Journal of Applied Mathematics, Statistics and Informatics (JAMSI), **4** (2008), No. 2

Order preserving similarities

JANA DOBRAKOVOVÁ

Abstract

In this paper we want to discuss some properties of similarity relations on the universe of real numbers, namely how their continuity and compatibility with real numbers lattice structure depend on the nature of corresponding fuzzy quantities.

Mathematics Subject Classification 2000: 03E72, 54A40
Additional Key Words and Phrases: Fuzzy equivalence relation, Similarity relation, Nearness, S-generating quantity

1. INTRODUCTION

The purpose of this article is to continue the study of relationship between properties of real numbers similarities and the way of their construction, by means of associate fuzzy quantities, so called similarity generating (briefly S-generating) quantities.

Fuzzy relations have been studied since 1965, when Lotfi Zadeh, in his first paper on fuzzy sets introduced the notion of a fuzzy binary relation. The most frequently appearing type of binary fuzzy relations is the so called **fuzzy equivalence relation**, playing a significant role in fuzzy set theory. They model in a sense proximity, indistinguishability, or similarity of elements from an arbitrary universe X. In what follows we will deal with one particular kind of fuzzy equivalence relations, introduced by Lotfi Zadeh (1971), called **similarity relations**.

We will study similarity relations defined on the universe of real numbers $\mathbb{R}$ by means of fuzzy quantities. We want to discuss several problems concerning correspondence between properties of a fuzzy quantity and properties of the associate similarity relation. Especially their continuity and compatibility with ordering of real numbers.

Since from another point of view, real binary fuzzy relations are in fact real functions of two real variables with some special properties, we will prefer this attitude in our study and we will pay attention mainly to geometric properties of such functions and their graphs.

First, let us briefly recall some basic concepts. For more details see for instance [6].

2. PRELIMINARIES

Definition 1.
A binary fuzzy relation S on the universe $\mathbb{R}$ is called a similarity relation (or briefly similarity) on $\mathbb{R}$ if and only if it is reflexive, symmetric and transitive w.r.t.minimum, it means, if and only if for any $x, y, z \in \mathbb{R}$:

(S1) $S(x,x) = 1$

(S2) $S(x,y) = S(y,x)$

(S3) $\min(S(x,z), S(y,z)) \leq S(x,y)$.

In this article we will discuss only a special type of real similarity relations, constructed in the following way:

$$S(x,y) = \begin{cases} 1, & \text{if } x = y, \\ \min(h(x), h(y)), & \text{if } x \neq y \end{cases}, \tag{1}$$

where h is a fuzzy quantity, that is to say a function defined on $\mathbb{R}$, with values from the interval $\langle 0, 1 \rangle$.

It is one of the most effective and natural methods (see [3]) to construct similarity relations on $\mathbb{R}$.

In what follows we will call such a fuzzy quantity h the **S-generating quantity** and we will consider only similarities defined by means of (1). It is obvious that properties of such a similarity S depend on properties of the associate S-generating quantity h.

3. CONTINUITY OF SIMILARITIES

Proposition 1.
Let h be a continuous S-generating quantity. Then the only points of discontinuity of the similarity S are points $[x_0, x_0]$, if $h(x_0) < 1$.

Proof. In [2] it is proved that if x_0 is a real number such that $h(x_0) < 1$, then the similarity S is not continuous at the point $[x_0, x_0]$. On the other hand, continuity at a point $[x_0, x_0]$, where $h(x_0) = 1$, or at a point $[x_0, y_0]$, where $x_0 \neq y_0$, simply follows from the definition of S and continuity of h.

Example 1.
Consider, for example, the fuzzy quantity $h(x) = e^{-x^2}$, as the S-generating quantity. Then we have

$$S(x,y) = \begin{cases} 1, & \text{if } x = y, \\ \min(e^{-x^2}, e^{-y^2}), & \text{if } x \neq y \end{cases}$$

Let us notice that, in accordance with Proposition 1, the similarity S is not continuous at any point $[x_0, x_0]$, excepting point $[0, 0]$, despite of the fact, that the S-generating quantity is a continuous function.

