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Generalized R-norm Information Measures

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Abstract

In the present communication, a new generalized R-norm information measure has been defined and characterized by infimum operation. Axiomatic characterization of the generalized measure of inaccuracy has been discussed. Joint and conditional cases have also been studied in detail.

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1. INTRODUCTION

Let us consider the set of positive real numbers, not equal to 1 and denote this by R^+ defined as $R^+ = \{R : R > 0, \ \ \mathrm{R} \neq 1\}$. Let Δ_n with $n \geq 2$ be a set of all probability distributions

$$P = \left\{(p_1, p_2 ,......................., p_n),\ p_i \geq 0, \text{for each i and} \sum_{i=1}^{n} p_i = 1\right\}.$$

Boekee and Lubbe (1980) studied R – norm information of the distribution P defined for $R \in R^+$ by

$$H_R(P) = \frac{R}{R-1}\left[1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right] \tag{1}$$

The R–norm information measure (1) is a real function $\Delta_n \rightarrow R^+$ defined on Δ_n, where $n \geq 2$ and R^+ is the set of real positive numbers. This measure is different from entropies of Shannon (1948), Renyi (1961), Havrda and Charvat (1967) and Daroczy (1970).The most interesting property of this measure is that when $R \rightarrow 1$, R-norm information measure approaches to Shannon's entropy and in case $R \rightarrow \infty$, $H_R(P) \rightarrow (1 - \max p_i)$, where $\mathrm{i} = 1,2,................,\mathrm{n}$.

The measure (1) can be generalized in so many ways. Hooda and Ram (2002) proposed and characterized the following parametric generalization of (1).

$$H_R^{\beta}(P)=\frac{R}{R+\beta-2}\left[1-\left(\sum_{i=1}^{n}p_i^{\frac{R}{2-\beta}}\right)^{\frac{2-\beta}{R}}\right],\quad 0<\beta\le 1,\ \mathrm{R}(>0)\ne 1 \tag{2}$$

They called (2) as the generalized R-norm entropy of degree β which reduces to (1), when $\beta=1$.

In case R = 1, (2) reduces to

$$H_1^{\beta}(P)=\frac{1}{\beta-2}\left[1-\left(\sum_{i=1}^{n}p_i^{\frac{1}{2-\beta}}\right)^{2-\beta}\right],\quad 0<\beta\le 1. \tag{3}$$

When $\gamma=\dfrac{1}{2-\beta}$, (3) reduces to

$$H^{\gamma}(P)=\frac{\gamma}{\gamma-1}\left[1-\left(\sum_{i=1}^{n}p_i^{\gamma}\right)^{\frac{1}{\gamma}}\right],\quad \frac{1}{2}<\gamma\le 1. \tag{4}$$

This is an information measure which has been mentioned by Arimoto (1971). It may be noted that (4) also approaches to Shannon's entropy when $\gamma\to 1$. Thus (1) measure can be generalized further parametrically in so many ways and consequently, we consider the following R-normed measure:

$$H_R^{\alpha,\beta}(P)=\frac{R}{R+\beta-2\alpha}\left[1-\left(\sum_{i=1}^{n}p_i^{\frac{R}{2\alpha-\beta}}\right)^{\frac{2\alpha-\beta}{R}}\right],$$

$$\alpha\ge 1,\ 0<\beta\le 1,\ \mathrm{R}(>0)\ne 1, 0<\mathrm{R}+\beta\ne 2\alpha. \tag{5}$$

It may be noted that the generalized R-norm information measure (5) of type α and degree β gives a family of R-norm information measures. It reduces to (2) when $\alpha=1$, which further reduces to (1) when $\beta=1$.Thus the generalized information measures have more flexibility for the application point of view that's why the measures are generalized parametrically.

In the present paper we characterize a non-additive generalized R-norm information measure (5) by applying the infimum operation in section 2. In section 3 we give axiomatic characterization of a generalized measure of R-norm accuracy through functional equation. In section 4 we study joint and conditional generalized R-norm information measures.

2. CHARACTERIZATION BY APPLYING INFIMUM OPERATION

We can consider the generalized R-norm entropy (5) as weighted arithmetic mean representation of elementary R-norm entropies of type α and degree β of occurrences of various single outcomes.

