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Oscillation of n^{th} order forced functional differential equations

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Abstract

The authors show that under the same conditions given in Theorem 2.6 of [1], every solution of 1 is either oscillatory or nonoscillatory and $\liminf_{t\to\infty} |x(t)| = 0$. The proof is much shorter and does not require Lemma 2.5 of [1] or similar results.

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1. INTRODUCTION

In an article with the same title as the current one, Grace and Hamedani [1], study the oscillatory behavior of the n^{th} order forced functional differential equation

$$(x(t) - x(t-\tau))^{(n)} + a(t)f(x(q(t))) = e(t), \quad t \geq t_0. \tag{1}$$

In proving their results they used, among other things, a lemma [1, Lemma 2.5]. This lemma was stated for any positive integer and without a proof. However, it appears that Lemma 2.5 of [1] does not work for n even and consequently the proof of Theorem 2.6 of [1] is incomplete.

In Section 2, we will show that under the same conditions given in Theorem 2.6 of [1], every solution of 1 is either oscillatory or nonoscillatory and $\liminf_{t\to\infty} |x(t)| = 0$. The proof is much shorter and does not require Lemma 2.5 of [1] or similar results.

2. RESULTS

For the sake of completeness, we shall state the following well-known lemmas which are basic in our discussion.

LEMMA 2.1. *Let $y \in C^n([T,\infty),\mathbb{R})$ be such that $y^{(n)}(t) \leq 0$ for $t \geq T$ and such that $y^{(n)}(t)$ is not identically zero in any interval $[t_1,\infty)$. Then there exist signs $s_\ell \in \{-1,1\}$, $\ell = 0,1,\ldots,n-1$ and $T_1 \geq T$ such that $s_\ell y^{(\ell)}(t) > 0$ for $t \geq T_1$, $\ell = 0,1,\ldots,n-1$. There is $j \in \{0,1,\ldots,n\}$ such that $s_0 = s_1 = \ldots = s_j$ and $s_\ell = (-1)^{n+\ell+1}$ for $j \leq \ell \leq n$, where $s_n := -1$.*

LEMMA 2.2. *Let $q \in C([T,\infty),\mathbb{R})$ be such that $q(t) \leq t$, $\lim_{t\to\infty} q(t) = \infty$, and let $y \in C^2([T,\infty),\mathbb{R})$ be such that $y(t) > 0$, $y'(t) > 0$ and $y''(t) \leq 0$ for $t \geq T$.*

Then for each $k_1 \in (0,1)$ *there exists a* $T_{k_1} \geq T$ *such that*

$$y(q(t)) \geq k_1 \frac{q(t)}{t} y(t), \quad t \geq T_{k_1}.$$

LEMMA 2.3. *Let* $y \in C^2([T,\infty),\mathbb{R})$ *be such that* $y(t) > 0$, $y'(t) > 0$, *and* $y''(t) \leq 0$ *for* $t \geq T$. *Then for each* $k_2 \in (0,1)$ *there is a* $T_{k_2} \geq T$ *such that*

$$y(t) \geq k_2 t y'(t), \quad t \geq T_{k_2}.$$

We consider equation 1 under the following assumptions:

(i) $n \geq 2$ is an integer;
(ii) $\tau > 0$;
(iii) $a : [t_0,\infty) \to [0,\infty)$ is continuous;
(iv) $q : [t_0,\infty) \to \mathbb{R}$ is continuous, $q(t) \leq t$ for all $t \geq t_0$, and $\lim_{t\to\infty} q(t) = \infty$;
(v) there is $\eta \in C^n([t_0,\infty),\mathbb{R})$ such that $\eta^{(n)}(t) = e(t)$ and $\lim_{t\to\infty} \eta(t) = 0$;
(vi) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $uf(u) > 0$ for $u \neq 0$;
(vii) there is a constant $c > 0$ such that

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} \geq c;$$

and

$$\limsup_{t\to\infty} t \int_t^\infty a(s) \frac{q(s)}{s} ds > c^{-1}.$$

THEOREM 2.4. *Under the above assumptions every solution* $x(t)$, $t \geq t_1$, *of* (1.1) *is either oscillatory or nonoscillatory and* $\liminf_{t\to\infty} |x(t)| = 0$.

PROOF. The proof is by contradiction. If the conclusion of the theorem is false, then there is a solution $x(t)$ of 1, $t_1 \geq t_0$ and $\epsilon > 0$ such that $|x(t)| \geq \epsilon$ for all $t \geq t_1$. Without loss of generality, we can assume that

$$x(t) \geq \epsilon, \quad t \geq t_1. \tag{2}$$

We define

$$y(t) = x(t) - x(t-\tau) - \eta(t). \tag{3}$$

Then, by (v),

$$y^{(n)}(t) + a(t) f(x(q(t))) = 0. \tag{4}$$

It follows from 4 and (iii), (iv), (vi) that $y^{(n)}(t) \leq 0$ for $t \geq t_2 \geq t_1$, and by (vii), for every T there is $t \geq T$ such that $y^{(n)}(t) < 0$. By Lemma 2.1, there is $t_3 \geq t_2$ and signs $s_\ell \in \{-1,1\}$ such that $s_\ell y^{(\ell)}(t) > 0$ for $t \geq t_3$ and $\ell = 0,1,..,n-1$.

