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Approximate Solutions of Twelfth-order Boundary Value Problems

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Abstract

In this paper, we implement a relatively new analytical technique which is called the variational iteration method for solving the twelfth-order boundary value problems by converting the problems into a system of integral equations. The analytical results of the problems have been obtained in terms of convergent series with easily computable components. Comparisons are made to verify the reliability and accuracy of the proposed algorithm. Several examples are given to check the efficiency of the suggested technique. The fact that variational iteration method solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

Mathematics Subject Classification 2000: 34B15
Additional Key Words and Phrases: Variational iteration method, nonlinear problems, higher order boundary value problems, error estimates.

1. INTRODUCTION

In this paper, we consider the general twelfth-order boundary value problem of the type:

$$y^{(xii)}(x) = f(x, y), \qquad a < x < b, \tag{1}$$

with boundary conditions

$$y(a) = A_0, \quad y^{(2)}(a) = A_1, \quad y^{(4)}(a) = A_2, \quad y^{(6)}(a) = A_3, \quad y^{(8)}(a) = A_4, \quad y^{(10)}(a) = A_5,$$
$$y(b) = B_0, \quad y^{(2)}(b) = B_1 \quad y^{(4)}(b) = B_2, \quad y^{(6)}(b) = B_3, \quad y^{(8)}(b) = B_4, \quad y^{(10)}(b) = B_5,$$

where $f = f(x, y)$ is assumed real and as many times differentiable as required for $x \in [0, b]$ and $A_i, i = 0, 1, 2, 3, 4, 5$ *and* $B_i, i = 0, 1, 2, 3, 4, 5$ are real finite constants. These problems arise in hydrodynamic and hydro magnetic stability. Moreover, when a uniform magnetic field is applied across the fluid in the same direction as that of gravity than instability may sets in as over stability which can be modeled by a twelfth-order boundary value problem, see [1-5, 13, 22, 23, 25-27] and the references therein. The boundary value problems of higher order have been investigated due to their mathematical importance and the potential for applications in hydrodynamic, astronmy and hydro magnetic stability; see [1-5, 13, 22, 23, 25-27]. Several techniques including finite-difference, polynomial and non polynomial spline and decomposition have been employed for solving such problems, see [2, 5, 25-27] and the references therein. All these techniques have their inbuilt deficiencies, like divergence of the results at the points adjacent to the boundary and calculation of the so-called Adomian's polynomials. Moreover, the performance of most of the methods used so far is well known that they provide the solution at grid points only. Recently, Noor and Mohyud-Din employed homotopy perturbation, variational iteration, variational iteration method using He's polynomials and variational decomposition methods for solving higher-order boundary value problems, see [13, 18, 19, 22, 23]. He [6-10] developed the variational iteration method (VIM) for solving linear, nonlinear, initial and boundary value problems. It is worth mentioning that the variational iteration method (VIM) was originated by Inokuti, Sekine and Mura [11], but the real potential of the VIM was explored and exploited by by He. The basic motivation of this paper is to apply the variational iteration method (VIM) for solving the twelfth-order boundary value problems. It is shown that the twelfth-order boundary value problems are equivalent to a system of integral equations by introducing a transformation which plays a pivotal role in the implementation of the proposed technique. We write the correct functional for the system of twelfth-order boundary value problems and calculate the Lagrange multiplier optimally via variational theory. Several examples are given to illustrate the reliability and performance of the proposed method. Numerical results and graphical representations clearly indicate the reliability and accuracy of the proposed variational iteration method (VIM).

2. VARIATIONAL ITERATION METHOD

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x), \tag{2}$$

where L is a linear operator, N a nonlinear operator and g(x) is the inhomogeneous term. According to variational iteration method [6-22, 24, 28], we can construct a correct functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(s) + N\tilde{u}_n(s) - g(s))\, ds, \tag{3}$$

where λ is a Lagrange multiplier [6-11], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e. $\delta\tilde{u}_n = 0;$ (3) is called as a correct functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [6-10]. In this method, it is required first to determine the Lagrange multiplier λ optimally via variational theory. The successive approximation $u_{n+1},\ n \geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by $u = \lim_{n \to \infty} u_n$. For the convergence criteria and error estimates of variational iteration method, see Ramos [24]. Now, we shall develop the proposed variational iteration method for a system of integral equations [19, 20]. For the sake of simplicity and to convey the idea of the technique, we consider the following system of differential equations:

$$x_i^{/}(t) = f_i(t, x_i), \quad i = 1,2,3,....,n, \tag{4}$$

subject to the boundary conditions $x_i(0) = c_i,\ i = 1,2,3,...,n$. To solve the system by means of the variational iteration method, we rewrite the system (4) in the following form:

$$x_i^{/}(t) = f_i(x_i) + g_i(t), \quad i = 1,2,3,.....,n, \tag{5}$$

subject to the boundary conditions. $x_i(0) = c_i,\ i = 1,2,3,...,n$ and g_i is defined in (2). The correct functional for the nonlinear system (5) can be approximated as:

$$x_1^{(k+1)}(t) = x_1^{(k)}(t) + \int_0^t \lambda_1 \left(x_1^{/(k)}(T), f_1(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - g_1(T) \right) dT,$$

$$x_2^{(k+1)}(t) = x_2^{(k)}(t) + \int_0^t \lambda_2 \left(x_2^{/(k)}(T), f_2(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - g_2(T) \right) dT,$$

$$\vdots$$

$$x_n^{(k+1)}(t) = x_n^{(k)}(t) + \int_0^t \lambda_n \left(x_n^{/(k)}(T), f_n(\tilde{x}_1^{(k)}(T), \tilde{x}_2^{(k)}(T), \ldots, \tilde{x}_n^{(k)}(T)) - g_n(T) \right) dT,$$

(6)

where $\lambda_i = \pm 1, \; i = 1,2,3,\ldots,n$ are Lagrange multipliers, $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ denote the restricted variations.

For $\lambda_i = -1, \quad i = 1,2,3,\ldots,n$ we have the following iterative schemes:

$$x_1^{(k+1)}(t) = x_1^{(k)}(t) - \int_0^t \left(x_1'^{(k)}(T), f_1(x_1^{(k)}(T), x_2^{(k)}(T), \ldots, x_n^{(k)}(T)) - g_1(T) \right) dT,$$

$$x_2^{(k+1)}(t) = x_2^{(k)}(t) - \int_0^t \left(x_2'^{(k)}(T), f_2(x_1^{(k)}(T), x_2^{(k)}(T), \ldots, x_n^{(k)}(T)) - g_2(T) \right) dT,$$

$$\vdots$$

$$x_n^{(k+1)}(t) = x_n^{(k)}(t) - \int_0^t \left(x_n'^{(k)}(T), f_n(x_1^{(k)}(T), x_2^{(k)}(T), \ldots, x_n^{(k)}(T)) - g_n(T) \right) dT.$$

(7)

If we start with the initial approximations $x_i(0) = c_i, \quad i = 1,2,3,\ldots,n$ then the approximations can be completely determined; finally we approximate the solution $x_i(t) = \lim_{k \to \infty} x_i^{(n)}(t)$ by the nth term $x_i^{(n)}(t)$ for $i = 1,2,3,\ldots,n.$

3. NUMERICAL APPLICATIONS

In this section, we apply the variational iteration method for solving the twelfth-order boundary value problems by converting the problem into a system of integral equations. The proposed variational iteration method is applied to the resultant system of integral equations.

Example 3.1 [13, 23, 26] Consider the following nonlinear boundary value problem of twelfth order

$$y^{(xii)}(x) = \frac{1}{2} e^{-x} y^2(x), \qquad\qquad 0 < x < 1$$

with boundary conditions

$$\begin{aligned} &y(0) = y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = y^{(x)}(0) = y^{(x)}(0) = 2, \\ &y(1) = y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) = y^{(x)}(1) = y^{(x)}(1) = 2e. \end{aligned}$$

