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Some Properties of The Normalizer of $\Gamma_o(N)$ on Graphs

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Abstract

In this paper we give some properties of the normalizer $\aleph$ of $\Gamma_o(N)$ in PSL(2, **i**) and show that the number of edges are related to the periods of group elements.

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1. INTRODUCTON

Let PSL(2, **i**) denote the group of all linear fractional transformations

$T : z \to \dfrac{az + b}{cz + d}$, where a,b,c,d are real and $ad - bc = 1$. For any natural number *N*, $\Gamma_o(N)$ denotes the subgroup of PSL(2, **i**) with integral coefficients and property that $c \equiv 0 \bmod N$. We shall in the sequel freely use 2x2 matrices to represent transformations, so that *T* is represented by the pair of the matrices

$$\pm\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ (a,b,c,d \in \textbf{i} \ , ad - bc = 1) \qquad (1.1)$$

We shall write these matrices as equal where convenient in matrix calculations.

Following [1], we denote by $\aleph$ the normalizer of $\Gamma_o(N)$ in PSL(2, **i**) consisting of all the matrices

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$$

where all letters are integers, $e \,//\, N/h^2$ and h $(= h(N))$ is the largest divisor of 24 for which $h^2 \mid N$ with the understandings that the determinant of the matrix is $e > 0$, and that $r \,//\, s$ means that r / s and $(r, r/s) = 1$ (r is called an exact divisor of s).

We now define $\Gamma_0(N/h : h)$ to be the set

$$\left\{\begin{pmatrix} a & b/h \\ cN & d \end{pmatrix}, ad-(bcN)/h = 1\right\},$$

which is a subgroup of PSL(2, $\mathbb{Z}$). It can be easily seen that this group is generated by $\Gamma_o(N)$ and $\begin{pmatrix} 1 & 1/h \\ 0 & 1 \end{pmatrix}$.

2. THE ACTION OF $\aleph$ ON $\hat{\mathbb{Q}}$

The element $\frac{x}{y}$ in $\hat{\mathbb{Q}}$ will be in reduced form, that is $(x,y)=1$. Since $\frac{x}{y}=\frac{-x}{-y}$, this representation is not unique. The action of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\frac{x}{y}$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \frac{x}{y} \to \frac{ax+by}{cx+dy}$$

and, it is the same on $\frac{-x}{-y}$.

THEOREM 2.1 [2]**.** Let N be any integer and $N = 2^{\alpha_1}3^{\alpha_2}p_3^{\alpha_3}\ldots p_n^{\alpha_n}$, the prime power decomposition of N. Then $\aleph$ is transitive on $\hat{\mathbb{Q}}$ if and only if $\alpha_1 \le 7$, $\alpha_2 \le 3$ and $\alpha_i \le 1$ for $i = 3,\ldots,n$.■

From now on N will be as in Theorem 2.1.

LEMMA 2.2. Let N be as above. Then the stabilizer of a point of $\hat{\mathbb{Q}}$ is an infinite cyclic group of $\aleph$**.**

PROOF. Since the action is transitive, the stabilizer of any two points in $\hat{\mathbb{Q}}$ are conjugate in $\aleph$**.** So it is sufficient to consider the stabilizer $\aleph_\infty$ of ∞. This is easily seen to consist of the elements of the form $\begin{pmatrix} 1 & b/h \\ 0 & 1 \end{pmatrix}$ and therefore $\aleph_\infty$ is the infinite cyclic group generated by $\begin{pmatrix} 1 & 1/h \\ 0 & 1 \end{pmatrix}$.■

We consider the imprimitivity of the action of $\aleph$ on $\hat{\mathbb{Q}}$ as follows: Generaly, let (G, Ω) be a transitive permutation group, consisting of a group G acting on a set Ω transitively. We will call (G, Ω) imprimitive if Ω admits some G-invariant equivalence relation ≈ different from

(i) the identity relation, $\alpha \approx \beta$ if and only if $\alpha = \beta$;

ii) the universal relation, $\alpha \approx \beta$ for all $\alpha, \beta \in \Omega$.