Suppose $s_{n-1} = -1$. By Lemma 2.1, $s_\ell = -1$ for $\ell = 0,1,...,n-1$. Set $\delta = -y(t_3) > 0$. Since $s_1 = -1$, $y(t) \leq -\delta$ for $t \geq t_3$. Using (v) and 3 we see that $x(t) - x(t-\tau) \leq -\frac{1}{2}\delta$ for large t. By (ii) this implies that $\lim_{t\to\infty} x(t) = -\infty$ which is impossible. Therefore, $s_{n-1} = 1$.

Integrating 4 from t to ∞ we have

$$y^{(n-1)}(t) = y^{(n-1)}(\infty) + \int_t^\infty a(s) f(x(q(s)))\, ds \geq \int_t^\infty a(s) f(x(q(s)))\, ds. \tag{5}$$

Since $n \geq 2$ we can integrate again from α to t and obtain

$$y^{(n-2)}(t) \geq y^{(n-2)}(\alpha) + \int_{\alpha}^{t} \int_{u}^{\infty} a(s) f(x(q(s)))\, ds\, du.$$

By 2 and (iv), $x(q(s)) \geq \epsilon$ for large s. By (vi), (vii) there is $\delta > 0$ such that $f(x(q(s))) \geq \delta$ for large s. Therefore, for large $\alpha \geq t_3$ and $t \geq \alpha$,

$$y^{(n-2)}(t) \geq y^{(n-2)}(\alpha) + \delta \int_{\alpha}^{t} \int_{u}^{\infty} a(s)\, ds\, du. \tag{6}$$

From (iv) and (vii) we obtain that

$$\limsup_{t\to\infty} t \int_{t}^{\infty} a(s)\, ds > 0. \tag{7}$$

Since

$$\int_{t}^{\infty} \int_{u}^{\infty} a(s)\, ds\, du = \int_{t}^{\infty} (s-t) a(s)\, ds \geq t \int_{2t}^{\infty} a(s)\, ds,$$

7 yields

$$\limsup_{t\to\infty} \int_{t}^{\infty} \int_{u}^{\infty} a(s)\, ds\, du > 0.$$

We conclude that

$$\int_{\alpha}^{\infty} \int_{u}^{\infty} a(s)\, ds\, du = \infty.$$

Therefore, from 6 we obtain $\lim_{t\to\infty} y^{(n-2)}(t) = \infty$. Hence, $s_{n-2} = 1$. By Lemma 2.1, $s_\ell = 1$ for every $\ell = 0, 1, ..., n-1$.

Since $s_0 = s_1 = 1$, there is $\delta > 0$ such that $y(t) \geq \delta$ for large t. Therefore, 3, (ii) and (v) show that

$$\lim_{t\to\infty} x(t) = \infty. \tag{8}$$

By Lemma 2.3, for $k_2 \in (0,1)$, there exists $t_4 \geq t_3$ such that

$$y^{(n-2)}(t) \geq k_2 t y^{(n-1)}(t) > 0, \quad t \geq t_4. \tag{9}$$

From 5 and 9 and the fact that for $t \geq t_5 \geq t_4$, $x(q(t)) \geq y(q(t))$, we have

$$y^{(n-2)}(t) \geq k_2 \left(t \int_{t}^{\infty} a(s) y(q(s))\, ds \right) \inf_{s\geq t} \frac{f(x(q(s)))}{x(q(s))}. \tag{10}$$

It follows from 9 that

$$y^{(n-2)}(t) \geq k_2 (t - t_4) y^{(n-1)}(t), \quad t \geq t_4.$$

If $n > 2$ we integrate both sides from t_4 to t integrating by parts on the right-hand side. We obtain

$$y^{(n-3)}(t) - y^{(n-3)}(t_4) \geq k_2 (t - t_4) y^{(n-2)}(t) - k_2 y^{(n-3)}(t) + k_2 y^{(n-3)}(t_4)$$

which gives

$$y^{(n-3)}(t) \geq \frac{k_2}{1+k_2} (t - t_4) y^{(n-2)}(t), \quad t \geq t_4.$$

If $n > 3$ we repeat this procedure until we obtain

$$y(t) \geq \frac{k_2}{1+(n-2)k_2}(t-t_4)y'(t), \quad t \geq t_4.$$

Therefore, there is $\delta > 0$ such that

$$y(t) \geq \delta(t-t_4)^{n-2}y^{(n-2)}(t), \quad t \geq t_4.$$

Therefore, using again (iv), there exists $t_6 \geq t_5$ such that

$$y(q(t)) \geq y^{(n-2)}(q(t)), \quad t \geq t_6. \tag{11}$$

Of course, 11 is also true if $n = 2$. By Lemma 2.2, for $k_1 \in (0,1)$, there exists $t_7 \geq t_6$ such that

$$y^{(n-2)}(q(t)) \geq k_1 \frac{q(t)}{t} y^{(n-2)}(t), \quad t \geq t_7. \tag{12}$$