THEOREM 2.1. Let

$$f_R^{\alpha,\beta}(*p_i) = \frac{R}{R+\beta-2\alpha}\left[1 - *p_i^{\frac{R+\beta-2\alpha}{R}}\right], \quad \alpha \geq 1,\ 0 < \beta \leq 1,\ \mathrm{R}(>0) \neq 1, 0 < \mathrm{R} + \beta \neq 2\alpha, \tag{6}$$

then

$$H_R^{\alpha,\beta}(P) = \inf_{*p_i} \sum_{i=1}^{n} p_i f_R^{\alpha,\beta}(*p_i)\ , \tag{7}$$

where the operation infimum is taken over the probability distribution

$(*p_1, *p_2., ..., *p_n) \in \Delta_n$.

PROOF. Let us consider

$$\sum_{i=1}^{n} p_i f_R^{\alpha,\beta}(*p_i) = \frac{R}{R+\beta-2\alpha}\sum_{i=1}^{n} p_i\left[1 - *p_i^{\frac{R+\beta-2\alpha}{R}}\right]. \tag{8}$$

We minimize (8) subject to natural constraint

$$\sum_{i=1}^{n} *p_i = 1 \tag{9}$$

For this we consider Lagrangian

$$L = \frac{R}{R+\beta-2\alpha}\left[1 - \sum_{i=1}^{n} p_i (*p_i)^{\frac{R+\beta-2\alpha}{R}}\right] + \lambda\left(\sum_{i=1}^{n} *p_i \text{ -1}\right)$$

Differentiating with respect to $*p_i$, we have

$$\frac{\partial \mathrm{L}}{\partial *\mathrm{p_i}} = -p_i (*p_i)^{\frac{\beta-2\alpha}{R}} + \lambda \tag{10}$$

For extremum value we put $\frac{\partial L}{\partial(*p_i)}=0$, which gives

$$*\,p_i=\lambda^{\frac{R}{\beta-2\alpha}}\;p_i^{\frac{R}{2\alpha-\beta}} \tag{11}$$

We see that $\frac{\partial^2 L}{\partial *p_i^{\,2}}>0,$ when $*\,p_i=\lambda^{\frac{R}{\beta-2\alpha}}\;p_i^{\frac{R}{2\alpha-\beta}}$. Hence the value of $*\,p_i$ given by (11) is minimum and using (9) in (11), we can find the value of λ and consequently, we have

$$*\,p_i=\frac{p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n}p_i^{\frac{R}{2\alpha-\beta}}},\qquad \alpha\ge 1,\ 0<\beta\le 1,\ R(>0)\ne 1. \tag{12}$$

Now we consider R.H.S of (7)

$$\inf_{*p_i}\sum_{i=1}^{n}p_i f_R^{\alpha,\beta}(*p_i)=\frac{R}{R+\beta-2\alpha}\left[1-\frac{\sum_{i=1}^{n}p_i\,p_i^{\frac{R+\beta-2\alpha}{2\alpha-\beta}}}{\left(\sum_{i=1}^{n}p_i^{\frac{R}{2\alpha-\beta}}\right)^{\frac{R+\beta-2\alpha}{R}}}\right]$$

$$=\frac{R}{R+\beta-2\alpha}\left[1-\frac{\sum_{i=1}^{n}\ p_i^{\frac{R}{2\alpha-\beta}}}{\left(\sum_{i=1}^{n}p_i^{\frac{R}{2\alpha-\beta}}\right)^{\frac{R+\beta-2\alpha}{R}}}\right]$$

$$=\frac{R}{R+\beta-2\alpha}\left[1-\left(\sum_{i=1}^{n}p_i^{\frac{R}{2\alpha-\beta}}\right)^{\frac{2\alpha-\beta}{R}}\right]$$

$$=H_R^{\alpha,\beta}(P)\ . \tag{13}$$

Further without any loss of generality, we may assume that corresponding to the observed probability distribution $P \in \Delta_n$, there is a prior probability distribution $Q \in \Delta_n$ and replacing $f_R^{\alpha,\beta}(*p_i)$ by $f_R^{\alpha,\beta}(q)$ in (7) we have

$$H_R^{\alpha,\beta}(P) = \inf_{q_i} \sum_{i=1}^{n} p_i f_R^{\alpha,\beta}(q_i) \tag{14}$$

In case we do not apply the operation of infimum to (14), then it depends on two probability distributions P and Q. For R = 1 and $\alpha = 1$, $f_R^{\alpha,\beta}(q_i)$ is analogous to

$\frac{1}{\beta - 1}\left(1 - q_i^{\beta-1}\right)$ which reduces to $\log\frac{1}{q_i}$ in case $\beta \to 1$.