The exact solution of the problem is

$$y(x) = 2e^x.$$

Using the transformation

$$\frac{dy}{dx} = a(x), \quad \frac{da}{dx} = b(x), \quad \frac{db}{dx} = e(x), \quad \frac{de}{dx} = f(x), \quad \frac{df}{dx} = g(x), \quad \frac{dg}{dx} = h(x),$$
$$\frac{dh}{dx} = q(x), \quad \frac{dq}{dx} = l(x), \quad \frac{dl}{dx} = m(x), \quad \frac{dm}{dx} = n(x), \quad \frac{dn}{dx} = z(x),$$

the above boundary value problem can be written as the following system of differential equations

$$\begin{cases} \dfrac{dy}{dx} = a(x), & \dfrac{da}{dx} = b(x), \quad \dfrac{db}{dx} = e(x), \quad \dfrac{de}{dx} = f(x), \quad \dfrac{df}{dx} = g(x), \quad \dfrac{dg}{dx} = h(x), \\ \dfrac{dh}{dx} = q(x), & \dfrac{dq}{dx} = l(x), \quad \dfrac{dl}{dx} = m(x), \quad \dfrac{dm}{dx} = n(x), \quad \dfrac{dn}{dx} = z(x), \quad \dfrac{dz}{dx} = \dfrac{1}{2} e^{-x} y^2(x), \end{cases}$$

with boundary conditions

$$y(0) = 2, \quad a(0) = A, \quad b(0) = 2, \quad e(0) = B, \quad f(0) = 2, \quad g(0) = C,$$
$$h(0) = 2, \quad q(0) = D, \quad l(0) = 2, \quad m(0) = E, \quad n(0) = 2, \quad z(0) = F.$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = +1, \qquad i = 1, 2, \cdots, 12.$

$$\begin{cases} y^{(k+1)}(x) = 2 + \int_0^x a^{(k)}(t)dt, & a^{(k+1)}(x) = A + \int_0^x b^{(k)}(t)dt, \quad b^{(k+1)}(x) = 2 + \int_0^x e^{(k)}(t)dt, \\ e^{(k+1)}(x) = B + \int_0^x f^{(k)}(t)dt, & f^{(k+1)}(x) = 2 + \int_0^x g^{(k)}(t)dt, \quad g^{(k+1)}(x) = C + \int_0^x h^{(k)}(t)dt, \\ h^{(k+1)}(x) = 2 + \int_0^x q^{(k)}(t)dt, & q^{(k+1)}(x) = D + \int_0^x l^{(k)}(t)dt, \quad l^{(k+1)}(x) = 2 + \int_0^x m^{(k)}(t)dt. \\ m^{(k+1)}(x) = E + \int_0^x n^{(k)}(t)dt, & n^{(k+1)}(x) = 2 + \int_0^x z^{(k)}(t)dt, \quad z^{(k+1)}(x) = F + \int_0^x e^{-t} y^2(t)dt. \end{cases}$$

Consequently, the following approximants are obtained

$$\begin{cases} y^{(0)}(x) = 2, & a^{(0)}(x) = A, & b^{(0)}(x) = 2, & e^{(0)}(x) = B, & f^{(0)}(x) = 2, & g^{(0)}(x) = C, \\ h^{(0)}(x) = 2, & q^{(0)}(x) = D, & l^{(0)}(x) = 2, & m^{(0)}(x) = E, & n^{(0)}(x) = 2, & z^{(0)}(x) = F, \end{cases}$$

$$\begin{cases} y^{(1)}(x) = 2 + Ax, & a^{(1)}(x) = A + 2x, & b^{(1)}(x) = 2 + Bx, & e^{(1)}(x) = B + 2x, \\ f^{(1)}(x) = 2 + Cx, & g^{(1)}(x) = C + 2x, & h^{(1)}(x) = 2 + Dx, & q^{(1)}(x) = D + 2x, \\ l^{(1)}(x) = 2 + Ex, & m^{(1)}(x) = E + 2x, & n^{(1)}(x) = 2 + Fx, & z^{(1)}(x) = F + 2 - 2e^{-x}, \end{cases}$$

