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Differential equations with property M^+ and M^-

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Abstract

In this paper a new classification for some linear and quasilinear differential equations named as property M^+ and M^- is given. It is shown the necessary and sufficient condition for property M^+ and M^- as well as the existence of some types of the monotonic solutions according with property M^+ or M^-.

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1. INTRODUCTION

In this paper a similar problematic is studied which we can find in [3] for the third order differential equations or in [2], [6] for the fourth order differential equations and in [1] for – n th order one. We study some kinds of differential equations where asymptotic behavior of some solution depends on its initial conditions. These equations will be called differential equations with property M^+ and M^-. In [1]- [3], [6] is easily seen that propert*ies* M^+ and M^- are very suitable for proving an oscillation character of some equations, or asymptotic behavior of its no*n* oscillation solutions, but in this paper one method will be presented how to use property M^- at proving the monotonic solution existence for the quasilinear differential equation.

2. LINEAR DIFFERENTIAL EQUATIONS

2.1 Monotonic solutions existence.

We will consider the linear differential equation in the form:

$$x^{(n)} + p_1(t)x^{(n-1)} + \ldots + p_n(t)x = 0 \tag{1}$$

on interval $I = [a, \infty)$, for $a > -\infty$ with continuous coefficients p_i, $i = 1, \ldots, n$ and, p_n is not identically equal to zero on each subinterval of I.

Now, basic definitions.

DEFINITION 2.1. We said, that the equation (1) has property M^{+}, if for each its solution u, which in some point $t_0 \in I$ satisfies initial conditions

(2) $$u^{(i)}(t_0) \geq 0,\ i = 0,1,2,\ldots,n-1 \text{ and } \sum_{i=0}^{n-1} u^{(i)}(t_0) > 0,$$

then holds

(3) $$u^{(i)}(t) > 0,\ i = 0,1,2,\ldots,n-1 \text{ for each } t > t_0.$$

DEFINITION 2.2. We said, that the equation (1) has property M^{-}, if for each its solution u, which in some point $t_0 > a$ satisfies initial conditions

(4) $$(-1)^i u^{(i)}(t_0) \geq 0,\ i = 0,1,2,\ldots,n-1 \text{ and } \sum_{i=0}^{n-1} \left| u^{(i)}(t_0) \right| > 0,$$

then holds

(5) $$(-1)^i u(t) \cdot u^{(i)}(t) > 0,\ i = 0,1,2,\ldots,n-1 \text{ for each } t \in [a, t_0).$$

If for the solution u of the differential equation (1) exists $t_0 \in I$ so, that for each $t > t_0$ holds $(-1)^i u(t) \cdot u^{(i)}(t) > 0$, $i = 0,1,2,\ldots,n-1$ and the differential equation (1) has property M^{-}, then for the solution u holds $(-1)^i u(t) \cdot u^{(i)}(t) > 0$, $i = 0,1,2,\ldots,n-1$ for every $t \in I$.

Let u_m, $m = 0,1,2,\ldots,n-1$ are solutions of *the* differential equation (1), with initial conditions:

(6) $$u_m^{(i)}(t_0) = \langle \begin{matrix} 1 & m = i & i = 0,1,2,\ldots,n-1 \\ 0 & m \neq i & i = 0,1,2,\ldots,n-1 \end{matrix} \quad t_0 \geq a.$$

(1)

The next two lemmas show the existence of monotonic solutions of linear differential equations with property M^{+} and M^{-}.

LEMMA 2.1. Let the differential equation (1) has property M^+, then it has the solution w, which has property $w(t)\cdot w^{(i)}(t)>0$, $i=0,1,2,...,n-1$ on whole interval I.

PROOF. Let w is *the* solution of $Lx = 0$, with initial conditions $w^{(i)}(a)>0$, $i=0,1,2,...,n-1$. Then property M^+ implies that for the given solution w is $w(t)\cdot w^{(i)}(t)>0$, $i=0,1,2,...,n-1$ on whole interval I.

LEMMA 2.2. Let the differential equation (1) has property M^-, then it has solution z with property $(-1)^i z(t)z^{(i)}(t)>0$, $i=0,1,2,...,n-1$ on whole interval I. .

