n-2}z(t)L_n y(t) \\
&= L_{n-2}y(t)f\big(t, z(t), L_1 z(t), ..., L_{n-3}z(t)\big) \\
&\quad -L_{n-2}z(t)\big(L_n y(t) + P(t)L_{n-2}y(t)\big) \\
&= L_{n-2}y(t)f\big(t, z(t), L_1 z(t), ..., L_{n-3}z(t)\big).
\end{aligned}$$

From the assumptions of the theorem it implies that the function $f\big(t, z(t), L_1 z(t), ..., L_{n-3}z(t)\big)$ is still positive on some subinterval I_b of I_a or is still negative on it. This and the fact $L_{n-2}y(t) \neq 0$ on I_a it imply that $W(t)$ is either an increasing or decreasing continuous function on I_b. Hence $W(t)$ does not change the sign on some I_c, $c \geq b$. Thus for any $t \in I_c$ it holds that

$$\begin{aligned}
\left(\frac{L_{n-2}z(t)}{L_{n-2}y(t)}\right)' &= \frac{L'_{n-2}z(t)L_{n-2}y(t) - L_{n-2}z(t)L'_{n-2}y(t)}{L^2_{n-2}y(t)} \\
&= \frac{W(t)}{p_{n-1}(t)L^2_{n-2}y(t)}.
\end{aligned}$$

From this we have $\frac{W(t)}{p_{n-1}(t)L^2_{n-2}y(t)}$ as well as $\left(\frac{L_{n-2}z(t)}{L_{n-2}y(t)}\right)'$ does not change the sign on I_c. Hence $\frac{L_{n-2}z(t)}{L_{n-2}y(t)}$ is either still positive or still negative on some I_d, $d \geq c$. Thus $L_{n-2}z(t)$ does not change the sign on some I_e, $e \geq d$. The following formula

$$L_k z(t) = L_k z(e) + \int_e^t \frac{L_{k+1}z(s)}{p_{k+1}(s)}\,ds$$

holds for $k = 0, 1, 2, ..., n-3$, i.e. $L_k z(t)$ does not change the sign on a proper unbounded interval for $k = 0, 1, 2, ..., n-3$. It yields that $z(t)$ is nonoscillatory. Since $z(t)$ is any solution of (1) on I_a, the theorem is proved. □

Remark 1. If the equation (1) admits no solution on any I_a, then this equation is, according to Definition 2, nonoscillatory, too. From this it implies, that we need not verify whether (1) admits at least one solution on some unbounded interval.

Example 1. The differential equation (1) of the form ($n = 3$, $a = 1$)

$$\Big(t\left(t^3 y'\right)'\Big)' - \frac{1}{t}\left(t^3 y'\right) = t^2 + y^4$$

is, according to Theorem 1, nonoscillatory, because the competent equation (3) of the form

$$\Big(t\left(t^3 y'\right)'\Big)' - \frac{1}{t}\left(t^3 y'\right) = 0$$

admits a solution $y(t) = \frac{1}{t}$ on I_1 such that $L_1 y(t) = -t \neq 0$ on I_1.

Remark 2. The assumption (2) is essential. If (2) is not fulfilled, then Theorem 1 need not be valid as it shows the following example:

Example 2. The differential equation (1) of the form ($n = 3$, $a = 1$)

$$y''' + y' = \sqrt[3]{t^6 \left(y - \sin t\right)^2}$$

is oscillatory, because it admits an oscillatory solution $y(t) = \sin t$. The assumption (2) is not fulfilled because $f(t, \sin t) = 0$ on the interval I_1.

It need not be easy to verify an existence of such a solution $y(t)$ as it is demanded in Theorem 1. In the case $P(t) \leq 0$ on I_a this problem is fully solved in the following assertion:

THEOREM 2. Let (2) hold. If it holds that $P(t) \leq 0$ on I_a, then the equation (1) is nonoscillatory.

Proof If $P(t) \leq 0$ on I_a, then Lemma 2 it yields that exists a solution $y(t)$ of the equation (3) on I_a such that $L_{n-2}y(t) > 0$ on I_a. Then Theorem 1 it gives that the equation (1) is nonoscillatory. □

EXAMPLE 3. The differential equation (1) of the form ($n = 3$, $a = 1$)

$$\left(t^2\left(t^4y'\right)'\right)' - \left(20 - \frac{2}{t^3}\right)\left(t^4y'\right) = \sqrt{3t^2 + y^2}$$

is, according to Theorem 2, nonoscillatory. This equation admits at least one solution on an unbounded interval; for example, a function $y(t) = t$ is a solution on I_1. It means that any other solution of this equation on I_1 is nonoscillatory, too.

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