Some Properties of The Normalizer of $\Gamma_0(N)$ on Graphs

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Abstract

In this paper we give some properties of the normalizer $\mathbb{K}$ of $\Gamma_0(N)$ in $\text{PSL}(2, \mathbb{A}G_1)$ and show that the number of edges are related to the periods of group elements.

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1. INTRODUCTION

Let $\text{PSL}(2, \mathbb{A}G_1)$ denote the group of all linear fractional transformations

$$T : z \rightarrow \frac{az + b}{cz + d},$$

where $a, b, c, d$ are real and $ad - bc = 1$. For any natural number $N$, $\Gamma_0(N)$ denotes the subgroup of $\text{PSL}(2, \mathbb{A}G_1)$ with integral coefficients and property that $c \equiv 0 \mod N$. We shall in the sequel freely use 2x2 matrices to represent transformations, so that $T$ is represented by the pair of the matrices

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}, (a, b, c, d \in \mathbb{A}G_1, \text{ad} - bc = 1) \quad (1.1)$$

We shall write these matrices as equal where convenient in matrix calculations.

Following [1], we denote by $\mathbb{K}$ the normalizer of $\Gamma_0(N)$ in $\text{PSL}(2, \mathbb{A}G_1)$ consisting of all the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$$

where all letters are integers, $e \parallel N/h^2$ and $h (= h(N))$ is the largest divisor of 24 for which $h^2 \mid N$ with the understandings that the determinant of the matrix is $e > 0$, and that $r \parallel s$ means that $r \mid s$ and $(r, s) = 1$ ($r$ is called an exact divisor of $s$).

We now define $\Gamma_0(N/h: h)$ to be the set
which is a subgroup of $\text{PSL}(2, \mathbb{R})$. It can be easily seen that this group is generated by $\Gamma_0(N)$ and $\begin{pmatrix} 1 & \frac{1}{h} \\ 0 & 1 \end{pmatrix}$.

2. **THE ACTION OF $\mathbb{R}$ ON $\hat{\mathbb{R}}$**

The element $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\hat{\mathbb{R}}$ will be in reduced form, that is $(x, y) = 1$. Since $\mathbb{R}$ is transitive, the stabilizer of any two points in $\hat{\mathbb{R}}$ are conjugate in $\mathbb{R}$. So it is sufficient to consider the stabilizer $\mathbb{R}_\infty$ of $\infty$. This is easily seen

THEOREM 2.1 [2]. Let $N$ be any integer and $N = 2^\alpha_1 3^\alpha_2 \cdots p_i^{\alpha_i}$, the prime power decomposition of $N$. Then $\mathbb{R}$ is transitive on $\hat{\mathbb{R}}$ if and only if $\alpha_i \leq 7$, $\alpha_2 \leq 3$ and $\alpha_i \leq 1$ for $i = 3, \ldots, n$.

From now on $N$ will be as in Theorem 2.1.

LEMMA 2.2. Let $N$ be as above. Then the stabilizer of a point of $\hat{\mathbb{R}}$ is an infinite cyclic group of $\mathbb{R}$.

PROOF. Since the action is transitive, the stabilizer of any two points in $\hat{\mathbb{R}}$ are conjugate in $\mathbb{R}$. So it is sufficient to consider the stabilizer $\mathbb{R}_\infty$ of $\infty$. This is easily seen.
to consist of the elements of the form \( \begin{bmatrix} 1 & b/h \\ 0 & 1 \end{bmatrix} \) and therefore \( \mathbb{K}_\infty \) is the infinite cyclic group generated by \( \begin{bmatrix} 1 & 1/h \\ 0 & 1 \end{bmatrix} \).

We consider the imprimitivity of the action of \( \mathbb{K} \) on \( \hat{\Omega} \) as follows: Generally, let \( (G, \Omega) \) be a transitive permutation group, consisting of a group \( G \) acting on a set \( \Omega \) transitively. We will call \( (G, \Omega) \) imprimitive if \( \Omega \) admits some \( G \)-invariant equivalence relation different from

1. the identity relation, \( \alpha = \beta \) if and only if \( \alpha = \beta \);
2. the universal relation, \( \alpha = \beta \) for all \( \alpha, \beta \in \Omega \).

Otherwise \( (G, \Omega) \) is called primitive. The equivalence classes are called blocks.