Thus (14) becomes

$$H_R^{\alpha,\beta}(P) \frac{1}{1-\beta} \sum_{i=1}^{n} p_i \left(q_i^{\beta-1} - 1\right), \tag{15}$$

which is a generalized inaccuracy measure of degree β characterized by Sharma and Taneja (1975). Therefore, we can represent (7) via $f_R^{\alpha,\beta}(q_i)$.

Hence $H_R^{\alpha,\beta}(P/Q) = \sum_{i=1}^{n} p_i f_R^{\alpha,\beta}(q_i)$

$$= \frac{R}{R+\beta-2\alpha}\left[\sum_{i=1}^{n} p_i \left(1 - q_i^{\frac{R+\beta-2\alpha}{R}}\right)\right], \quad \alpha \geq 1,\ 0 < \beta \leq 1,\ \ \mathrm{R}(>0) \neq 1, 0 < \mathrm{R} + \beta \neq 2\alpha. \tag{16}$$

Actually, (16) can also be described as the average of elementary R-norm inaccuracies $f_R^{\alpha,\beta}(q_i), \quad \mathrm{i} = 1,2,\ldots\ldots\ldots\ldots\ldots, \mathrm{n}$ and so can be called as R-normed inaccuracy measure of type α and degree β. Thus it seems plausible that (16) may be characterized and then by taking its infimum we can arrive at (7).

In the next theorem we characterize the elementary information function $f_R^{\alpha,\beta}(q_i)$ by assuming only two axioms and applying infimum operation.

THEOREM 2.2. Let f be a real valued continuous self-information function defined on (0,1] satisfying the following axioms:

Axiom 2.1. $f(xy) = f(x) + f(y) - \frac{R+\beta-2\alpha}{R} f(x)f(y)$

Axiom 2.2. $f(\frac{1}{n}) = \frac{R}{R+\beta-2\alpha}\left(1-n^{\frac{2\alpha-R-\beta}{R}}\right), \quad \alpha \geq 1,\ 0<\beta \leq 1,\ \mathrm{R}(>0) \neq 1, 0<\mathrm{R}+\beta \neq 2\alpha$

and $n = 2,3,\ldots\ldots\ldots\ldots\ldots\ldots$ is a maximality constant. If $f_R^{\alpha,\beta}(*p_i)$ is replaced by $f_R^{\alpha,\beta}(q_i)$ in (13), the result holds.

PROOF. By taking $f(x) = \frac{R}{R+\beta-2\alpha}(1-\phi(x))$ in axiom 2.1, we get

$$\frac{R}{R+\beta-2\alpha}(1-\phi(xy)) = \frac{R}{R+\beta-2\alpha}(1-\phi(x)) + \frac{R}{R+\beta-2\alpha}(1-\phi(y)) - \frac{R}{R+\beta-2\alpha}(1-\phi(x))(1-\phi(y))$$

or

$$\phi(xy) = \phi(x) + \phi(y) \tag{17}$$

The relation (17) is well known Cauchy's functional equation (refer Aczel (1996)). The continuous solution of (17) is $\phi(x) = \mathrm{x}^{\mathrm{a}}$, where $\mathrm{a} \neq 0$ is an arbitrary constant. On using axiom 2.2, we get $a = \frac{R+\beta-2\alpha}{R}$ and hence

$$f(x) = \frac{R}{R+\beta-2\alpha}\left(1-x^{\frac{R+\beta-2\alpha}{R}}\right),$$

which is exactly of the form of (6). Next the measure (7) can be easily obtained by applying the operation infimum on the equation (16) on the lines of theorem 2.1.