$$\begin{cases} y^{(2)}(x) = 2 + Ax + x^2, \; a^{(2)}(x) = A + 2x + \frac{1}{2}Bx^2, \quad b^{(2)}(x) = 2 + Bx + x^2, \; e^{(2)}(x) = B + 2x + \frac{1}{2}Cx^2, \\ f^{(2)}(x) = 2 + Cx + x^2, \; g^{(2)}(x) = C + 2x + \frac{1}{2}Dx^2, \quad h^{(2)}(x) = 2 + Dx + x^2, \; q^{(2)}(x) = D + 2x + \frac{1}{2}Ex^2, \\ l^{(2)}(x) = 2 + Ex + x^2, \; m^{(2)}(x) = E + 2x + \frac{1}{2}Fx^2, \quad n^{(2)}(x) = 2 + Fx + 2 + + Fx + 2x - 2e^{-x}, \\ z^{(2)}(x) = F + 2 - 2e^{-x} + 1 - \frac{1}{2}A^2x^2e^{-x} - A^2xe^{-x} - e^{-x} + \frac{3}{2} - \frac{1}{4}x^4x^{-2x} - \frac{3}{4}x^2e^{-2x} - \frac{3}{4}xe^{-2x} - \frac{3}{4}e^{-2x}, \end{cases}$$

$$\begin{cases} y^{(3)}(x)=2+Ax+x^2+\frac{1}{3!}Bx^3, & a^{(3)}(x)=A+2x+\frac{1}{2}Bx^2+\frac{1}{3}x^3, \quad b^{(3)}(x)=2+Bx+x^2+\frac{1}{3!}Cx^3, \\ e^{(3)}(x)=B+2x+\frac{1}{2}Cx^2+\frac{1}{3}x^3, & f^{(3)}(x)=2+Cx+x^2+\frac{1}{3!}Dx^3, \quad g^{(3)}(x)=C+2x+\frac{1}{2}Dx^2+\frac{1}{3}x^3, \\ h^{(3)}(x)=2+Dx+x^2+\frac{1}{3!}Ex^3, & q^{(3)}(x)=D+2x+\frac{1}{2}Ex^2+\frac{1}{3}x^3, \quad l^{(3)}(x)=2+Ex+x^2+\frac{1}{3!}Fx^3, \\ m^{(3)}(x)=E+2x+\frac{1}{2}Fx^2-2x+x^2, \\ n^{(3)}(x)=2+Fx-2+2x-1+Ax^2-\frac{1}{2}(A^2-\frac{1}{2}A^2x^2e^{-x}-A^2xe^{-x}-e^{-x})-A^2(-1+e^{-x}-xe^{-x})+e^{-x}, \\ z^{(3)}(x)=F+2-2e^{-x}+1-\frac{1}{2}A^2x^2e^{-x}-A^2xe^{-x}-e^{-x}+\frac{3}{2}-\frac{1}{4}x^4x^{-2x}-\frac{3}{4}x^2e^{-2x}-\frac{3}{4}xe^{-2x}-\frac{3}{4}e^{-2x}, \\ \vdots \end{cases}$$

The series solution is given as

$$y(x)=2+Ax+\frac{2}{2!}x^2+\frac{1}{3!}Bx^3+\frac{2}{4!}x^4+\frac{1}{5!}Cx^5+\frac{1}{6!}x^6+\frac{1}{7!}Dx^7+\frac{2}{8!}x^8+\frac{1}{9!}Ex^9+\frac{2}{10!}x^{10}+\frac{1}{11!}Fx^{11}$$
$$+\frac{2}{12!}x^{12}+\frac{2}{13!}(A-1)x^{13}+\cdots.$$

Imposing the boundary conditions at x = 1 will yield

$A=2.002114383,\quad B=1.986107510,\quad C=2.333176702,$
$D=3.001515917,\quad E=1.986107510,\quad F=2.102525395.$

The series solution is given as

$$y(x)=2+2.002114783x+x^2+0.329322880x^3+\frac{1}{12}x^4+0.01944313918x^5+\frac{1}{720}x^6+0.0005955388724x^7$$
$$+\frac{1}{20160}x^8+0.5473179867\times10^{-5}x^9+\frac{1}{1814400}x^{10}+0.5267269408\times10^{-7}x^{11}+\frac{1}{239500800}x^{12}$$
$$+0.3218601046\times10^{-9}x^{13}+\cdots.$$