Otherwise (G, Ω) is called primitive. The equivalence classes are called blocks.

LEMMA 2.3 [3]. Let (G, Ω) be transitive. Then (G, Ω) is primitive if and only if G_α, the stabilizer of a point α, is a maximal subgroup of G for each $\alpha \in \Omega$.■

According to the above lemma if, for some α in Ω, $G_\alpha \quad H \quad G$, then the following relation ≈ on Ω makes (G, Ω) imprimitive:

$$g(\alpha) \approx g'(\alpha) \text{ if and only if } g^{-1}g' \in H. \qquad (2.1)$$

Now as a special case, since $\aleph_\infty \quad \Gamma_0(N/h : h) \quad \aleph$, we apply these ideas to the case where G is the normalizer $\aleph$, Ω is $\hat{\mathbb{Q}}$, and G_α is $\aleph_\infty$. In this case if $v = \frac{r}{s} \in \hat{\mathbb{Q}}$ then we find integers b and d such that the element $g = \begin{pmatrix} (re_2')e_2^0 & b/h \\ s_1 N/h & de_2^0 \end{pmatrix}$ is in $\aleph$ with determinant $e_2^0 \,\|\, N/h^2$ and sending ∞ to $\frac{r}{s}$, where

$$e_1 = \left(s, N/h\right),\ s = s_1 e_1,\ e_2 = N/e_1 h,\ e_2^0 = \left(e_2, N/h^2\right) \text{ and } e_2 = e_2' e_2^0.$$

Likewise, given $w = \frac{x}{y}$ in $\hat{\mathbb{Q}}$ there exists $g' \in \aleph$ such that $g'(\infty) = \frac{x}{y}$. Therefore by (2.1)

$$v \approx w = g(\infty) \approx g'(\infty) \in \aleph \;\Leftrightarrow\; ry_1 e_2' - xs_1 f_2 \equiv 0 \mod h,$$

where y_1, f_2 (for $\frac{x}{y}$) are defined as s_1, e_2' (for $\frac{r}{s}$) respectively. After some calculations it can be seen, if $\frac{r}{s} = \frac{1}{0}$, that $e_2' = 1$ and $y_1 \equiv 0 \mod h$. And here $y_1 \equiv 0 \mod h$ shows that $y_1 \equiv 0 \mod N$. Consequently, we have the block

$$[\infty] := \left\{ \frac{x}{y} \in \hat{¤} \mid y \equiv 0 \mod N \right\}. \qquad (2.2)$$

From the imprimitivity the number of blocks under $\approx$ is the index $|\aleph : \Gamma_0(N/h : h)|$, which is $2^r hs$, where h is as above, r is the number of distinct primes dividing N, and $s = s_2 s_3$ as in [1] as follows

$$s_1 = \begin{cases} \frac{3}{4} \;, & 2 \mid h(2^{\alpha_1})^2 \,\|\, N \\ 1 \;, & otherwise \end{cases} \;, \qquad s_2 = \begin{cases} \frac{2}{3} \;, & h(2^{\alpha_2})^2 = 9 \,\|\, N \\ 1 \;, & otherwise \end{cases} .$$

3. SUBORBITAL GRAPHS FOR ℵ ON $\hat{¤}$

Since (ℵ , $\hat{¤}$) is a transitive permutation group then (ℵ , $\hat{¤}^2$) is a permutation group with the action: for $(\alpha, \beta) \in \hat{¤}^2$ and $g \in$ ℵ , $g(\alpha, \beta) := (g(\alpha), g(\beta))$. The orbits of this action are called suborbitals of ℵ . From the orbit $\boldsymbol{O}(\alpha, \beta)$ we from a suborbital graph $\Delta(\alpha, \beta)$: its vertices are the elements of $\hat{¤}$ and there is a directed edge from a to b if $(a,b) \in \boldsymbol{O}(\alpha, \beta)$.