REMARKS: For an incomplete probability distribution scheme

$$P = (p_1, p_2, \ldots\ldots\ldots\ldots\ldots\ldots, p_n),\ \ \mathrm{p_i} \geq 0,\ \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{p_i} \leq 1,\ \ f_R^{\alpha,\beta}(q_i),\quad \mathrm{i} = 1,2,\ldots\ldots\ldots\ldots\ldots,$$

associated with individual events may be worked out. Then as in case of (16) we may define

$$H_R^{\alpha,\beta}(P/Q) = \left(p_1, p_2, \ldots\ldots\ldots\ldots, p_n; q_1, q_2, \ldots\ldots\ldots\ldots, q_n\right) = \frac{\sum_{i=1}^{n} p_i f_R^{\alpha,\beta}(q_i)}{\sum_{i=1}^{n} \mathrm{p_i}}$$

(18)

By using the operation infimum with respect to $\mathrm{q_i}'s$ the equation (18) gives

$$H_R^{\alpha,\beta}(P/Q) = \frac{R}{R+\beta-2\alpha}\left[1-\left(\frac{\sum_{i=1}^{n} p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^{n} p_i}\right)^{\frac{2\alpha-\beta}{R}}\right] \tag{19}$$

which is the R-norm entropy of type α and degree β of incomplete probability distribution. It is also worth mentioning that if we take arithmetic average with weights as continuous function w(.), then we get the general expression

$$H_R^{\alpha,\beta}(P/Q) = \frac{\sum_{i=1}^{n} w(p_i) f_R^{\alpha,\beta}(q_i)}{\sum_{i=1}^{n} w(p_i)} \tag{20}$$

By considering different weight w(.) satisfying the condition $w(pq) = w(p)w(q)$, where $w(.) \neq 0$, we can obtain various generalized R-normed entropies by using the operation infimum with respect to $q_i's$.

In next section we characterize the generalized measure of inaccuracy (16) for two distributions P and $\mathrm{Q} \in \Delta_\mathrm{n}$ through functional equation.

3. AXIOMATIC CHARACTERIZATION

Let $\mathrm{S_n} = \Delta_\mathrm{n} \times \Delta_\mathrm{n} \rightarrow R^+$, $\mathrm{n} = 2,3,\ldots\ldots\ldots$ and G_n be a sequence of functions of $\mathrm{p_i}'s$ and $\mathrm{q_i}'s$, i = 1,2,.............,n. over $\mathrm{S_n}$ satisfying the following axioms:

AXIOM 3.1. $G_n(P:Q) = a_1 + a_2 \sum_{i=1}^{n} h(p_i, q_i)$ where a_1 *and* a_2 are non-zero

constants, and
$p,q \in J = (0,1)\times(0,1) \quad \{(0,y);0 \le y \le 1\} \quad \{(1,y'):\ 0 \le y' \le 1\}.$

This axiom is also called sum property.

AXIOM 3.2. For $\mathrm{P,Q} \in \Delta_\mathrm{n}$ and $\mathrm{P',Q'} \in \Delta_\mathrm{m}, \mathrm{G_n}$ satisfies the following property

$$G_{mn}(PP':QQ') = G_n(P:Q) + G_m(P':Q') - \frac{1}{a_1} G_n(P:Q)G_m(P':Q').$$

AXIOM 3.3. h(p, q) is a continuous function of its arguments p and q.

AXIOM 3.4. Let all $\mathrm{p_i}'s$ and $\mathrm{q_i}'s$ are equiprobable posterior and prior probabilities of events respectively, then

$$G_n\left(\frac{1}{n},\ldots\ldots\ldots\ldots\ldots\frac{1}{n},\frac{1}{n},\ldots\ldots\ldots\ldots\ldots\frac{1}{n}\right) = \frac{R}{R+\beta-2\alpha}\left(1-n^{\frac{2\alpha-R-\beta}{R}}\right), \quad \text{where } \mathrm{n} = 2,3,\ldots\ldots\ldots\ldots\ldots,$$

$\alpha \ge 1,\ 0 < \beta \le 1,\ \mathrm{R}(>0) \ne 1, 0 < \mathrm{R} + \beta \ne 2\alpha.$

First of all we prove the following three lemmas to facilitate to prove the main theorem:

LEMMA 3.1. From axioms 3.1 and 3.2, it is very easy to arrive at the following functional equation:

$$\sum_{i=1}^{n}\sum_{j=1}^{m} h(p_i p'_j, q_i q'_j) = \left(\frac{-a_2}{a_1}\right)\sum_{i=1}^{n} h(p_i,q_i)\sum_{j=1}^{m} h(p'_j,q'_j), \tag{21}$$

where $(p_i,q_i),(p'_j,q'_j) \in J$ for i = 1,2,..............,n and j = 1,2,..............,m.