PROOF. We make this solution z. The constants $c_{0,k}$; $c_{1,k}$;…; $c_{n-1,\ k}$ evidently exist for each natural number $k > a$, for which is valid:

$$\sum_{m=0}^{n-1} c_{m,k}u_m(k)=0$$

$$\sum_{m=0}^{n-1} c_{m,k}u'_m(k)=0$$

$$\vdots$$

$$(-1)^n\sum_{m=0}^{n-1} c_{m,k}u_m^{(n-1)}(k)<0 \quad \text{and} \quad \sum_{m=0}^{n-1} c_{m,k}^2=1$$

Let's put $z_k(t)=c_{0,k}u_0(t)+c_{1,k}u_1(t)+...+c_{n-1,k}u_{n-1}(t)$. It is clear, that z_k is a nontrivial solution and from property M^- for solution $z_k(t)$ holds $(-1)^i z_k(t)z_k^{(i)}(t)>0$, $i=0,1,...,n-1$ for $t\in[a,k)$. Because sequences $\{c_{m,k}\}_{k\in N}$, $m=0,1,...,n-1$ are bounded, and it is possible to choose some convergent subsequences $\{c_{m,k_r}\}_{k_r\in N}$, $k_r>a$, which converge to c_m, $m=0,1,...,n-1$. It is true that $\sum_{m=0}^{n-1} c_m^2=1$.

Sequences $\{z_{k_r}^{(i)}(t)\}_{k_r\in N}$, $i=0,1,...,n-1$, $k_r>a$ converge uniformly on each finite subinterval $[a,\infty)$ to the functions $z^{(i)}$. From property M^- for z then holds $(-1)^i z(t)z^{(i)}(t)>0$ for each $t\in I$.

2.2 n – th order differential equations with property M^+ or M^-.

Consider the equation (1) on interval I for $n \geq 2$. And also let coefficients p_i, $i=2,3,\ldots,n$ hold:

(7) $$p_i(t) \le 0, \quad i = 2, 3, \ldots, n,$$

(1.1)

(8) $$(-1)^i p_i(t) \le 0, \quad i = 2, 3, \ldots, n,$$

(1.2)

(1.3)

Putting $x = x_1, x_i' = x_{i+1}, \quad i = 1, 2, \ldots, n-1$, then (1) goes to the system form:

(9) $$\begin{aligned} x_1' &= x_2, \\ x_2' &= x_3, \\ &\vdots \\ x_{n-1}' &= x_n, \\ x_n' &= -p_1(t)x_n - p_2(t)x_{n-1} - \cdots - p_{n-1}(t)x_2 - p_n(t)x_1. \end{aligned}$$

If we put $E(t, \tau) = \exp \int_\tau^t p_1(s)\, ds, \quad (t, \tau) \in I \times I; \quad x_i = y_i, i = 1, 2, \ldots, n-1$

and $x_n E(t, \tau) = y_n$, so system (9) goes to the form

(9*) $$\begin{aligned} y_1' &= y_2, \\ y_2' &= y_3, \\ &\vdots \\ y_{n-1}' &= y_n E(a, t), \\ y_n' &= -p_2(t)E(t, a)y_{n-1} - \cdots - p_n(t)E(t, a)y_1. \end{aligned}$$

THEOREM 2.1. The necessary and sufficient condition for property M^+ is that coefficients of differential equation (1) hold (7).

PROOF.
Necessary condition:

Let τ is *an* arbitrary number from interval *I*. Consider solutions u_m, $m = 0, 1, \ldots, n-2$, which in τ have initial conditions (6). For arbitrary, but fixed *m* has the differential equation (6) in τ form $u_m^{(n)}(\tau)+p_{n-m}(\tau)=0$. The property M^+ implies that for every $t>\tau$ is $u^{(n-1)}(t)>0$ and then $u_m^{(n)}(\tau)\geq 0$. It means that $p_{n-m}(\tau)\leq 0$. Because τ is arbitral, coefficients p_i, $i=2,3,\dots,n$ are not positive on whole interval I and property M^+ implies that p_n is not identical to zero on any subinterval of I.

