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Primal-Dual Approach to Solve Linear Fractional Programming Problem

VISHWAS DEEP JOSHI, EKTA SINGH AND NILAMA GUPTA

Abstract

In this paper a new method is suggested for solving the problem in which the objective function is a linear fractional function, and where the constraint functions are in the form of linear inequalities. The proposed method is based mainly upon revised primal dual simplex algorithm (RPDSA).The algorithm can be combined with interior-point methods to move from an interior point to a basic optimal solution. The advantages of RPDSA algorithm are simplicity of implementation and less computational effort.

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General Terms: linear fractional programming, RPDSA algorithm.
Additional Key Words and Phrases: primal-dual simplex method, interior-point method.

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1. INTRODUCTION

Linear fraction maximum problems (i.e. ratio objective that have numerator and denominator) have attracted considerable research and interest, since they are useful in production planning, financial and corporate planning, health care and hospital planning.

The field of LFP, largely developed by Hungarian mathematician B. Martos and his associates in the 1960's, is concerned with problem of optimization. Several methods to solve this problem are proposed in (1962), Charnes and Cooper have proposed their method depends on transforming this (LFP) to an equivalent linear program. Another method is called up dated objective function method derived from Bitran and Novaes (1973) is used to solve this linear fractional program by solving a sequence of linear programs only re-computing the local gradient of the objective function .Also some aspects concerning duality and sensitivity analysis in linear fraction program was discussed by Bitran and Magnant I (1976), and Singh .C. (1981) in his paper made a useful study about the optimality condition in fractional programming.

The suggested method in this paper depends mainly on the solving linear fractional functions, and where the constraint functions are in the form of linear inequalities, the proposed method is based mainly upon RPDSA method given by Paparrizos et. al. in 2003 [9]. We use the concept of duality to solve this problem.. Since the earlier methods based on the vertex information may have difficulties as the problem size increases, our method appears to be less sensitive to the problem size. An example is given to clarify the developed theory and the proposed method.

In section 2 proposed algorithm and definitions of the (LFP) problem is given while in section 3 a numerical example is given to clarify the proposed algorithm. In section 4 we give a conclusion remarks about this proposed method and in section 5 we given the references.

2. Proposed Algorithm:

The optimal solution to a general Linear Fractional Programming Problem, if it exists, can be obtained by using the following steps.

Step 0:

LFP problem can be formulated mathematically as follows

Minimize $F(u) = \dfrac{\overline{c}^T u + \alpha}{\overline{d}^T u + \beta}$

Subject to $\overline{A}u \geq \overline{b}$

$$u \geq 0 \qquad \ldots(i)$$

where $u \in U(\in R^n)$, $\overline{A}$ is (m×n) matrix, also $\overline{c}$ and $\overline{d}$ are n-vectors, $\overline{b} \in R^m$ and α , β are scalars. U is compact set. Moreover $\overline{d}^T u + \beta > 0$ everywhere in U

Step I:

In (i) we shall assume that $\beta \neq 0$, than an equivalent form of (i) can be formulated as

Minimize $F(x) = (\overline{c}^T - \dfrac{\alpha}{\beta}\overline{d}^T)\dfrac{u}{\overline{d}^T + \beta} + \dfrac{\alpha}{\beta}$

Subject to $(\overline{A}+\frac{\overline{b}^T}{\beta}\overline{d}^T)\frac{u}{\overline{d}^T+\beta} \geq \frac{\overline{b}}{\beta}$ .....(ii)

If we define $y = \frac{u}{\overline{d}^T+\beta} \geq 0$ than (ii) can be written in the form of Linear Program

Minimize $Z = c^T x + \frac{\alpha}{\beta}$

Subject to $Ax = b$

$x \geq 0$ ....(iii)

Where

$$c^T = (\overline{c}^T - \frac{\alpha}{\beta}\overline{d}^T)$$

$$A = (\overline{A} + \frac{\overline{b}}{\beta}\overline{d}^T)$$

and $b = \frac{\overline{b}}{\beta}$

Step II:

(**Interior - point method & RPDSA Algorithm**)

Step 0 (initialization): Start with a primal feasible solution y for problem (LP) and a dual feasible basic partition (B,N) to problem (LP). Set

$$x_B = (A_B)^{-1}b \;,\quad w^T = (c_B)^T(A_B)^{-1} \;,\; (s_N)^T = (c_N)^T - w^T A_N \;,\; d = y - x$$

Step 1 (general step):

While $(\exists i \in \{1,2,.....m\} : x_{B(i)} < 0)$

$\lambda = \frac{x_{B(r)}}{-d_{B(r)}} = \max\left\{\frac{x_{B(r)}}{-d_{B(r)}} : x_{B(i)} < 0\right\}$ /*computation of the leaving variable*/

$y = x + \lambda d$ /*computation for the next feasible point*/

$H_{rN} = (B^{-1})_r A_N$ /*computation of pivoting row*/

$\mu = \frac{s_{N(t)}}{H_{rN(t)}} = \min\left\{\frac{s_{N(j)}}{-H_{rN(j)}} : H_{rN(j)} < 0\right\}$