**Lemma 2.3** [3]. Let \( (G, \Omega) \) be transitive. Then \( (G, \Omega) \) is primitive if and only if the stabilizer of a point \( \alpha \), is a maximal subgroup of \( G \) for each \( \alpha \in \Omega \).

According to the above lemma if, for some \( \alpha \) in \( \Omega \), \( G_\alpha H G \), then the following relation \( \approx \) on \( \Omega \) makes \( (G, \Omega) \) imprimitive:

\[
(2.1) \quad g(\alpha) = g'(\alpha) \text{ if and only if } g^{-1}g' \in H.
\]

Now as a special case, since \( \mathbb{K}_\infty \sim \Gamma_{\infty}(N/h: h) \), we apply these ideas to the case where \( G \) is the normalizer \( \mathbb{K} \), \( \Omega \) is \( \hat{\Omega} \), and \( G_\alpha \) is \( \mathbb{K}_\infty \). In this case if \( v = \frac{r}{s} \in \hat{\Omega} \) then we find integers \( b \) and \( d \) such that the element \( g = \begin{pmatrix} (re_z)^0 & b/h \\ s/N/h & d e_z^0 \end{pmatrix} \) is in \( \mathbb{K} \) with determinant \( e_z^0 \parallel N/h^2 \) and sending \( \infty \) to \( \frac{r}{s} \), where

\[
\begin{align*}
e_1 &= (s, N/h), \quad s = s, \quad e_z = N/e_z h, \quad e_z^0 = (e_z, N/h^2) \quad \text{and} \quad e_z = e_z e_z^0.
\end{align*}
\]

Likewise, given \( w = \frac{x}{y} \) in \( \hat{\Omega} \) there exists \( g' \in \mathbb{K} \) such that \( g'(\infty) = \frac{x}{y} \). Therefore by (2.1)
where \( y_1, f_2 \) (for \( \frac{x}{y} \)) are defined as \( s_1, e_2 \) (for \( \frac{r}{s} \) ) respectively. After some calculations it can be seen, if \( \frac{r}{s} = 1 \), that \( e_2' = 1 \) and \( y_1 \equiv 0 \mod h \). And here \( y_1 \equiv 0 \mod h \) shows that \( y_1 \equiv 0 \mod N \). Consequently, we have the block

\[
[\infty] = \left\{ \frac{x}{y} \in \hat{\cal G} \mid y \equiv 0 \mod N \right\}. \tag{2.2}
\]

From the imprimitivity the number of blocks under \( = \) is the index \( |\hat{\cal G} : \hat{\cal G} \cap \frac{N}{h} : h| \), which is \( 2^r h s \), where \( h \) is as above, \( r \) is the number of distinct primes dividing \( N \), and \( s = s_1 s_2 \) as in [1] as follows

\[
s_1 = \begin{cases} \frac{3}{4}, & 2 \mid h(2^\alpha)^2 \parallel N, \\ 1, & \text{otherwise} \end{cases}, \quad s_2 = \begin{cases} \frac{2}{3}, & h(2^\alpha)^2 = 9 \parallel N, \\ 1, & \text{otherwise} \end{cases}
\]

3. SUBORBITAL GRAPHS FOR \( \hat{\cal G} \) ON \( \hat{\cal G} \)

Since \((\hat{\cal G}, \hat{\cal G})\) is a transitive permutation group then \((\hat{\cal G}, \hat{\cal G}^+)\) is a permutation group with the action: for \((\alpha, \beta) \in \hat{\cal G}^+\) and \(g \in \hat{\cal G} \), \(g(\alpha, \beta) := (g(\alpha), g(\beta))\). The orbits of this action are called suborbitals of \( \hat{\cal G} \). From the orbit \( O(\alpha, \beta) \) we from a suborbital graph \( \Delta(\alpha, \beta) \) : its vertices are the elements of \( \hat{\cal G} \) and there is a directed edge from \( a \) to \( b \) if \((a, b) \in O(\alpha, \beta)\).