LEMMA 3.2. The continuous solution that satisfies (21) is the continuous solution of the functional equation:

$$h(pp',qq') = \left(\frac{-a_2}{a_1}\right) h(p,q)h(p',q'). \tag{22}$$

PROOF. Let a,b,c,d and a',b',c',d' be positive integers such that

$1 \le a' \le a, 1 \le b' \le b, 1 \le c' \le c,$ and $1 \le d' \le d$.

Setting $n = a - a' + 1 = b - b' + 1$ and $\mathrm{m} = c - c' + 1 = d - d' + 1,$

$$p_i = \frac{1}{a}(i = 1,2,\ldots\ldots\ldots\ldots\ldots, a - a'), \quad \mathrm{p_{a-a'+1}} = \frac{a'}{a},$$

$$q_i = \frac{1}{b}(i = 1,2,\ldots\ldots\ldots\ldots\ldots, b - b'), \quad \mathrm{q_{b-b'+1}} = \frac{b'}{b},$$

$$p'_j = \frac{1}{c}(j = 1,2,................,c - c'), \quad \mathrm{p}'_{\mathrm{c-c'+1}} = \frac{c'}{c},$$

$$q'_j = \frac{1}{d}(j = 1,2,................,d - d'), \quad \mathrm{q}'_{\mathrm{d-d'+1}} = \frac{d'}{d},$$

From equation (21) we have

$$(a-a')(c-c')h\left(\frac{1}{ac},\frac{1}{bd}\right)+(c-c')h\left(\frac{a'}{ac},\frac{b'}{bd}\right)+(a-a')h\left(\frac{c'}{ac},\frac{d'}{bd}\right)+h\left(\frac{a'c'}{ac},\frac{b'd'}{bd}\right)$$

$$=\left(\frac{-a_2}{a_1}\right)\left[(a-a')h\left(\frac{1}{a},\frac{1}{b}\right)+h\left(\frac{a'}{a},\frac{b'}{b}\right)\right]\left[(c-c')h\left(\frac{1}{c},\frac{1}{d}\right)+h\left(\frac{c'}{c},\frac{d'}{d}\right)\right] . \tag{23}$$

Taking $a' = b' = c' = d' = 1$ in (23), we get

$$h\left(\frac{1}{ac},\frac{1}{bd}\right)=\left(\frac{-a_2}{a_1}\right)h\left(\frac{1}{a},\frac{1}{b}\right)h\left(\frac{1}{c},\frac{1}{d}\right). \tag{24}$$

Taking $a' = b' = 1$ in (23) and using (24), we have

$$h\left(\frac{c'}{ac},\frac{d'}{bd}\right)=\left(\frac{-a_2}{a_1}\right)h\left(\frac{1}{a},\frac{1}{b}\right)h\left(\frac{c'}{c},\frac{d'}{d}\right). \tag{25}$$

Again taking $c' = d' = 1$ in (23) and using (24), we get

$$h\left(\frac{a'}{ac},\frac{b'}{bd}\right)=\left(\frac{-a_2}{a_1}\right)h\left(\frac{1}{c},\frac{1}{d}\right)h\left(\frac{a'}{a},\frac{b'}{b}\right). \tag{26}$$

Now (23) together with (24), (25) and (26) reduces to

$$h\left(\frac{a'c'}{ac},\frac{b'd'}{bd}\right)=\left(\frac{-a_2}{a_1}\right)h\left(\frac{a'}{a},\frac{b'}{b}\right)h\left(\frac{c'}{c},\frac{d'}{d}\right). \tag{27}$$

Putting $\frac{a'}{a} = p,\ \frac{b'}{b} = q,\ \frac{c'}{c} = p',\ \frac{d'}{d} = q'$ in (27), we get the required results (22) for rational numbers which by continuity of h holds for all real $p,\ q,\ p',\ q' \in J$.

Next we obtain the most general solution of (22).

LEMMA 3.3. The most general continuous solutions of equation (22) are given by

$$h(p,q)=\left(\frac{-a_1}{a_2}\right)p^{\mu}q^{\nu}\text{, where } \mu\neq 0,\ \nu\neq 0 \tag{28}$$

and $h(p,q)=0$ (29)

PROOF. Taking $g(p,q)=\left(\frac{-a_2}{a_1}\right)h(p,q)$ in (22), we have

$$g(pp',qq')=g(p,q)g(p',q') \tag{30}$$

The most general continuous solution of (30) [cf Aczel (1996)] is given by

$$g(p,q)=p^{\mu}q^{\nu},\ \mu\neq 0 \text{ and } \nu\neq 0 \tag{31}$$

and

g (p, q) = 0 (32)

On substituting $g(p,q)=\left(\frac{-a_2}{a_1}\right)h(p,q)$ in (31) and (32) we get (28) and (29)

respectively. This proves the lemma 3.3.