Sufficient condition:

Let $(x_1, x_1', \dots, x_1^{(n-2)}, x_1^{(n-1)}E(t,a))$ is the solution of system (10*), which in point $t_0\in I$ satisfies initial conditions $x_1^{(i)}(t_0)\geq 0,\quad i=0,1,\dots,n-1$ and $\sum_{i=0}^{n-1} x_1^{(i)}(t_0)>0$. All (9*) system's coefficients are nonnegative and $x_1^{(i)}(t_0)\geq 0,\quad i=0,1,\dots,n-1$. From this implies that $x_1^{(i)}(t)\geq 0$ on $(t_0,+\infty)$, $x_1^{(i)}$ are not decreasing in $[t_0,+\infty)$, $i=0,1,\dots,n-1$ and also $x_1(t)>0$ on $(t_0,+\infty)$. Because function p_n is not identical to zero on any subinterval of I, it is $x_1^{(n-1)}E(.,a)$ increasing on $[t_0,+\infty)$. From this we have $x_1(t)>0, x_1'(t)>0,\dots, x^{(n-1)}(t)>0$ for $t\in(t_0,+\infty)$.

Now we will consider our problem with a different substitution.

The differential equation (1) will be again rewritten as the system in two steps. Putting $x=x_1$, $(-1)^i x_1^{(i)}=x_{i+1}$, $i=1,2,\dots,n-1$ and thereafter

$$E(t,\tau)=\exp\int_\tau^t p_1(s)\,ds,\quad (t,\tau)\in I\times I;\quad x_i=y_i, i=1,2,\dots,n-1 \qquad \text{and}$$

$x_nE(t,a)=y_n$, so we get the system:

$$\text{(9**)}\qquad \begin{aligned} y_1' &= -y_2,\\ y_2' &= -y_3,\\ &\vdots\\ y_{n-1}' &= -y_n,\\ y_n' &= p_2(t)E(t,a)y_{n-1}-p_3(t)E(t,a)y_{n-2}+\dots+p_{n-1}(t)E(t,a)y_2-p_n(t)E(t,a)y_1. \end{aligned}$$

THEOREM 2.2. The necessary and sufficient condition for property M^- is that coefficients of the differential equation (1) hold (8).

PROOF.
Necessary condition:

Let τ is *an* arbitrary number from interval *I*. Consider solutions u_m, $m = 0, 1, \ldots, n-2$, that satisfy initial conditions

(6*) $$u_m^{(i)}(t_0) = \langle \begin{matrix} (-1)^i & m = i & i = 0,1,2,\ldots,n-1 \\ 0 & m \neq i & i = 0,1,2,\ldots,n-1 \end{matrix} \quad t_0 \geq a.$$

For arbitrary, but fixed m has the differential equation (1) in τ form $u_m^{(n)}(\tau) + (-1)^{n-m} p_{n-m}(\tau) = 0$. The property M^- implies that for every $t < \tau$ is $u_m^{(n-1)}(t) > 0$ and then $u_m^{(n)}(\tau) \leq 0$. It means, that $(-1)^{n-m} p_{n-m}(\tau) \geq 0$. Because τ is arbitrary, coefficients p_i, $i = 2, 3, \ldots, n$ hold (8) on whole interval *I* and property M^- implies that p_n is not identical to zero on any subinterval of *I*.

Sufficient condition:

Let $(x_1, -x_1', \ldots, -x_1^{(n-2)}, x_1^{(n-1)} E(., a))$ is the solution of system (10**), which has no positive coefficients. Because $(-1)^i x_1^{(i)}(t_0) \geq 0$, $i = 0, 1, \ldots, n-1$; $\sum_{i=0}^{n-1} \left| x^{(i)}(t) \right| > 0$ and p_n is not identical to zero on any subinterval of *I* $[c, d] \subset I$, from system (9**) for $x_1, -x_1', \ldots, x_1^{(n-1)}$ we get that $(-1)^i x_1^{(i)}(t) > 0$ for $t \in [a, t_0)$ and $i = 0, 1, \ldots, n-1$.

From conditions (7) amd (8) we get the next theorem.

THEOREM 2.3. Let *n* is odd. The differential equation (1) cannot have together property M^+ and M^-.