If \( O(\alpha, \beta) = O(\beta, \alpha) \) then the graph consists of pairs of oppositely directed edges; we will replace such edges by undirected edges. In this case we have an undirected graph which will be called self-paired. If \( O(\alpha, \beta) \neq O(\beta, \alpha) \) then \( \Delta(\alpha, \beta) \) is just \( \Delta(\alpha, \beta) \) with the arrows reversed. In this case we call \( \Delta(\alpha, \beta) \) and \( \Delta(\beta, \alpha) \) paired graphs. Since \( \hat{\cal G} \) is transitive on \( \hat{\cal G} \), each suborbital contains a pair \((\infty, v)\) for some \( v \in \hat{\cal G} \); writing
SOME PROPERTIES OF THE NORMALIZER OF $\Gamma_d(N)$ ON GRAPHS

$v = \frac{u}{n}$ with $n > 0$ and $(u,n) = 1$, we denote the suborbital $O(\infty, \frac{u}{n})$ by $O_{\infty, u/n}$ and the graph by $\Delta_{\infty, u/n}$.

These ideas were first introduced by Sims [8], and are also described in a paper by Neumann [7] and in books by Tsuzuku [9] and by Biggs and White [4], the emphasis being on applications to finite groups.

Using Theorem 4.1 in [2] we see that there exist integers $u, m$ such that $O(\infty, \frac{u}{mN}) = O(\infty, \frac{u}{mN})$. Therefore we can derive the following.

**THEOREM 3.1.** $r \stackrel{x}{\rightarrow} s \rightarrow y$ is an edge in $\Delta_{\infty, mN}$ if and only if there exists some $e \in \mathbb{C}$ with $e \parallel \frac{N}{h^2}, \frac{N}{eh} \parallel s$ then either

(a) $ry - sx = mN/e$ and $x = ur \mod \left(\frac{mN}{eh}\right)$, $y = us \mod mN$, or

(b) $ry - sx = -mN/e$ and $x = -ur \mod \left(\frac{mN}{eh}\right)$, $y = -us \mod mN$.

**PROOF.** Let $r \stackrel{x}{\rightarrow} s \rightarrow y$ is an edge in $\Delta_{\infty, mN}$. Then there is some element

$$A = \begin{pmatrix} a & b/\frac{N}{h} \\ cN/\frac{N}{h} & d \end{pmatrix} \text{ in } \Re, \text{ of determinant } e \parallel \frac{N}{h^2}, \text{ sending } \infty \text{ to } \frac{r}{s} \text{ and } \frac{u}{mN} \text{ to } \frac{x}{y} \text{ and}
$$

therefore $\frac{ae}{cN/\frac{N}{h}} = \frac{r}{s}$ and $\frac{(au + (bmN)/\frac{N}{h})}{(cuN/\frac{N}{h} + dmN)} = \frac{x}{y}$. Since the determinant of the matrix $A$ is $e$ and that $(1,0) = 1$ then $a = r$ and $s = cN/eh$. That is $\frac{N}{eh} | s$. Likewise we can see that

$$x = \pm(au + (bmN)/\frac{N}{h}), \quad y = \pm(cuN/\frac{N}{h} + dmN).$$

From this we get
\[
\begin{pmatrix}
\alpha e & b/h \\
\gamma N/h & d e
\end{pmatrix}
\begin{pmatrix}
1 & u \\
0 & mN
\end{pmatrix}
= \begin{pmatrix}
\alpha e & \alpha \nu e + (b\gamma \nu N) / h \\
\gamma N/h & \gamma \nu N + d\nu e \nu N
\end{pmatrix}
\begin{pmatrix}
(-1)\nu e r & (-1)\nu e x \\
(-1)\nu e s & (-1)\nu e y
\end{pmatrix}
\] (3.1)

where \( i, j = 0,1 \). If \( i = j = 0 \), then \( x \equiv ur \mod \left(\frac{mN}{eh}\right) \), \( y \equiv us \mod mN \) and taking determinant of (3.1) we see that \( r y - s x = \frac{mN}{e} \). Similarly if \( i = 1, j = 0 \) (or \( i = 0, j = 1 \)) we obtain (b). Conversely, if (a) holds with the given conditions, then exist integers \( b \) and \( d \) such that \( x = ur + \left(\frac{b\gamma \nu N}{eh}\right) \) and \( y = us + dmN \). It can be easily shown that the element \( \begin{pmatrix}
re & b/h \\
se & d e
\end{pmatrix} \) is in \( \mathfrak{K} \) and sends \( \infty \) to \( \frac{r}{s} \) and \( \frac{u}{mN} \) to \( \frac{x}{y} \). If (b) holds the proof follows similarly. \( \blacksquare \)

From now on, for the sake of simplicity, suppose that \( m \) is 1.

THEOREM 3.2. If \( uv \equiv -1 \mod N \), then the suborbital graphs \( \Delta_{\alpha,\gamma} \) and \( \Delta_{\gamma,\alpha} \) are paired.