Table 3.1 (Error estimates)

x	Exact solution	Series solution	*Errors
0.0	2.000000000	2.000000000	0.00000000000
0.1	2.2103418362	2.2105496398	0.0002078037
0.2	2.4428055163	2.4431996667	0.0003941504
0.3	2.6997176152	2.7002578996	0.0005402845
0.4	2.9836493953	2.9842815093	0.0006321141
0.5	3.2974425414	3.2981039180	0.0006613766
0.6	3.6442376008	3.6448637056	0.0006261048
0.7	4.0275054149	4.0280359083	0.0005304933
0.8	4.4510818570	4.4514661221	0.0003842651
0.9	4.9192062223	4.9194078617	0.0002016394
1.0	5.4365636569	5.4365636559	0.0002016394

*Error = Exact solution-Series solution.

Table 3.1 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of y(x).

Figure 3.1

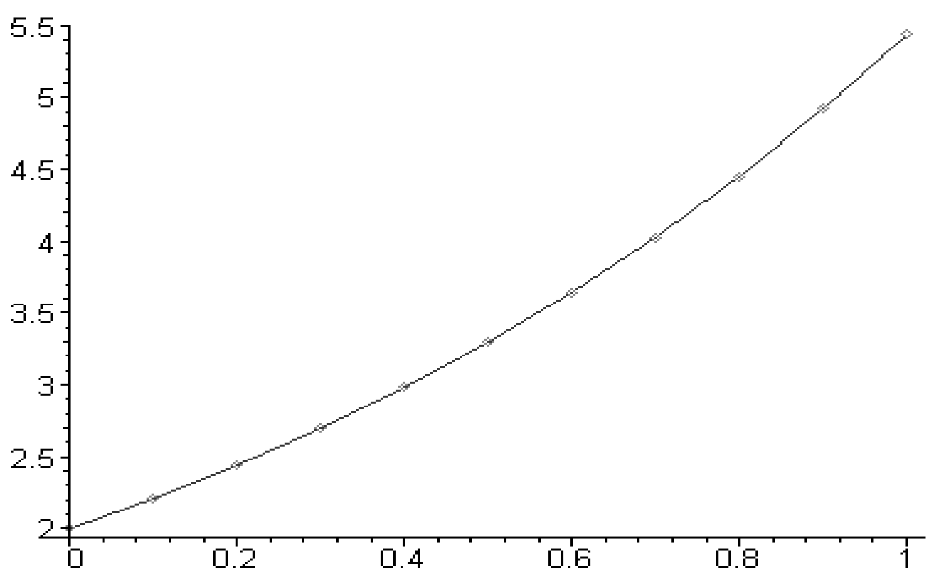

Fig. 3.1 clearly indicates the accuracy of the proposed variational iteration method (VIM).

Example 3.2 [13, 23, 26] Consider the following nonlinear boundary value problem of twelfth order,

$$y^{(xii)}(x) = 2e^{x}y^{2}(x) + y'''(x), \qquad 0 < x < 1$$

with boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = y^{(x)}(0) = 1,$$
$$y(1) = y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) = y^{(x)}(1) = e^{-1}.$$

The exact solution of the problem is

$$y(x) = e^{-x}.$$

Using the transformation

$$\frac{dy}{dx} = a(x), \quad \frac{da}{dx} = b(x), \quad \frac{db}{dx} = e(x), \quad \frac{de}{dx} = f(x), \quad \frac{df}{dx} = g(x), \quad \frac{dg}{dx} = h(x),$$
$$\frac{dh}{dx} = q(x), \quad \frac{dq}{dx} = l(x), \quad \frac{dl}{dx} = m(x), \quad \frac{dm}{dx} = n(x), \quad \frac{dn}{dx} = z(x),$$

the above boundary value problem can be written as the following system of differential equations

$$\begin{cases} \frac{dy}{dx} = a(x), \quad \frac{da}{dx} = b(x), \quad \frac{db}{dx} = e(x), \quad \frac{de}{dx} = f(x), \quad \frac{df}{dx} = g(x), \quad \frac{dg}{dx} = h(x), \\ \frac{dh}{dx} = q(x), \quad \frac{dq}{dx} = l(x), \quad \frac{dl}{dx} = m(x), \quad \frac{dm}{dx} = n(x), \quad \frac{dn}{dx} = z(x), \quad \frac{dz}{dx} = 2e^{x} y^{2}(x) + y'''(x), \end{cases}$$

with boundary conditions

$$y(0) = 1, \quad a(0) = A, \quad b(0) = 1, \quad e(0) = B, \quad f(0) = 1, \quad g(0) = C,$$
$$h(0) = 1, \quad q(0) = D, \quad l(0) = 1, \quad m(0) = E, \quad n(0) = 1, \quad z(0) = F.$$