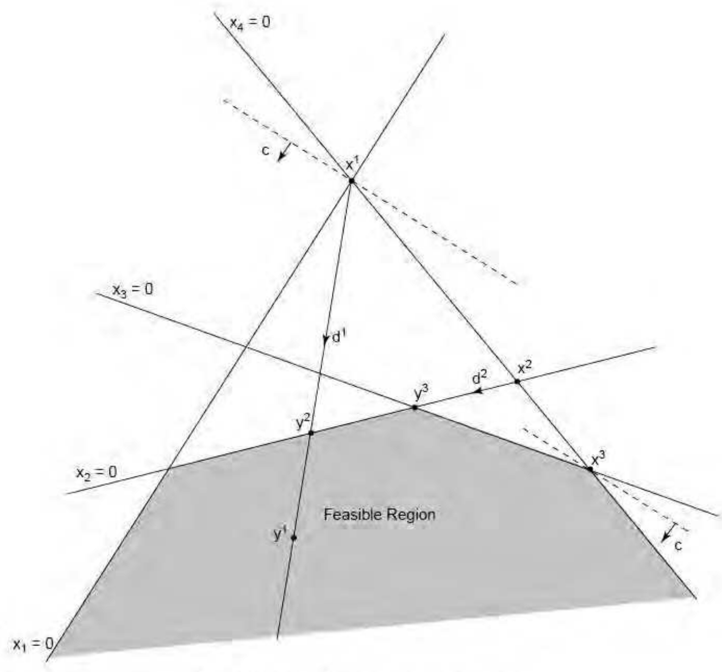
/*choice of the entering variable $x_{N(j)} = x_p$*/

$k = B(r), p = N(t), B(r) = p, N(t) = k$ /*update sets B and P*/

$x_B = (A_B)^{-1}b,\; w^T = (c_B)^T(A_B)^{-1} \;,(s_N)^T = (c_N)^T - w^T A_N$

$d = y - x$ /*computation of the next direction*/

end



Graphical representation of algorithm

3. NUMERICAL EXAMPLE

Linear fractional program

Minimize $$\frac{x_1 + \frac{1}{2}x_2}{-\frac{3}{2}x_1 - \frac{1}{2}x_2 + 3}$$

Subject to:

$$\frac{5}{2}x_1 + \frac{3}{2}x_2 \geq 3$$

$$\frac{7}{2}x_1 + \frac{1}{2}x_2 \geq 3$$

$$x_1, x_2 \geq 0$$

By transformation formula we can change linear fractional programming problem to linear problem.

After transformation we got following LPP

Primal

$$\text{Minimize} \quad x_1 + \frac{1}{2} x_2$$

Subject to:

$$\begin{aligned} x_1 + x_2 - x_3 \quad &= 1 \\ 2x_1 + \quad - x_4 &= 1 \\ x_1 , x_2 , x_3 , x_4 &\geq 0 \end{aligned}$$

Dual

$$\text{Maximize} \quad w_1 + w_2$$

Subject to:

$$\begin{aligned} w_1 + 2w_2 + s_1 &= 1 \\ w_1 + s_2 &= \frac{1}{2} \\ -w_1 + s_3 &= 0 \\ - w_2 + s_4 &= 0 \\ w_1 , w_2 , s_1 , s_2 , s_3 , s_4 &\geq 0 \end{aligned}$$

We can see that for primal problem

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix} , \quad c = \left(1 , \frac{1}{2} , 0 , 0\right)^T , \quad b = (1 , 1)^T$$

Initialization with dual feasible basic partition $B^1 = \{3 , 4\}$, $N^1 = \{1 , 2\}$.

$$w^1 = (0 , 0) , \; s^1 = \left(1 , \frac{1}{2} , 0 , 0\right)$$

Let $\quad y^1 = \left(\frac{2}{3} , \frac{7}{12} , \frac{1}{4} , \frac{1}{3}\right)$

We get

$$x_B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} , \; x^1 = (0 , 0 , -1 , -1),$$

$$w^{1T} = (0 , 0) \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = (0 , 0)$$

$$s_N^T = \left(1 , \frac{1}{2}\right)^T - (0 , 0) \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = \left(1 , \frac{1}{2}\right)^T , \; s^1 = \left(1 , \frac{1}{2} , 0 , 0\right),$$

$$d^1 = \left(\frac{2}{3}, \frac{7}{12}, \frac{1}{4}, \frac{1}{3}\right) - (0, 0, -1, -1) = \left(\frac{2}{3}, \frac{7}{12}, \frac{5}{4}, \frac{4}{3}\right)$$

Choice of the leaving variable

$$\lambda = \max\left\{\frac{x_3}{-d_3}, \frac{x_4}{-d_4}\right\} = \frac{4}{5} = \frac{x_3}{-d_3},$$