PROOF. Suppose that \( uv \equiv -1 \mod N \) and \( \frac{r}{s} \rightarrow \frac{x}{y} \) is an edge in \( \Delta_{\alpha,\gamma} \). Then, using Theorem 3.1, there exists some integer that \( e \parallel \frac{N}{h} \cdot \frac{N}{eh} \parallel s \) and (a) or (b) is satisfied. Suppose (a) holds. Then \( r y - s x \equiv \frac{N}{\alpha} \), and \( x \equiv ur \mod \left(\frac{N}{eh}\right) \), \( y \equiv us \mod N \). Since \( uv \equiv -1 \mod N \), then \( r \equiv -\nu x \mod \frac{N}{eh} \), \( s \equiv -\nu y \mod N \). Then \( \frac{x}{y} \rightarrow \frac{r}{s} \) is an edge in \( \Delta_{\gamma,\alpha} \) that is, \( \Delta_{\alpha,\gamma} \) and \( \Delta_{\gamma,\alpha} \) are paired. \( \blacksquare \)

COROLLARY 3.3. \( \Delta_{\alpha,\gamma} \) is self-paired if and only if \( u^2 \equiv -1 \mod \frac{N}{h} \).

PROOF. Suppose \( \mathcal{O}(\infty, u_N) = \mathcal{O}(u_N, \infty) \). Then there exists some \( \varphi = \begin{pmatrix}
\alpha e & b/h \\
\gamma N/h & d e
\end{pmatrix} \) in \( \mathfrak{K} \) such that sending \( \infty \) to \( \frac{u}{N} \) and \( \frac{u}{N} \) to \( \infty \). From this, \( \varphi \) must be \( \begin{pmatrix}
ue & b/h \\
\epsilon N & -ue
\end{pmatrix} \) and then \( e \) is equal to 1. So \( u^2 \equiv -1 \mod \frac{N}{h} \).
Conversely, \( u^2 \equiv -1 \mod \frac{N}{N/h} \), then exists some integer \( b \) such that \( u^2 = -1 + b\frac{N}{N/h} \).
Therefore, \( \begin{pmatrix} u & b \frac{N}{N/h} \\ N & -u \end{pmatrix} \) is in \( \mathbb{R} \) and satisfied desired condition.

4. THE GRAPH \( F_{u,N} \)

We let \( F_{u,N} \) be the subgraph of \( \Delta_{u,N} \) whose vertices form the block \( \{\infty\} \). So, by Theorem 3.1 we have

THEOREM 4.1. \( \frac{x}{y} \rightarrow \frac{r}{s} \) is an edge in \( F_{v,N} \) if and only if either

(a) \( x \equiv ur \mod \frac{N}{N/h} \) and \( ry - sx = N \), or
(b) \( x \equiv -ur \mod \frac{N}{N/h} \) and \( ry - sx = -N \).

An automorphism of the graph \( F_{u,N} \) is a permutation of \( \{\infty\} \) which takes edges to edges.

In view of this it can easily seen that \( \Gamma_v(N/h:h) < \text{Aut} F_{u,N} \).

THEOREM 4.2. \( \Gamma_v(N/h:h) \) permutes the vertices and the edges of \( F_{v,N} \) transitively.

PROOF. We do calculations only for edges. Suppose \( \frac{a}{bN} \rightarrow \frac{c}{dN} \) and \( \frac{k}{mN} \rightarrow \frac{s}{tN} \) are two edges in \( F_{v,N} \). Then there exist \( T_0, T_1 \) in \( \Gamma_v(N/h:h) \) such that

\[
\begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} T_0 \begin{pmatrix} a & c \\ bN & dN \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & u \\ 0 & N \end{pmatrix} T_1 \begin{pmatrix} k & s \\ mN & tN \end{pmatrix} .
\]

Therefore the element \( T_2 := T_1 T_0^{-1} \) is a required transformation in \( \Gamma_v(N/h:h) \).