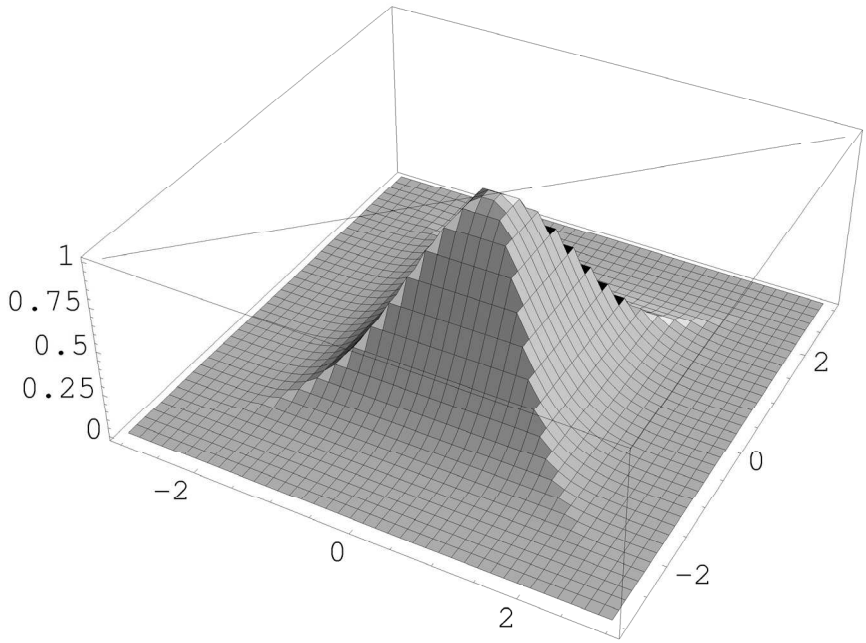

Fig. 1. Figure 1.

Problem of continuity or more precisely, discontinuity of a similarity S, in the case its S-generating quantity is not continuous, is not so simple and we are not going to study it in details. In the following proposition we consider only one special type of discontinuity of h, the so called jump discontinuity, and its consequence to the discontinuity of S.

Proposition 2.
Let x_0 be a real number and h be a fuzzy quantity such that

$$\lim_{x \to x_0^-} h(x) = b_1 < b_2 = \lim_{x \to x_0^+} h(x).$$

Then the associate similarity S is discontinuous at each point $[x_0, y]$, if $h(y] > b_1$.

Proof. Let $\{x'_n\}$ and $\{x''_n\}$ be two sequences of real numbers such that $\lim\limits_{n\to\infty} x'_n = \lim\limits_{n\to\infty} x''_n = x_0$ and for each natural n is $x'_n < x_0 < x''_n$. Then

$$\lim_{n\to\infty} S(x'_n, y) = \lim_{n\to\infty} \min(h(x'_n), h(y)) = \min(b_1, h(y)) = b_1$$

and

$$\lim_{n\to\infty} S(x''_n, y) = \lim_{n\to\infty} \min(h(x''_n), h(y)) = \min(b_2, h(y)) > b_1.$$

It means that despite of the fact that both sequences of points $\{[x'_n, y]\}$ and $\{[x''_n, y]\}$ in E_2 converge to the point $[x_0, y]$, at least one of the sequences $\{S(x'_n, y)\}$ and $\{S(x''_n, y)\}$ does not converge to the value $S(x_0, y)$.

It is clear, that the situation is analogical for any point of jump discontinuity, disregarding $b_1 < b_2$ or $b_2 < b_1$, as well as the value $h(x_0)$.

Example 2. Let us consider similarity with the S-generating quantity h defined by

$$h(x) = \begin{cases} \frac{1}{4}, & \text{if } x < 1, \\ \frac{3}{4}, & \text{if } x \geq 1 \end{cases}$$

From the foregoing proposition it follows, that S is discontinuous at all points $[1, y]$, for $y > 1$ and as well as at points $[y, 1]$. Let us show it on the graph of S on the square $\langle -3, 5\rangle \times \langle -3, 5\rangle$ (see Figure 2.).

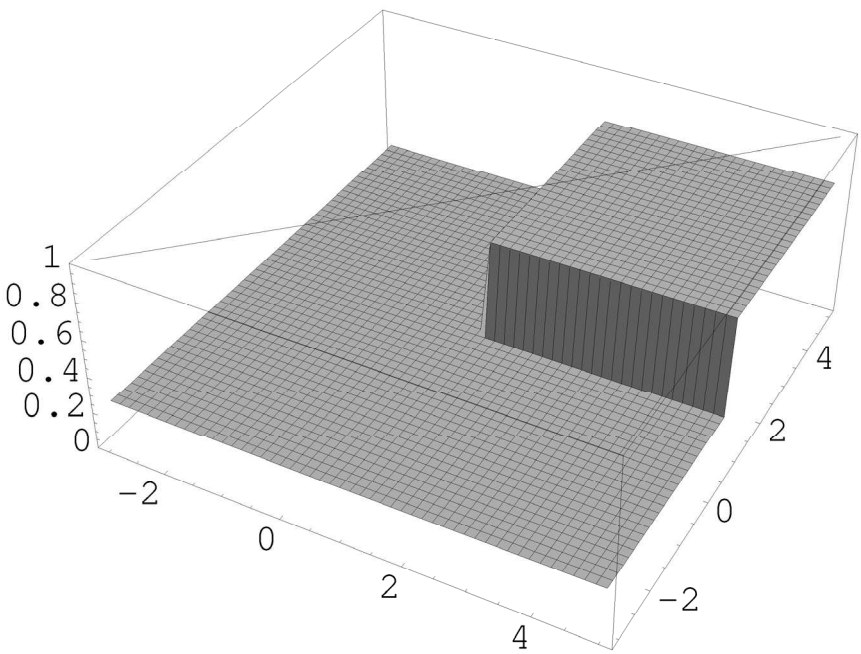

Fig. 2. Figure 2.