Computation for next feasible point

$$y^2 = x^1 + \lambda d^1 = (0, 0, -1, -1) + \frac{4}{5}\left(\frac{2}{3}, \frac{7}{12}, \frac{5}{4}, \frac{4}{3}\right)$$

$$= \left(\frac{8}{15}, \frac{7}{12}, 0, \frac{1}{15}\right)$$

Computation for the pivoting row

$$H_{rN} = (-1, 0)\begin{pmatrix}1 & 1\\ 2 & 0\end{pmatrix} = \begin{pmatrix}-1\\ -1\end{pmatrix} \begin{matrix}\sim x_1 = x_{N(1)}\\ \sim x_2 = x_{N(2)}\end{matrix}$$

Choice of the entering variable

$$\mu = \min\left\{\frac{s_1}{-H_{rN(1)}}, \frac{s_2}{-H_{rN(2)}}\right\} = \frac{1}{2} = \frac{s_2}{-H_{rN(2)}},$$

So x_3 leave the basis and x_2 enter

Second iteration

$$B^2 = (2, 4), N^2 = (1, 3)$$

We get

$$w^{2T} = \left(\frac{1}{2}, 0\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{-1} = \left(\frac{1}{2}, 0\right),$$

$$x_B = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{-1}\begin{pmatrix}1\\ 1\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}, \quad x^2 = (0, 1, 0, -1)$$

$$s_N^T = (1, 0) - \left(\frac{1}{2}, 0\right)\begin{pmatrix}1 & -1\\ 2 & 0\end{pmatrix} = \left(\frac{1}{2}, \frac{1}{2}\right), \quad s^2 = \left(\frac{1}{2}, 0, \frac{1}{2}, 0\right),$$

$$d^2 = y^2 - x^2 = \left(\frac{8}{15}, \frac{-8}{15}, 0, \frac{16}{15}\right)$$

Choice of leaving variable

$$\lambda = \max\left\{\frac{x_4}{-d_4}\right\} = \frac{15}{16},$$

Computation for the next feasible point

$$y^3 = x^2 + \frac{15}{16}d^2 = \left(\frac{1}{2}\ ,\ \frac{1}{2}\ ,\ 0\ ,\ 0\right),$$

Computation for pivoting row

$$H_{rN} = \begin{pmatrix} 0 & , & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \begin{matrix} \sim x_1 = x_{N(1)} \\ \sim x_3 = x_{N(2)} \end{matrix}$$

Choice of the new entering variable

$$\mu = \min\left\{ \frac{s_1}{-H_{rN(1)}}, \frac{s_3}{-H_{rN(2)}} \right\} = 0 = \frac{s_3}{-H_{rN(2)}}$$

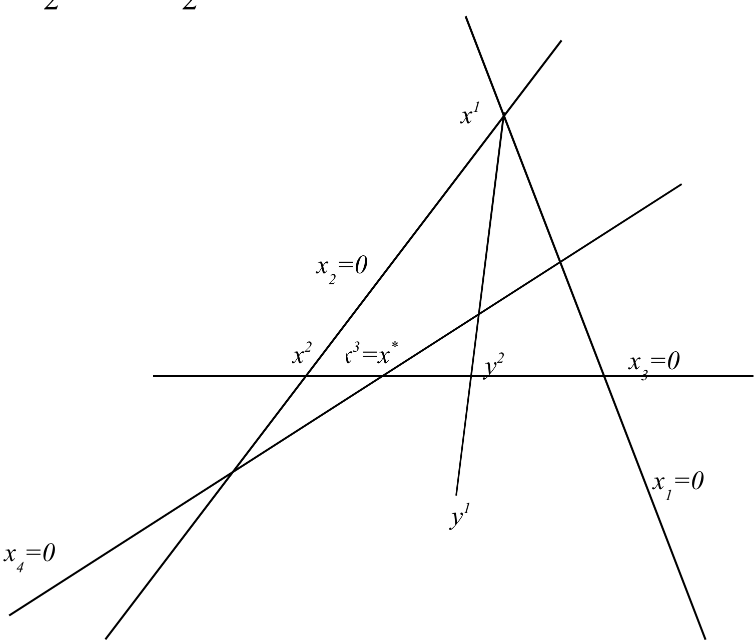
So x_3 leave the basis and x_2 enters

Third iteration

$$x_B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}\ ,\ x^3 = \left(\frac{1}{2}\ ,\ \frac{1}{2}\ ,\ 0\ ,\ 0\right)$$

After third iteration x^3 is feasible point of primal problem so we got the answer

$$x_1 = \frac{1}{2}\ ,\ x_2 = \frac{1}{2}$$



Geometric illustration of the problem

CONCLUSION

The proposed algorithm is developed to solve linear fractional programming problem and restriction is $\beta \neq 0$ and denominator value in objective function must be grater than zero. This algorithm may be extended to solve linear fractional programming problem whose objective function of the numerator and denominator value may be either positive or negative. Also this may be extended to solve linear fractional programming problems with bounded variables.

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VISHWAS DEEP JOSHI
Research Scholar
Department of Mathematics
Malaviya national institute of technology,
Jaipur, India

EKTA SINGH
Research scholar
Department of Mathematics
Malaviya national institute of technology,
Jaipur, India

NILAMA GUPTA
Department of Mathematics
Malaviya national institute of technology,
Jaipur, India

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