DEFINITION 4.3. By a directed circuit in \( \Delta_{u,N} \) we mean a finite sequence \( v_1, v_2, \ldots, v_m \) of different vertices such that \( v_i \rightarrow v_j \rightarrow \ldots \rightarrow v_m \rightarrow v_1 \), where \( m \geq 3 \); an anti-directed circuit denotes the configuration like the above with at least one arrow (not all) reversed. If \( m = 3 \) then the circuit is called a triangle. And more we call the configuration \( v_i \rightarrow v_j \rightarrow v_i \) a self-paired edge.
THEOREM 4.4. (a) The graph $F_{u,N}$ contains directed triangles if and only if

$$u^2 \pm u + 1 \equiv 0 \mod \frac{N}{h}.$$ 

(b) $F_{u,N}$, $N > 1$, contains no anti-directed triangles.

PROOF. (a) Let $\frac{a}{b} \rightarrow \frac{c}{d} \rightarrow \frac{k}{l} \rightarrow \frac{a}{b}$ be a directed triangle in $F_{u,N}$. By Theorem 4.2 we may assume that the triangle has the form $\frac{1}{0} \rightarrow u \frac{x}{y} N \rightarrow \frac{x}{y} N \rightarrow 1 \frac{1}{0}$. Since $\frac{x}{y} N \rightarrow 1 \frac{1}{0}$, then $1 \equiv -ux \mod \frac{N}{h}$ and $-yN = -N$, that is $y = 1$. From $\frac{u}{N} \rightarrow \frac{x}{N}$, we get that either $x \equiv u^2 \mod \frac{N}{h}$ and $u-x \equiv 1$ or $x \equiv -u^2 \mod \frac{N}{h}$ and $u-x \equiv -1$. Therefore $x = u-1$ or $x = u+1$. Then the triangle is $\infty \rightarrow \frac{u}{N} \rightarrow \frac{u \pm 1}{N} \rightarrow \infty$. From the triangle we obtain $u^2 \pm u + 1 \equiv 0 \mod \frac{N}{h}$.

Now suppose that $u^2 \pm u + 1 \equiv 0 \mod \frac{N}{h}$. Then, by Theorem 4.1, the circuit $\infty \rightarrow \frac{u}{N} \rightarrow \frac{u \pm 1}{N} \rightarrow \infty$ is a directed triangle in $F_{u,N}$.

(b) By Theorem 4.2, without loss of generality, we may assume that we have an anti-directed triangle as $\infty \rightarrow \frac{u}{N} \leftrightarrow \frac{u \pm 1}{N} \rightarrow \infty$. Then, either $u \equiv bu \mod \frac{N}{h}$, $b-u = 1$ or $u \equiv -bu \mod \frac{N}{h}$, $b-u = -1$. On the other hand, $\frac{b}{N} \rightarrow \infty$ gives $1 \equiv -bu \mod \frac{N}{h}$.

So if $b = u+1$ then $u \equiv u(u+1) \mod \frac{N}{h}$, and if $b = u-1$ then $u \equiv -u(u-1) \mod \frac{N}{h}$.

Therefore we get $u \equiv 0 \mod \frac{N}{h}$, a contradiction.

COROLLARY 4.5. The subgroup $\Gamma_u(N/h : h)$ of $\mathbb{K}$ does not contain elliptic elements of orders 4 or 6.

PROOF. This follows from Theorem 2 in [3].
SOME PROPERTIES OF THE NORMALIZER OF $\Gamma_0(N)$ ON GRAPHS

COROLLARY 4.6. $\Gamma_g(N/h:h)$ contains an elliptic element of order 3 if and only if there exists $u$ in the group $U_N$ of units $mod\ N$ such that $F_{u,N}$ contains a triangle.

Proof. Suppose that $\varphi = \begin{pmatrix} a & b/h \\ cN/h & d \end{pmatrix}$ is an elliptic element in $\Gamma_g(N/h:h)$ of order 3. Then $a + d = \pm 1$ and $ad \equiv 1 \mod N/h$. And $a(\pm 1 - a) \equiv 1 \mod N/h$, that is $a^2 \pm a + 1 \equiv 0 \mod N/h$. As $(a, N) = (a, N/h) = 1$, there exists $u$ in $U_N$ such that $u \equiv a \mod N$. So $u^2 \pm u + 1 \equiv 0 \mod N/h$. By Theorem 4.4, $F_{u,N}$ contains a triangle.

Conversely suppose that $F_{u,N}$ contains a triangle. By Theorem 4.4 we get the triangle $\infty \to \frac{u}{N} \to \frac{u \pm 1}{N} \to \infty$. From this we obtain $\varphi = \begin{pmatrix} -u & u^2 \pm u + 1 \\ -N & u \pm 1 \end{pmatrix}$ in $\Gamma_g(N/h:h)$ of order 3. ■

COROLLARY 4.7. Let $2 \mid N$ or $3 \mid N$. Then $F_{u,N}$ does not contain any triangles. That is, $\Gamma_g(N/h:h)$ does not have elliptic elements of order 3.