Now let hold for coefficients p_i

(10) $$\begin{aligned} &p_i = 0, \text{ if } i \text{ is odd} \\ &p_i \leq 0, \text{ if } i \text{ is even}, \quad \mathrm{i} = 2,3,\ldots,\mathrm{n} \end{aligned}$$

(1.4)

We can easily verify, that it is the situation when holds conditions (7) and (8) in *n* even case. This fact gives us the possibility for next theorem formulation.

THEOREM 2.4. Let *n* is even. The necessary and sufficient condition for properties M^+ and M^- is that coefficients of differential equation (1) hold (10).

3. Application for some quasilinear differential equations

In this part will be shown the existence of a solution with property

(11) $$(-1)^i x^{(i)}(t) > 0\,,\ i = 0, 1, \dots, n \quad \text{for each } t \in I$$

for some quasilinear differential equation of *n*- th order.

We will consider the quasilinear equation

(12) $$x^{(n)} + f(t, x, x', \dots, x^{(n-1)})x = 0\,,$$

where f is continuous function, which is uniformly bounded on each interval $\langle c,d\rangle \times R^n \subset I \times R^n$ and $f(\cdot, u(\cdot), u'(\cdot), \dots, u^{n-1}(\cdot))$ for every function $u \in C^{n-1}(I)$ is not identically equal to zero on any interval $\langle c,d\rangle \subset I$
with corresponding linear equation:

(13) $$x^{(n)} + f(t, u, u', \dots, u^{(n-1)})x = 0\,,$$

where $u \in C^n(I)$.

$C^0(I)$ means the set of continues function on *I* and $C^{(n-1)}(I)$ means the set of functions with continuous (*n* – 1) differentiation on interval *I*.

The topology on $C^{n-1}(I)$ is introduced by the countable family of semi norms

$$p_m(x) = \max_{0 \le i \le n-1} \max_{t \in [a, a+m]} \left| x^{(i)}(t) \right|,\ m \in N$$

In this topology $C^{n-1}(I)$ is a Fréchet and the convergence $x_p \to x$ in this space means the locally uniform convergence $x_p^{(i)} \to x^{(i)}$ up to the order *n* – 1. In a similar way the Fréchet space $C^0(I)$ is defined.

For our next theorems proving we will need these two lemmas. The first is the Fan and Glickberg fixed point theorem.

LEMMA 3.1. **[7]** If *S* is closed, convex and nonempty subset of a Fréchet space *X* and if a mapping *T* satisf*ies*:

i.) for each $u \in S$ is *T*(*u*) nonempty, compact and convex subset of *X*,

ii.) *T* is a closed mapping,

iii.) *T*(*S*) is contained in a compact subset of *S*, then there is an $u \in S$, such that $u \in T(u)$.

LEMMA 3.2. [7] Let $P_{k,m} \in C^0(I)$, $Q_m \in C^0(I)$, $k = 1, 2, \ldots, n$; $m, n = 1, 2, \ldots$, are bounded function in topology $C^0(I)$, i.e. on each compact subinterval of *I* are sequences $\{P_{k,m}\}_{m=1}^{\infty}$ a $\{Q_m\}_{m=1}^{\infty}$ uniformly bounded. Then the following statement holds:

If $\{x_m\}_{x=1}^{\infty}$ is a sequence of solutions the equation

$$x^{(n)} + \sum_{n=1}^{n} P_{k,m}(t) x^{(n-k)} = Q_m(t),$$

which is bounded in $C^0(I)$ topology, then it is relatively compact in the topology of $C^{n-1}(I)$. That means, that there exists a subsequence $\{x_{m(r)}^{(i)}\}$ $i = 0, 1, \ldots, n-1$ which uniformly converges on each compact subinterval of *I*.

THEOREM 3.1. Let *n* is odd and and function where *f* is continuous nonnegative function, which is uniformly bounded on each interval $\langle c, d\rangle \times R^n \subset I \times R^n$ and for every function $u \in C^{n-1}(I)$ is not identically equal to zero on any interval $\langle c, d\rangle \subset I$. Then differential equation (12) has a solution with property (11).