If $\boldsymbol{O}(\alpha, \beta) = \boldsymbol{O}(\beta, \alpha)$ then the graph consists of pairs of oppositely directed edges; we will replace such edges by undirected edges. In this case ve have an undirected graph which will be called self-paired. If $\boldsymbol{O}(\alpha, \beta) \neq \boldsymbol{O}(\beta, \alpha)$ then $\Delta(\beta, \alpha)$ is just $\Delta(\alpha, \beta)$ with the arrows reversed. In this case we call $\Delta(\alpha, \beta)$ and $\Delta(\beta, \alpha)$ paired graphs. Since ℵ is transitive on $\hat{¤}$, each suborbital contains a pair (∞ , v) for some $v \in$ ¤ ; writing

$v = \frac{u}{n}$ with $n > 0$ and $(u,n) = 1$, we denote the suborbital $\boldsymbol{O}(\infty, \frac{u'}{n})$ by $\boldsymbol{O}_{u',n}$ and the graph by $\Delta_{u',n}$.

These ideas were first introduced by Sims [8], and are also described in a paper by Neumann [7] and in books by Tsuzuku [9] and by Biggs and White [4], the emphasis being on applications to finite groups.

Using Theorem 4.1 in [2] we see that there exist integers u, m such that $\boldsymbol{O}(\infty, \frac{u'}{n}) = \boldsymbol{O}(\infty, \frac{u}{mN})$. Therefore we can drive the following.

THEOREM 3.1. $\frac{r}{s} \to \frac{x}{y}$ is an edge in $\Delta_{u,mN}$ if and only if there exists some $e \in ¢$ with $e \parallel N/h^2$, $N/eh \mid s$ then either

(a) $ry - sx = mN/e$ and $x \equiv ur \bmod \left(mN/eh\right)$, $y \equiv us \bmod mN$, or

(b) $ry - sx = -mN/e$ and $x \equiv -ur \bmod \left(mN/eh\right)$, $y \equiv -us \bmod mN$.

PROOF. Let $\frac{r}{s} \to \frac{x}{y}$ is an edge in $\Delta_{u,mN}$. Then there is some element

$A = \begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$ in ℵ , of determinant $e \parallel N/h^2$, sending ∞ to $\frac{r}{s}$ and $\frac{u}{mN}$ to $\frac{x}{y}$ and

therefore $\frac{ae}{cN/h} = \frac{r}{s}$ and $\frac{(aeu + (bmN)/h)}{(cuN/h + demN)} = \frac{x}{y}$. Since the determinant of the matrix A is

e and that (1,0) = 1 then $a = r$ and $s = cN/eh$. That is $N/eh \mid s$. Likewise we can see that

$$x = \pm(au + (bmN)/eh), \quad y = \pm(cuN/eh + dmN) .$$

From this we get

$$\begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}\begin{pmatrix} 1 & u \\ 0 & mN \end{pmatrix} = \begin{pmatrix} ae & aeu + (bmN)/h \\ cN/h & cuN/h + demN \end{pmatrix} = \begin{pmatrix} (-1)^i er & (-1)^j ex \\ (-1)^i es & (-1)^j ey \end{pmatrix} \quad (3.1)$$

where $i, j = 0,1$. If $i = j = 0$, then $x \equiv ur \bmod (mN/eh)$, $y \equiv us \bmod mN$ and taking determinant of (3.1) we see that $ry - sx = mN/e$. Similarly if $i = 1, j = 0$ (or $i = 0, j = 1$) we obtain (b). Conversely, if (a) holds with the given conditions, then exist integers b and d such that $x = ur + (bmN)/eh$ and $y = us + dmN$. It can be easily shown that the element $\begin{pmatrix} re & b/h \\ se & de \end{pmatrix}$ is in $\aleph$ and sends ∞ to $\frac{r}{s}$ and $\frac{u}{mN}$ to $\frac{x}{y}$. If (b) holds the proof follows similarly. ■

From now on, for the sake of simplicity, suppose that m is 1.