The above system of differential equations can be written as the following system of integral equations with Lagrange multipliers $\lambda_i = +1, \qquad i = 1, 2, \cdots, 12.$

$$\begin{cases} y^{(k+1)}(x) = 1 + \int_0^x a^{(k)}(t)dt, \quad a^{(k+1)}(x) = A + \int_0^x b^{(k)}(t)dt, \quad b^{(k+1)}(x) = 1 + \int_0^x e^{(k)}(t)dt, \\ e^{(k+1)}(x) = B + \int_0^x f^{(k)}(t)dt, \quad f^{(k+1)}(x) = 1 + \int_0^x g^{(k)}(t)dt, \quad g^{(k+1)}(x) = C + \int_0^x h^{(k)}(t)dt, \\ h^{(k+1)}(x) = 1 + \int_0^x q^{(k)}(t)dt, \quad q^{(k+1)}(x) = D + \int_0^x l^{(k)}(t)dt, \quad l^{(k+1)}(x) = 1 + \int_0^x m^{(k)}(t)dt, \\ m^{(k+1)}(x) = E + \int_0^x n^{(k)}(t)dt, \quad n^{(k+1)}(x) = 1 + \int_0^x z^{(k)}(t)dt, \quad z^{(k+1)}(x) = F + \int_0^x 2e^{t}\left(\tilde{y}^{(k)}(t)\right)^2 + \frac{d^3\tilde{y}^{(k)}(t)}{dt^3}dt. \end{cases}$$

Consequently, the following approximants are obtained

$$\begin{cases} y^{(0)}(x)=1, & a^{(0)}(x)=A, & b^{(0)}(x)=1, & e^{(0)}(x)=B, & f^{(0)}(x)=1, & g^{(0)}(x)=C, \\ h^{(0)}(x)=1, & q^{(0)}(x)=D, & l^{(0)}(x)=1, & m^{(0)}(x)=E, & n^{(0)}(x)=1, & z^{(0)}(x)=F, \end{cases}$$

$$\begin{cases} y^{(1)}(x)=1+Ax, & a^{(1)}(x)=A+x, & b^{(1)}(x)=1+Bx, & e^{(1)}(x)=B+x, \\ f^{(1)}(x)=1+Cx, & g^{(1)}(x)=C+x, & h^{(1)}(x)=1+Dx, & q^{(1)}(x)=D+x, \\ l^{(1)}(x)=1+Ex, & m^{(1)}(x)=E+x, & n^{(1)}(x)=1+Fx, & z^{(1)}(x)=F+2-2e^{x}, \end{cases}$$

$$\begin{cases} y^{(2)}(x)=1+Ax+\frac{1}{2!}x^2, & a^{(2)}(x)=A+x+\frac{1}{2!}Bx^2, & b^{(2)}(x)=1+Bx, & e^{(2)}(x)=B+x+\frac{1}{2!}Cx^2, \\ f^{(2)}(x)=1+Cx+\frac{1}{2!}x^2, & g^{(2)}(x)=C+x+\frac{1}{2!}Dx^2, & h^{(2)}(x)=1+Dx+\frac{1}{2!}x^2, & q^{(2)}(x)=D+x+\frac{1}{2!}Ex^2, \\ l^{(2)}(x)=1+Ex+\frac{1}{2!}x^2, & m^{(2)}(x)=E+x+\frac{1}{2!}Fx^2, & n^{(2)}(x)=3+(F+2)x-2e^{x}, & \\ z^{(2)}(x)=4(A-A^2)+F-4Ax(1+A)-2A^2x^2-4Ae^{x}+4A(1-A)xe^{x}-4A^2x^2e^{x}, \\ \vdots \end{cases}$$