THEOREM 3.1. The inaccuracy measure (16) is uniquely determined by the axioms 3.1 to 3.4.

PROOF. Substituting the solution (28) in axiom 3.1 we have

$$G_n(P/Q)=a_1\left(1-\sum_{i=1}^{n}p_i^{\mu}q_i^{\nu}\right),\ \mu\nu\neq 0 \tag{33}$$

Using axiom 3.4 in (33), we get

$$G_n\left(\frac{1}{n},\dots\dots\dots\dots\dots\frac{1}{n},\frac{1}{n},\dots\dots\dots\dots\dots\frac{1}{n}\right)=\frac{R}{R+\beta-2\alpha}\left[1-n^{\frac{2\alpha-\beta-R}{R}}\right],\ \text{n = 2, 3,...........,} \tag{34}$$

From (33) we have

$$G_n\left(\frac{1}{n},\dots\dots\dots\dots\dots\frac{1}{n},\frac{1}{n},\dots\dots\dots\dots\dots\frac{1}{n}\right)=a_1\left(1-n^{1-\mu-\nu}\right),\quad \text{n = 2, 3,...........,} \tag{35}$$

From (34) and (35) we have

$$a_1\left(1-n^{1-\mu-\nu}\right)=\frac{R}{R+\beta-2\alpha}\left[1-n^{\frac{2\alpha-\beta-R}{R}}\right],\quad \text{n = 2,3,...........,}$$

It implies

$$a_1 = \frac{R}{R+\beta-2\alpha}, \quad 1 - \mu - \nu = \frac{2\alpha-\beta-R}{R}$$

or

$$a_1 = \frac{R}{R+\beta-2\alpha}, \quad \mu = 1, \;\; \nu = \frac{R+\beta-2\alpha}{R}$$

Now from (33) we have

$$G_n(P/Q) = \frac{R}{R+\beta-2\alpha}\left(1-\sum_{i=1}^{n} p_i q_i^{\frac{R+\beta-2\alpha}{R}}\right) = \frac{R}{R+\beta-2\alpha}\left[\sum_{i=1}^{n} p_i\left(1-q_i^{\frac{R+\beta-2\alpha}{R}}\right)\right]$$

$$= H_R^{\alpha,\beta}(P/Q).$$

Hence this completes the proof of the theorem 3.1.

REMARKS: In the equation (28) if $\mu = 0$ and $\nu = 0$, then

$$h(p,q) = \left(\frac{-a_1}{a_2}\right),$$

which is a trivial solution and is of no interest. The solution (29) does not even contain any variable and hence it is to be discarded.

4. JOINT AND CONDITIONAL GENERALIZED R-NORM INFORMATION MEASURES

In this section, we consider joint and conditional probability distributions of two random variables ξ and η having probability distributions P and Q over the sets $X = \{x_1, x_2, ..., x_n\}$ and $Y = \{y_1, y_2, ..., y_m\}$ respectively. Then the generalized R- norm information measure of type α and degree β of the random variables

ξ, η are respectively given by

$H_R^{\alpha,\beta}(\xi) = H_R^{\alpha,\beta}(P)$ and $H_R^{\alpha,\beta}(\eta) = H_R^{\alpha,\beta}(Q)$,

where $p_i = \Pr(\xi = x_i), i = 1,2...........n$ and $q_j = \Pr(\eta = y_j), j = 1,2......m$ are the probabilities of the possible values of the random variables. Similarly we consider a two-dimensional discrete random variable (ξ,η) with the joint probability distribution $\pi=\{\pi_{11},....\pi_{nm}\}$ where $\pi_{ij}=\Pr(\xi=x_i,\eta=y_j),\ i=1,2,...n; j=1,2,.....m$ is the joint probability for the values $(x_i y_j)\ of\ (\xi,\eta)$.