THEOREM 3.2. If $uv \equiv$ -1 mod N, then the suborbital graphs $\Delta_{u,N}$ and $\Delta_{v,N}$ are paired.

PROOF. Suppose that $uv \equiv -1 \bmod N$ and $\frac{r}{s} \to \frac{x}{y}$ is an edge in $\Delta_{u,N}$. Then, using Theorem 3.1, there exists some integer that $e \parallel N/h^2$, $N/eh \mid s$ and (a) or (b) is satisfied. Suppose (a) holds. Then $ry - sx = N/e$, and $x \equiv ur \bmod (N/eh)$, $y \equiv us \bmod N$. Since $uv \equiv -1 \bmod N$, then $r \equiv -vx \bmod N/eh$, $s \equiv -vy \bmod N$. Then $\frac{x}{y} \to \frac{r}{s}$ is an edge in $\Delta_{v,N}$ that is, $\Delta_{u,N}$ and $\Delta_{v,N}$ are paired. ■

COROLLARY 3.3. $\Delta_{u,N}$ is self-paired if and only if $u^2 \equiv -1 \bmod N/h$.

PROOF. Suppose $\boldsymbol{O}(\infty, \frac{u}{N}) = \boldsymbol{O}(\frac{u}{N}, \infty)$. Then there exists some $\varphi = \begin{pmatrix} ae & b/h \\ cN/h & de \end{pmatrix}$ in $\aleph$ such that sending ∞ to $\frac{u}{N}$ and $\frac{u}{N}$ to ∞. From this, φ must be $\begin{pmatrix} ue & b/h \\ eN & -ue \end{pmatrix}$ and then e is equal to 1. So $u^2 \equiv -1 \bmod N/h$.

Conversely $u^2 \equiv -1 \bmod N/h$, then exists some integer b such that $u^2 = -1 + bN/h$.

Therefore $\begin{pmatrix} u & b/h \\ Ne & -u \end{pmatrix}$ is in $\aleph$ and satisfied desired condition. ■

4. THE GRAPH $F_{u,N}$

We let $F_{u,N}$ be the subgraph of $\Delta_{u,N}$ whose vertices form the block [∞]. So, by Theorem 3.1 we have

THEOREM 4.1. $\frac{r}{s} \to \frac{x}{y}$ is an edge in $F_{u,N}$ if and only if either

(a) $x \equiv ur \bmod N/h$ and $ry - sx = N$, or

(b) $x \equiv -ur \bmod N/h$ and $ry - sx = -N$. ■

An automorphism of the graph $F_{u,N}$ is a permutation of [∞] which takes edges to edges. In view of this it can easily seen that $\Gamma_o(N/h:h) < Aut\ F_{u,N}$.

THEOREM 4.2. $\Gamma_o(N/h:h)$ permutes the vertices and the edges of $F_{u,N}$ transitively.

PROOF. We do calculations only for edges. Suppose $\frac{a}{bN} \to \frac{c}{dN}$ and $\frac{k}{mN} \to \frac{s}{tN}$ are two edges in $F_{u,N}$. Then there exist T_0, T_1 in $\Gamma_o(N/h:h)$ such that $\left(\frac{1}{0}, \frac{u}{N}\right) \xrightarrow{T_0} \left(\frac{a}{bN}, \frac{c}{dN}\right)$ and $\left(\frac{1}{0}, \frac{u}{N}\right) \xrightarrow{T_1} \left(\frac{k}{mN}, \frac{s}{tN}\right)$. Therefore the element $T_2 := T_1 o T_0^{-1}$ is a required transformation in $\Gamma_o(N/h:h)$. ■

DEFINITION 4.3. By a directed circuit in $\Delta_{u,N}$ we mean a finite sequence $v_1, v_2, ..., v_m$ of different vertices such that $v_1 \to v_2 \to ... \to v_m \to v_1$, where $m \geq 3$; an anti-directed circuit denotes the configuration like the above with at least one arrow (not all) reversed. If $m = 3$ then the circuit is called a triangle. And more we call the configuration $v_1 \to v_2 \to v_1$ a self-paired edge.