3. REAL NUMBERS SIMILARITY AND ORDERING

There are more approaches to the fuzzification of real numbers proximity. One of them is based on the concept of **nearness** (see for example [1], [4], [5]). It is a

type of binary fuzzy relations on $\mathbb{R}$ with some natural properties, corresponding to the algebraic, topological and lattice structures of real axis. There is not complete unification in definition of nearness. Of course, every nearness, likewise as similarity, is supposed to be reflexive and symmetrical.

Another quite natural and frequently appearing property of a nearness, reflected in some sense the ordering of real axis, has the following meaning: If we have two different real numbers, x, y and z is a real number lying between them, then nearness of the couple x, y cannot be greater than nearness of couples x, z and y, z. Let us denote a nearness by N. Then this örder preserving"property can be formulated as follows.

(OP) $\quad N(x,y) \leq \min(N(x,z), N(y,z))$ for each $x, y, z \in \mathbb{R},\ x \leq z \leq y$

Similarities can, but need not be order preserving. It depends on the corresponding S-generating quantity, namely on whether this fuzzy quantity is or is not **convex**. Let us recall, that a fuzzy quantity is said to be convex, if all its α-cuts are intervals. It can be proved ([6]), that a fuzzy quantity h is convex iff

$$h(z) \geq \min(h(x), h(y)) \quad \text{for each} \quad x, y, z \in \mathbb{R},\ x \leq z \leq y$$

Proposition 3.
A similarity S is order preserving if and only if its S-generating quantity h is convex.

Proof. First, let S be order preserving and let $x, y, z \in \mathbb{R},\ x \leq z \leq y$. Then
$\min(h(x), h(y)) = S(x,y) \leq \min(S(x,z), S(y,z)) = \min(\min(h(x), h(z)), \min(h(y), h(z)))$
$\leq h(z)$, hence h is convex.

Now let h be convex and let $x, y, z \in \mathbb{R},\ x < z < y$ (for $x = z$ or $z = y$ it is trivial). Then
$\min(S(x,z), S(y,z)) = \min(\min(h(x), h(z)), \min(h(y), h(z))) = h(z) \geq \min(h(x), h(y)) =$
$S(x,y)$.

From the previous proposition it follows, that both similarities in Examples 1 and 2 are order preserving, contrary to the similarity in the further example.

Example 3. Let the S-generating quantity h be defined by $h(x) = 1 - e^{-x^2}$. Then

$$S(x,y) = \begin{cases} 1, & \text{if } x = y, \\ \min(1 - e^{-x^2}, 1 - e^{-y^2}), & \text{if } x \neq y \end{cases}$$

It is easy to see that h is not convex and therefore S is not order preserving. For example $-1 < 0 < 1$, but $S(-1,1) = 1 - \frac{1}{e}$ while $S(-1,0) = S(1,0) = 0$.

Graph of S on the square $\langle -4, 4\rangle \times \langle -4, 4\rangle$ is in Figure 3.

There is one characteristic property of the order preserving similarities, having interesting geometric consequence, presented in the following proposition.

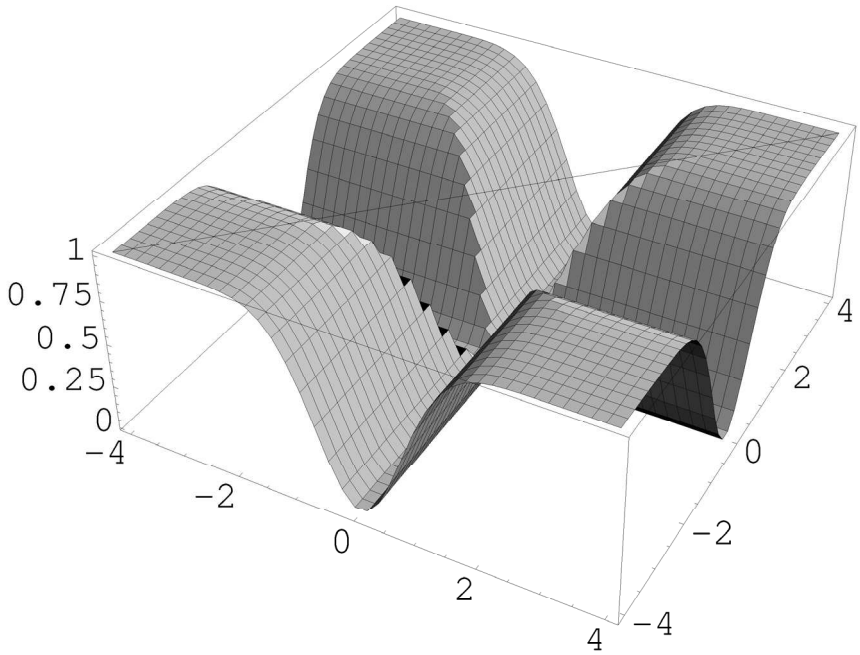

Fig. 3. Figure 3.