The series solution is given as

$$y(x)=1+Ax+\frac{1}{2!}x^2+\frac{1}{3!}Bx^3+\frac{1}{4!}x^4+\frac{1}{5!}Cx^5+\frac{1}{6!}x^6+\frac{1}{7!}Dx^7+\frac{1}{8!}x^8+\frac{1}{9!}Ex^9+\frac{1}{10!}x^{10}+\frac{1}{11!}Fx^{11}$$
$$+\frac{1}{12!}(2+B)x^{12}+\cdots.$$

Imposing the boundary conditions at x = 1 will yield

$$\begin{aligned} &A=-0.9999983604, \quad B=-1.600016174, \quad C=-0.9998407313, \\ &D=-1.001558298, \quad E=-0.9851011393, \quad F=-1.132112472. \end{aligned}$$

Consequently, the series solution is given by

$$y(x)=1-0.9999983604x+\frac{1}{2!}x^2-0.1666693624x^3+\frac{1}{4!}x^4-0.008332006094x^5+\frac{1}{6!}x^6-0.0001987218845x^7$$
$$+\frac{1}{8!}x^8-2.715\times10^{-9}x^9+\frac{1}{10!}x^{10}-2.836\times10^{-8}x^{11}+2.087\times10^{-9}x^{11}+2.087\times10^{-9}x^{12}+\cdots.$$

Table 3.2 (Error estimates)

x	Exact solution	Series solution	*Errors
0.0	1.000000000	1.000000000	0.00000
0.1	0.904837418	0.904837579	-1.61E-7
0.2	0.818730753	0.818731060	-3.07 E-7
0.3	0.740818221	0.740818643	-4.22 E-7
0.4	0.670320046	0.670320543	-4.97 E-7
0.5	0.606530659	0.606531182	-5.22 E-7
0.6	0.548811636	0.548812133	-4.97 E-7
0.7	0.496585304	0.496585726	-4.22 E-7
0.8	0.449328964	0.449329271	-3.07 E-7
0.9	0.406569659	0.406569821	-1.61 E-7
1.0	0.367879441	0.367879441	2.00 E-10

*Error = Exact solution-Series solution.

Table 3.2 exhibits the exact solution and the series solution along with the errors obtained by using the variational iteration method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of y(x).

Figure 3.2

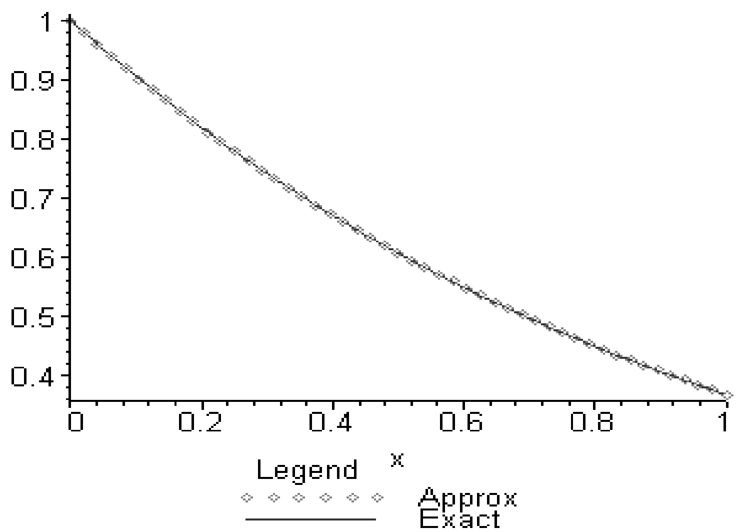

Fig. 3.2 clearly indicates the accuracy of the proposed variational iteration method (VIM).

4. CONCLUSIONS

In this paper, we have used the variational iteration method (VIM) for solving boundary value problems for twelfth-order by converting the problem into a system of integral equations. The method is applied in a direct way without using linearization, perturbation or restrictive assumptions. It may be concluded that VIM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that converge very rapidly in physical problems. The fact that the VIM solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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