THEOREM 4.4. (a) The graph $F_{u,N}$ contains directed triangles if and only if $u^2 \pm u + 1 \equiv 0 \mod N/h$.

(b) $F_{u,N}$, *N >1,* contains no anti-directed triangles.

PROOF. (a) Let $\frac{a}{b} \to \frac{c}{d} \to \frac{k}{\ell} \to \frac{a}{b}$ be a directed triangle in $F_{u,N}$. By Theorem 4.2 we may assume that the triangle has the form $\frac{1}{0} \to \frac{u}{N} \to \frac{x}{yN} \to \frac{1}{0}$. Since $\frac{x}{yN} \to \frac{1}{0}$, then $1 \equiv -ux \mod N/h$ and *-yN = -N,* that is $y = 1$. From $\frac{u}{N} \to \frac{x}{N}$, we get that either $x \equiv u^2 \mod N/h$ and $u - x = 1$ or $x \equiv -u^2 \mod N/h$ and $u - x = -1$. Therefore $x = u-1$ or $x = u+1$. Then the triangle is $\infty \to \frac{u}{N} \to \frac{u \pm 1}{N} \to \infty$. From the triangle we obtain $u^2 \pm u + 1 \equiv 0 \mod N/h$.

Now suppose that $u^2 \pm u + 1 \equiv 0 \mod N/h$. Then, by Theorem 4.1, the circuit $\infty \to \frac{u}{N} \to \frac{u \pm 1}{N} \to \infty$ is a directed triangle in $F_{u,N}$.

(b) By Theorem 4.2, without loss of generality, we may assume that we have an anti-directed triangle as $\infty \to \frac{u}{N} \leftarrow \frac{u \pm 1}{N} \to \infty$. Then, either $u \equiv bu \mod N/h$, $b - u = 1$ or $u \equiv -bu \mod N/h$, $b - u = -1$. On the other hand, $\frac{b}{N} \to \infty$ gives $1 \equiv -bu \mod N/h$. So if $b = u+1$ then $u \equiv u(u+1) \mod N/h$, and if $b = u-1$ then $u \equiv -u(u-1) \mod N/h$. Therefore we get $u \equiv 0 \mod N/h$, a contradiction. ■

COROLLARY 4.5. The subgroup $\Gamma_0(N/h:h)$ of $\aleph$ does not contain elliptic elements of orders 4 or 6.

PROOF. This follows from Theorem 2 in [3]. ■

COROLLARY 4.6. $\Gamma_o(N/h:h)$ contains an elliptic element of order 3 if and only if there exists *u* in the group U_N of units *mod N* such that $F_{u,N}$ contains a triangle.

PROOF. Suppose that $\varphi=\begin{pmatrix} a & b/h \\ cN/h & d \end{pmatrix}$ is an elliptic element in $\Gamma_o(N/h:h)$ of order 3. Then $a+d=\pm1$ and $ad\equiv 1 \bmod N/h$. And $a(\pm1-a)\equiv 1 \bmod N/h$, that is $a^2\pm a+1\equiv 0 \bmod N/h$. As $(a,N)=(a, N/h)=1$, then there exists *u* in U_N such that $u\equiv a \bmod N$. So $u^2\pm u+1\equiv 0 \bmod N/h$. By Theorem 4.4, $F_{u,N}$ contains a triangle.

Conversely suppose that $F_{u,N}$ contains a triangle. By Theorem 4.4 we get the triangle $\infty\rightarrow\frac{u}{N}\rightarrow\frac{u\pm1}{N}\rightarrow\infty$. From this we obtain $\phi=\begin{pmatrix} -u & \frac{u^2\pm u+1}{N} \\ -N & u\pm1 \end{pmatrix}$ in $\Gamma_o(N/h:h)$ of order 3.■

COROLLARY 4.7. Let $2\mid N$ or $3^3\mid N$. Then $F_{u,N}$ does not contain any triangles. That is, $\Gamma_o(N/h:h)$ does not have elliptic elements of order 3.