Proposition 4.

Let S be an order preserving similarity and let $x, y \in \mathbb{R}$, $x < y$. Then in the interval $\langle x, y\rangle$ there exists a sequence of real numbers couples $\{x_n, y_n\}$, such that $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ and $S(x_n, y_n) = S(x, y)$ for each natural n.

Proof. Let S be an order preserving similarity and $x, y \in \mathbb{R}$, $x < y$. Put $z_1 = \dfrac{x+y}{2}$. Since $x < z_1 < y$, by (OP)

$$\min(S(x, z_1), S(y, z_1)) \geq S(x, y).$$

But at the same time it is valid (S3), hence

$$\min(S(x, z_1), S(y, z_1)) \leq S(x, y).$$

Therefore

$$\min(S(x, z_1), S(y, z_1)) = S(x, y).$$

If $S(x, z_1) = S(x, y)$ we put $x_1 = x, y_1 = z_1$, otherwise $x_1 = z_1, y_1 = y$.

In the next step we put $z_2 = \dfrac{x_1 + y_1}{2}$. Again by combining (OP) with (S3) we obtain

$$\min(S(x_1, z_2), S(y_1, z_2)) = S(x_1, y_1) = S(x, y).$$

Repeating the procedure if $S(x_1, z_2) = S(x, y)$ we put $x_2 = x_1,\ y_2 = z_2$, otherwise $x_2 = z_2,\ y_2 = y_1$.

Proceeding analogously by the ïnterval halving method"we get desired sequence of couples $\{x_n, y_n\}$, such that $|x_n - y_n| = \dfrac{|x-y|}{2n}$ and $S(x_n, y_n) = S(x, y)$ for each natural n.

We illustrate geometrical meaning of the preceding statement by the following figure, displaying contour lines of graphs in Figures 1 and 3.

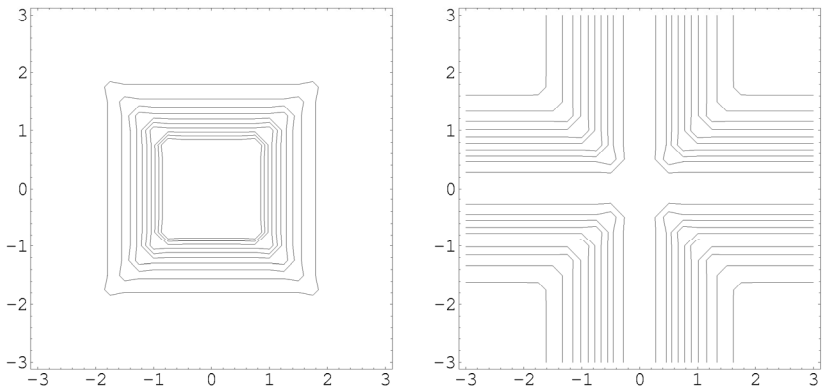

Fig. 4. Figure 4.

Let us notice that in the first picture in Figure 4 all contour lines are ïntersecting"the straight line $y = x$, contrary to the second picture. It is just due to the fact that the similarity in the first case is order preserving, contrary to the other similarity.

Acknowledgement

This work was supported by the grant Vega 1/3012/06 and by the grant Vega 1/3014/06

REFERENCES

Dobrakovová J.: *Indistinguishability and Nearness.* Busefal 83, 2000, pp. 46-52.

Dobrakovová J.: *Discontinuity of Similarity Relations.* (to appear)

Dubois D., Prade H.: *Fundamentals of Fuzzy sets.* Kluwer Academic Publishers, Dordrecht, 2000.

Janiš V.: *Resemblance is a nearness.* Fuzzy Sets and Systems, vol.133, 2003, pp. 171-173.

Kalina M.: *Nearness relations in linear spaces.* Kybernetika 40, 2004, pp. 441-458.

Klement E. P., Mesiar R., Pap E.: *Triangular Norms.* Kluwer Academic Publishers, Dordrecht, 2000.

Jana Dobrakovová,
Department of Mathematics, Slovak Technical University,
Nám. Slobody 17, 812 31 Bratislava
Slovak Republic
e-mail: jana.dobrakovova@stuba.sk

Received August 2008