we shall denote conditional probabilities by $p_{ij}\ and\ q_{ji}$ such that

$$\pi_{ij}=P_{ij}q_j=q_{ji}p_i,\ p_i=\sum_{j=1}^{m}\pi_{ij}\ and\ q_j=\sum_{i=1}^{n}\pi_{ij}$$

DEFINITION 4.1. The joint generalized R-norm information measure of type α and degree β for $R\in\Re^+$ and $\alpha\geq 1,\ 0<\beta\leq 1,\ \mathrm{R}(>0)\neq 1, 0<\mathrm{R}+\beta\neq 2\alpha$ is given by

$$H_R^{\alpha,\beta}(\xi,\eta)=\frac{R}{R+\beta-2\alpha}\left[1-\left[\sum_{i=1}^{n}\sum_{j=1}^{m}\pi_{ij}^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}}\right]. \tag{36}$$

It may be seen that $H_R^{\alpha,\beta}(\xi,\eta)$ is symmetric in ξ and η. If ξ and η are stochastically independent, then the following non-additive property holds:

$$H_R^{\alpha,\beta}(\xi,\eta)=H_R^{\alpha,\beta}(\xi)+H_R^{\alpha,\beta}(\eta)-\frac{R+\beta-2\alpha}{R}H_R^{\alpha,\beta}(\xi)H_R^{\alpha,\beta}(\eta). \tag{37}$$

DEFINITION 4.2. The average conditional generalized R-norm information measure of type α and degree β for $R\in\Re^+$ and

$\alpha\geq 1,\ 0<\beta\leq 1,\ \mathrm{R}(>0)\neq 1, 0<\mathrm{R}+\beta\neq 2\alpha$ is given by

$$^{*}H_R^{\alpha,\beta}(\eta/\xi)=\frac{R}{R+\beta-2\alpha}\left[1-\sum_{i=1}^{n}p_i\left[\sum_{j=1}^{m}q_{ji}^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}}\right] \tag{38}$$

or alternately

$$^{**}H_R^{\alpha,\beta}(\eta/\xi)=\frac{R}{R+\beta-2\alpha}\left[1-\left[\sum_{i=1}^{n}p_i\sum_{j=1}^{m}q_{ji}^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}}\right]. \tag{39}$$

The two conditional measures (38) and (39) differ according to the probabilities p_i have been taken. The expression (38) is a true mathematical expression over ξ, whereas the expression (39) is not.

In next theorem, we prove three results for the conditional generalized R-norm information measures given by (38) and (39).

THEOREM 4.1. If ξ and η are discrete random variables then for $R \in \Re^{+}$ and $\alpha \geq 1,\ 0 < \beta \leq 1,\ \mathrm{R}(>0) \neq 1, 0 < \mathrm{R} + \beta \neq 2\alpha$ then the following inequalities holds

$$^{*}H_R^{\alpha,\beta}(\eta/\xi) \leq H_R^{\alpha,\beta}(\eta) \tag{40}$$

$$^{**}H_R^{\alpha,\beta}(\eta/\xi) \leq H_R^{\alpha,\beta}(\eta) \tag{41}$$

$$^{**}H_R^{\alpha,\beta}(\eta/\xi) \leq {}^{*}H_R^{\alpha,\beta}(\eta/\xi) \tag{42}$$

The equality sign holds iff ξ and η are independent.

PROOF. We know [refer Beckenbach and Bellman (1971)] that for

$$\frac{R}{2\alpha-\beta} > 1 \ \ or \ \ R+\beta > 2\alpha$$

$$\left[\sum_{j=1}^{m}\left[\sum_{i=1}^{n} x_{ij}\right]^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}} \leq \sum_{i=1}^{n}\left[\sum_{j=1}^{m} x_{ij}^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}}. \tag{43}$$

Setting $x_{ij} = \pi_{ij} \geq 0$ in (43), we have

$$\left[\sum_{j=1}^{m}\left[\sum_{i=1}^{n} \pi_{ij}\right]^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}} \leq \sum_{i=1}^{n}\left[\sum_{j=1}^{m} \pi_{ij}^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}} \tag{44}$$

or

$$\left[\sum_{j=1}^{m} q_j^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}} \leq \sum_{i=1}^{n}\left[\sum_{j=1}^{m} \left(q_{ji}p_i\right)^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}}$$

or

$$\left[\sum_{j=1}^{m} q_j^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}} \le \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} q_{ji}^{\frac{R}{2\alpha-\beta}}\right)^{\frac{2\alpha-\beta}{R}}. \tag{45}$$