PROOF. Consider a space $C^{n-1}(I)$ topologized as above. Let $S = \{x \in C^{n-1}(I); x(t_0) = 1; (-1)^i x^{(i)}(t) \geq 0 \text{ for } i = 0, 1, \ldots, n-1; t \in I\}$, S is closed, convex and nonempty. For $u \in S$ let $T(u) = \{x \in S, x \text{ is solution of } L_u x = 0\}$. From linearity $L_u x$ is *T*(*u*) a convex set. By lemma 2.2 *T*(*u*) is nonempty.

Let $T(u) \subset S$ and *S* is bounded $C^0(I)$ then by lemma 3.2 *T*(*u*) is relatively compact in $C^{n-1}(I)$. But *T*(*u*) is also closed, and hence, compact in the $C^{n-1}(I)$ topology. Thus the so defined mapping $T : S \to 2^S$ satisfies the requirement (*i*) of lemma 3.1.

Let $u_p \in S, u_p \to u_0$ a $x_p \in T(u_p), x_p \to x_0$ v $C^{n-1}(I)$. Then functions $f\left(\cdot, u_p(\cdot), u_p'(\cdot), \ldots, u_p^{(n-1)}(\cdot)\right)$ converge locally uniformly on *I* to

$f\left(\cdot,u_0(\cdot),u_0'(\cdot),\ldots,u_0^{(n-1)}(\cdot)\right)$ and by corollary 4.1 [4, pg. 73] $x_p \to y_0$, where y_0 is the solution of the equation (12) for $u = u_0$ and it satisfies the same initial conditions as x_0. Therefore $x_0 = y_0$ and $x_0 \in T(u_0)$. Thus, T is a closed mapping.

As S is bounded in the topology of $C^0(I)$ and $T(S) \subset S$, lemma 3.2 guarantees that $T(S)$ is relatively compact in $C^{n-1}(I)$. Hence its closure $\overline{T(S)} \subset S$ is compact. Thus all assumptions of lemma 3.1 are satisfied and by this lemma there exists an $x \in S$ such that $x \in T(x)$, which is the searched solution.

Because in n even case, we can by using the same proof technique show the existence of the solution with property (11) for the differential equation (12), the proof of following theorem will be omitted.

THEOREM 3.2. Let n is even and function where f is continuous nonpositive function, which is uniformly bounded on each interval $\langle c,d\rangle \times R^n \subset I \times R^n$ and $f\left(\cdot,u(\cdot),u'(\cdot),\ldots,u^{(n-1)}(\cdot)\right)$ for every function $u \in C^{n-1}(I)$ is not identically equal to zero on any interval $\langle c,d\rangle \subset I$. Then the differential equation (12) has a solution with property (11).

EXAMPLE 3.1

For n odd has the equation

$x^{(n)} + \dfrac{x'^2+1}{x^2+1}x = 0$ has solution $x = e^{-t}$.

For n even has the equation

$x^{(n)} - n!x^n x = 0$ has solution $x = \dfrac{1}{t}$ on interval $[a,\infty)$ for $a > 0$.

We can use this method for general quasilinear differential equations case too in the form

$$(14)\qquad x^{(n)} + p_1(t,x,x',\ldots,x^{(n-1)})x^{(n-1)} + p_2(t,x,x',\ldots,x^{(n-1)})x^{(n-2)} + \cdots + p_n(t,x,x',\ldots,x^{(n-1)})x = 0,$$

with the next assumption which will be fulfilled to the end of this paper. The assumption is that coefficients p_1, ..., p_n are continuous functions, which are uniformly bounded on each interval $\langle c,d\rangle \times R^n \subset I\times R^n$ and $p_n\left(\cdot ,u(\cdot),u'(\cdot),\ldots ,u^{(n-1)}(\cdot)\right)$ for every function $u\in C^{n-1}(I)$ is not identically equal to zero on any interval $\langle c,d\rangle \subset I$ Moreover, we will consider following sign's statements:

$$(-1)^i\, p_i(t,x_1,x_2,\ldots ,x_n)\le 0\,,\ \ i=2,3,\ldots ,n;\quad (t,x_1,\ldots ,x_n)\in I\times R^n \tag{15}$$

Using the basis of lemma 2.2 we can formulate following theorems which proofs will be omitted because they are similar to theorem 3.1 prove.

THEOREM 3.3. Let for coefficients of the differential equation (14) is valid (15) then the equation (14) has a solution with property (11).

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