Combining 10, 11, and 12 yields

$$y^{(n-2)}(t) \geq y^{(n-2)}(t) \inf_{s \geq t} \frac{f(x(q(s)))}{x(q(s))} k_1 k_2 t \int_t^\infty a(s) \frac{q(s)}{s}\, ds, \quad t \geq t_7.$$

From this we obtain

$$1 \geq k_1 k_2 \left(t \int_t^\infty a(s) \frac{q(s)}{s}\, ds \right) \inf_{s \geq t} \frac{f(x(q(s)))}{x(q(s))}, \quad t \geq t_7.$$

and by 8, (vii) and the fact that $k_1, k_2 \in (0,1)$ are arbitrary, we obtain

$$c^{-1} < \limsup_{t\to\infty} t \int_t^\infty a(s)\, \frac{q(s)}{s}\, ds \leq c^{-1},$$

which is a contradiction. □

Remark 2.5. *If $x(t)$ oscillates then $\liminf_{t\to\infty} |x(t)| = 0$. Therefore, the statement of Theorem 2.4 is equivalent to: every solution $x(t)$, $t \geq t_1$, of 1 satisfies $\liminf_{t\to\infty} |x(t)| = 0$.*

Theorem 2.6. *Under the above assumptions, if n is odd and $e(t) = 0$, then every solution $x(t)$ of 1 is oscillatory.*

Proof. The proof is by contradiction. If the conclusion of the theorem is false, then there is a solution $x(t)$ of 1 such that $x(t) \neq 0$ for $t \geq t_1$. Without loss of generality, we can assume that

$$x(t) > 0, \quad t \geq t_1. \tag{13}$$

We define

$$y(t) = x(t) - x(t-\tau). \tag{14}$$

As before we see that $y^{(n)}(t) \leq 0$ for large t, and, for every T there is $t \geq T$ such that $y^{(n)}(t) < 0$. There are signs $s_\ell \in \{-1, 1\}$ such that $s_\ell y^{(\ell)}(t) > 0$ for large t, $\ell = 0, 1, ..., n-1$. If $s_0 = s_1 = -1$, then $\lim_{t\to\infty} x(t) = -\infty$ which is impossible. If $s_0 = s_1 = 1$, then $\lim_{t\to\infty} x(t) = \infty$ which, in view of Theorem 2.4, is impossible. It follows from Lemma 2.1 that $s_\ell = (-1)^\ell$ for $\ell = 0, 1, ..., n-1$, so $y(t) > 0$ for large t. Then 13, 14 imply that there is $\epsilon > 0$ such that $x(t) \geq \epsilon$ for large t. This is impossible by Theorem 2.4. □

Note that for n odd, Theorem 2.6 improves Theorem 2.4 when $e(t) = 0$.

EXAMPLE 2.7. *Take* $\tau = 1$, $n \geq 2$ *even,* $f(u) = u$, $q(t) = t$, $e(t) = 0$. *Let* $h \in C^n(\mathbb{R}, [0, \infty))$ *be a function with period* 1 *such that* $h(t) = 0$ *for* $0 \leq t \leq \frac{1}{2}$. *Define*

$$a(t) = \frac{(e-1)e^{-t}}{e^{-t} + h(t)}.$$

It is easy to check that assumptions (i) through (vi) are satisfied. Moreover, we have $\int_t^\infty a(s)\,ds = \infty$ *because* $a(t) = e - 1$ *for all* $t \in [m, m + \frac{1}{2}]$ *for every integer* m. *Therefore, assumption (vii) is also satisfied with* $c = 1$. *Now*

$$x(t) = e^{-t} + h(t)$$

is a solution of 1. This solution is positive, and $\liminf_{t\to\infty} x(t) = 0$, *but* $\limsup_{t\to\infty} x(t) > 0$ *unless* $h(t) = 0$ *for all* t.

This example shows that Theorem 2.4 with n *even is optimal even when* $e(t) = 0$ *in the sense that we cannot replace* $\liminf_{t\to\infty} |x(t)| = 0$ *with* $\lim_{t\to\infty} x(t) = 0$.

EXAMPLE 2.8. *Take* $\tau = 1$, $n \geq 3$ *odd,* $f(u) = u$, $q(t) = t$. *Let* a *and* h *be as in Example 2.7 and set* $e(t) = 2(e-1)\,e^{-t}$. *All the assumptions (i) through (vii) are satisfied. Again* $x(t) = e^{-t} + h(t)$ *is a solution of 1.*

This example shows that Theorem 2.4 with n *odd is optimal in the sense that we cannot replace* $\liminf_{t\to\infty} |x(t)| = 0$ *with* $\lim_{t\to\infty} x(t) = 0$.

REFERENCES

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