It implies

$$1-\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} q_{ji}^{\frac{R}{2\alpha-\beta}}\right)^{\frac{2\alpha-\beta}{R}} \le 1-\left(\sum_{j=1}^{m} q_j^{\frac{R}{2\alpha-\beta}}\right)^{\frac{2\alpha-\beta}{R}}.$$

Since $\frac{R}{R+\beta-2\alpha} > 0$, in view of $R+\beta > 2\alpha \ \ and \ \ 0<\beta\le 1$, therefore on multiplication we get

$$^{*}H_R^{\alpha,\beta}(\eta/\xi) \le H_R^{\alpha,\beta}(\eta). \tag{46}$$

On the same lines, we can prove that (46) holds for $0 < R+\beta < 2\alpha \ \ and \ \ 0<\beta\le 1$.

Hence (40) holds for all $R \in \Re^{+}$ and

$\alpha \ge 1,\ 0<\beta\le 1,\ \mathrm{R}(>0)\ne 1, 0<\mathrm{R}+\beta \ne 2\alpha$. The equality sign holds iff π_{ij} is separable in the sense that $\pi_{ij} = p_i q_j$.

From Jensen's inequality for $R+\beta > 2\alpha \ \ and \ \ 0<\beta\le 1$, we find

$$\sum_{i=1}^{n} p_i q_{ji}^{\frac{R}{2\alpha-\beta}} \ge \left[\sum_{i=1}^{n} p_i q_{ji}\right]^{\frac{R}{2\alpha-\beta}} = q_j^{\frac{R}{2\alpha-\beta}}. \tag{47}$$

After summation over j and raising both sides to power $\frac{2\alpha-\beta}{R}$, we have

$$\left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}} \ge \left[\sum_{j=1}^{m} q_j^{\frac{R}{2\alpha-\beta}}\right]^{\frac{2\alpha-\beta}{R}} \tag{48}$$

Using $\frac{R}{R+\beta-2\alpha} > 0,\ as\ R+\beta > 2\alpha$, we get

$$^{**}H_R^{\alpha,\beta}(\eta/\xi) \le H_R^{\alpha,\beta}(\eta). \tag{49}$$

Equality holds if for all i, $q_{ij} = q_j$, which is equivalent to the independent property.

For $0 < R + \beta < 2\alpha$ the inequality in (48) reverses. However, in view of

$\frac{R}{R+\beta-2\alpha} < 0, as\ R+\beta < 2\alpha$ and $0 < \beta \le 1$, (49) still holds.

Hence result (41) is proved.

Next for the proof of (42) we apply Jensen inequality and obtain

$$\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} q_j^{\frac{R}{2\alpha-\beta}} \right)^{\frac{2\alpha-\beta}{R}} \le \left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}^{\frac{R}{2\alpha-\beta}} \right]^{\frac{2\alpha-\beta}{R}} \quad (50)$$

It implies

$$1 - \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} q_j^{\frac{R}{2\alpha-\beta}} \right)^{\frac{2\alpha-\beta}{R}} \ge 1 - \left[\sum_{i=1}^{n} p_i \sum_{j=1}^{m} q_{ji}^{\frac{R}{2\alpha-\beta}} \right]^{\frac{2\alpha-\beta}{R}} \quad (51)$$

Since $\frac{R}{R+\beta-2\alpha} > 0$, in view of $R+\beta > 2\alpha\ and\ \ 0 < \beta \le 1$, therefore on multiplication we get

${}^{**}H_R^{\alpha,\beta}(\eta/\xi) \le {}^{*}H_R^{\alpha,\beta}(\eta/\xi)$, which is the required result.

Hence (42) is proved for all $R \in \Re^+$ and $0 < \beta \le 1$

This completes the proof of Theorem 4.1.

5. CONCLUSION

In the context of feature selection problem in pattern recognition, many authors have studied upper and lower bounds on the Bayesian property of error. A relation between the probability error and average conditional generalized R-norm information measures of type α and degree β can be established in the form of an inequality. In case $\alpha = 1, \beta = 1$ and $R \to \infty$, this inequality can be considered as the analog of well known Fano-bound inequality.

The mean code length due to Boekee and Lubbe (1980) can be generalized and the generalized R-norm information measure given by (5) can be applied in study of the lower and upper bounds of the generalized mean codeword length.

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