Proof. Suppose first that $2 \mid N$. Then $2 \not\mid N$. Hence $F_{u,N}$ does not contain any triangles. If $3 \mid N$ then $3 \not\mid N/h$. Since $u^2 \pm u + 1 \not\equiv 0 \mod 3^2$, $u^2 \pm u + 1 \not\equiv 0 \mod N/h$. Therefore the result follows. ■

COROLLARY 4.8. Let $\begin{array}{cccc} a & \to & c \to & e \\ b & \to & d & \to & a \\ f & \to & b \end{array}$ be a triangle in $F_{u,N}$. Suppose that $u^2 \pm u + 1 \equiv 0 \mod N/h$, but $u^2 \pm u + 1 \not\equiv 0 \mod N$. Then there exists one and only one
elliptic element $T$ in $\Gamma_5(N/h:h)\setminus\Gamma_5(N)$, of order 3, such that

$$T\left(\begin{array}{c}
\frac{a}{b}
\end{array}\right) = \left(\begin{array}{c}
\frac{e}{f}
\end{array}\right), \quad T\left(\begin{array}{c}
\frac{c}{d}
\end{array}\right) = \left(\begin{array}{c}
\frac{e}{f}
\end{array}\right), \quad T\left(\begin{array}{c}
\frac{e}{f}
\end{array}\right) = \left(\begin{array}{c}
\frac{a}{b}
\end{array}\right).$$

PROOF. By Theorem 4.2 there exists some $T_0$ in $\Gamma_5(N/h:h)$ sending the given triangle to the triangle $u\rightarrow\frac{u\pm1}{N}\rightarrow\frac{u\pm1}{N}$, then $U = \left\{\begin{array}{c}
-u
\end{array}\right. \frac{u\pm1}{N}\rightarrow\frac{u\pm1}{N}$ is an elliptic element in $\Gamma_5(N/h:h)\setminus\Gamma_5(N)$ of order 3. So the element $T := T_0^{-1}U_{U\rightarrow\frac{u\pm1}{N}}$ is an elliptic element of order 3. If $T$ is in $\Gamma_5(N)$ then $U$ is in $T_0\Gamma_5(N)\Gamma_5^{-1} = \Gamma_5(N)$, a contradiction. It is obvious that $T$ satisfies desired conditions.

The converse of the above corollary is also true. That is,

**COROLLARY 4.9.** Let $\phi = \left(\begin{array}{c}
a
\frac{b}{d}
\end{array}\right)$ be an elliptic element of order 3 in $\Gamma_5(N/h:h)\setminus\Gamma_5(N)$. Then there exists a unique $u$ in $U_N$, with $u \equiv a \mod N$, such that

$u^2 \pm u + 1 \equiv 0 \mod N/h$ and $u^2 \pm u + 1 \not\equiv 0 \mod N$ and $\rightarrow u \rightarrow \frac{u\pm1}{N}$ is a triangle in $F_u\rightarrow\frac{u\pm1}{N}$.

PROOF. Since $\phi$ in an elliptic element of order 3 and is in $\Gamma_5(N/h:h)\setminus\Gamma_5(N)$ then by Corollary 4.7, $2 \mid N$ and $3^2 \mid N$ and so $h$ is just 3. It may happen that $h \mid bc$. This does not occur because in that case from the determinant we get $ad \equiv 1 \mod N$, that is, $a(-a^2+1) \equiv 1 \mod N$ or $a^2x + a + 1 \equiv 0 \mod N$. So $a^2 + a + 1 \equiv 0 \mod 3^2$, a contradiction. Therefore $h \mid bc$. Hence we get $a^2 + a + 1 \equiv 0 \mod h$, but $a^2 + a + 1 \not\equiv 0 \mod N$. Since $(a,h) = (a, N) = 1$ there is some $u$ in $U_N$ with $u \equiv a \mod N$. Therefore

$u^2 \pm u + 1 \equiv 0 \mod h$. Consequently we get the triangle as $\rightarrow u \rightarrow \frac{u\pm1}{N}$.
SOME PROPERTIES OF THE NORMALIZER OF $\Gamma_0(N)$ ON GRAPHS

REFERENCES


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