PROOF. Suppose first that $2\mid N$. Then $2\,/\,N/h$. So $u^2\pm u+1\not\equiv 0 \bmod 2$, that is, $u^2\pm u+1\not\equiv 0 \bmod N/h$. Hence $F_{u,N}$ does not contain any triangles. If $3^3\mid N$ then $3^2\mid N/h$. Since $u^2\pm u+1\not\equiv 0 \bmod 3^2$, $u^2\pm u+1\not\equiv 0 \bmod N/h$. Therefore the result follows.■

COROLLARY 4.8. Let $\frac{a}{b}\rightarrow\frac{c}{d}\rightarrow\frac{e}{f}\rightarrow\frac{a}{b}$ be a triangle in $F_{u,N}$. Suppose that $u^2\pm u+1\equiv 0 \bmod N/h$, but $u^2\pm u+1\not\equiv 0 \bmod N$. Then there exists one and only one elliptic element T in $\Gamma_0(N/h:h)\backslash\Gamma_0(N)$, of order 3, such that

$$T\begin{pmatrix}\frac{a}{b}\end{pmatrix}=\begin{pmatrix}\frac{c}{d}\end{pmatrix},\ T\begin{pmatrix}\frac{c}{d}\end{pmatrix}=\begin{pmatrix}\frac{e}{f}\end{pmatrix},\ T\begin{pmatrix}\frac{e}{f}\end{pmatrix}=\begin{pmatrix}\frac{a}{b}\end{pmatrix}.$$

PROOF. By Theorem 4.2 there exists some T_0 in $\Gamma_0(N/h:h)$ sending the given triangle to the triangle $\infty\to\frac{u}{N}\to\frac{u\pm1}{N}\to\infty$, then $U=\begin{pmatrix}-u & \frac{u^2\pm u+1}{N}\\ -N & u\pm1\end{pmatrix}$ is an elliptic element in $\Gamma_0(N/h:h)\backslash\Gamma_0(N)$ of order 3. So the element $T:=T_0^{-1}\mathrm{o}U\mathrm{o}T_0$ is an elliptic element of order 3. If T is in $\Gamma_0(N)$ then U is in $T_0\mathrm{o}\Gamma_0(N)\mathrm{o}T_0^{-1}=\Gamma_0(N)$, a contradiction. It is obvious that T satisfies desired conditions.■

The converse of the above corollary is also true. That is,

COROLLARY 4.9. Let $\varphi=\begin{pmatrix}a & b/h\\ cN & d\end{pmatrix}$ be an elliptic element of order 3 in $\Gamma_0(N/h:h)\backslash\Gamma_0(N)$. Then there exists a unique u in U_N, with $u\equiv a \bmod N$, such that $u^2\pm u+1\equiv0 \bmod N/h$ and $u^2\pm u+1\not\equiv0 \bmod N$ and $\infty\to\frac{u}{N}\to\frac{u\pm1}{N}\to\infty$ is a triangle in $F_{u,N}$.

PROOF. Since φ in an elliptic element of order 3 and is in $\Gamma_0(N/h:h)\backslash\Gamma_0(N)$ then by Corollary 4.7, $2\mid N$ and $3^2\parallel N$ and so h is just 3. It may happen that $h\mid bc$. This does not occur because in that case from the determinant we get $ad\equiv1 \bmod N$, that is, $a(-a\pm1)\equiv1 \bmod N$ or $a^2\pm a+1\equiv0 \bmod N$. So $a^2\pm a+1\equiv0 \bmod 3^2$, a contradiction. Therefore h bc. Hence we get $a^2\pm a+1\equiv0 \bmod N/h$, but $a^2\pm a+1\not\equiv0 \bmod N$. Since $(a, N/h)$ = (a, N) = 1 there is some u in U_N with $u\equiv a \bmod N$. Therefore $u^2\pm u+1\equiv0 \bmod N/h$. Consequently we get the triangle as $\infty\to\frac{u}{N}\to\frac{u\pm1}{N}\to\infty$.■

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