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APPLICATION OF HE'S VARIATIONAL ITERATION METHOD TO NONLINEAR HELMHOLTZ AND FIFTH-ORDER KDV EQUATIONS

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Abstract

In this article, He's variational iteration method (VIM), is implemented to solve the linear Helmholtz partial differential equation and some nonlinear fifth-order Korteweg-de Vries (FKdV) partial differential equations with specified initial conditions. The initial approximations can be freely chosen with possible unknown constants which can be determined by imposing the boundary or initial conditions after few iterations. Comparison of the results with those obtained by Adomian's decomposition method reveals that VIM is very effective, convenient and quite accurate to both linear and nonlinear problems. It is predicted that VIM can be widely applied in engineering.

Mathematics Subject Classification 2000: 35Q53
Additional Key Words and Phrases: variational iteration method; Helmholtz equation; FKdV equation; nonlinear partial differential equations.

1. INTRODUCTION

It is well-known that there are many nonlinear partial equations in the study of physics, mechanics and biology. The solution of these equations can guide authors to know the described process deeply. But because of the complexity and limitations of classical mathematical methods, it is difficult for us to achieve exact solutions for the problems. In the recent decades, there has been great development in the numerical analysis [1] and exact solutions for nonlinear partial equations.

Recently, Many different methods have been introduced to solve linear and nonlinear problems, such as the homotopy analysis method [2], the variational iteration method (VIM) [3-7], the Adomian's decomposition method (ADM) [8,9], and homotopy perturbation method [10–13]. The VIM is capable for solving a large class of linear or nonlinear differential equations without the tangible restriction of sensitivity to the degree of the nonlinear term and also it reduces the size of calculations.

In this paper, we implement the VIM for finding the exact solutions of some derived linear Helmholtz and nonlinear fifth-order Korteveg-de Vries (FKdV) partial differential equations form Eqs. (1) and (4), respectively. The homogeneous and inhomogeneous solutions of the equations will be handled more easily and quickly by implementing the VIM rather than the traditional methods for the exact solutions as well as approximate solutions, without suffering from traditional difficulty. Unlike classical techniques, the nonlinear equations are solved easily without transforming the equation by using the VIM. The technique has many advantages over the classical ones. Mainly, it avoids linearization and perturbation in order to find solutions of a given nonlinear equations. On the other hand, the VIM provides explicit and numerical solutions with high accuracy, minimal calculations and strong operability, avoiding physically unrealistic assumptions.

The Helmholtz equation is:

$$\nabla^2 u + f(x,y)u = g(x,y) \tag{1}$$

With the boundary and initial conditions of:

$$u(0,y) = \psi_1(y), \qquad u_x(0,y) = \psi_2(y) \tag{2}$$

$$u(x,0) = \psi_3(x), \qquad u_y(x,0) = \psi_4(x) \tag{3}$$

Where $\psi_1(x), \psi_2(x), \psi_3(x)$ and $\psi_4(x)$ are given functions. The Helmholtz equation appears in such diverse phenomena as elastic waves in solids including vibrating string, bars, membranes, sound or acoustics, electromagnetic waves, and nuclear reactors [14, 15].

The FKdV equation is [14]:

$$u_t - u_{xxxxx} = F(x,t,u,u^2,u_x,u_{xx},u_{xxx}) \tag{4}$$

which occurs, for example in the theory of magneto- acoustic waves in plasmas [16] and in the theory of shallow water waves with surface tension [17]. The FKdV equation has been investigated extensively over the last decade. It has been shown that the traveling-wave solutions of this equation do not vanish at infinity [18,19].

2. BASIC IDEA OF THE VIM

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(t), \tag{5}$$

Where L is a linear operator, N is a nonlinear operator and $g(t)$ is a homogeneous term.

According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)\right)d\tau \tag{6}$$

Where λ is a general lagrangian multiplier which can be identified optimally via the variational theory. The subscript n indicates the nth approximation and u_n is considered as a restricted variation, i.e., $\delta\,\tilde{u}_n = 0$.

3. APPLICATIONS

In this section, we will apply the VIM to six linear and nonlinear examples

Example. 1.

Let's consider a special case of the Helmholtz equation, as follows:

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}-u(x,y)=0, \tag{7}$$

With the initial conditions of:

$$u(0,y)=y,\ \ u_x(0,y)=y+\cosh(y), \tag{8}$$

Now we start with an arbitrary initial approximation: $u_0(x,y)=A+Bx,$ where A and B are constants and x needs to be determined using the initial conditions (8), thus we have:

$$u_0(x,y)=y(1+x)+x\cosh(y), \tag{9}$$

According to the VIM, we can construct the correction functional of (7) as follows:

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^x \lambda(\ddot{u}_n(\tau,y)+u_n''(\tau,y)-u_n(\tau,y))d\tau, \tag{10}$$

where "dot" denotes differentiation with respect to x and "prime" denotes differentiation with respect to y and λ is general Lagrange multiplier.

Making the above correction functional stationary, we can obtain following stationary condition:

$$\begin{aligned}&\lambda''(\tau)-\lambda(\tau)=0,\\ &1-\lambda'(\tau)\big|_{\tau=x}=0,\\ &\lambda(\tau)\big|_{\tau=x}=0,\end{aligned} \tag{11}$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda=\frac{1}{2}e^{(\tau-x)}-\frac{1}{2}e^{(x-\tau)} \tag{12}$$

Substituting Eq. (12) into the correction functional Eq. (10) results in the following iteration formula:

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^x \left(\frac{1}{2}e^{(\tau-x)}-\frac{1}{2}e^{(x-\tau)}\right)\left(\ddot{u}_n(\tau,y)+u_n''(\tau,y)-u_n(\tau,y)\right)d\tau, \quad (13)$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,y)=y(1+x)+x\cosh(y), \quad (14)$$

Using the above variational formula (13), we have:

$$u_1(x,y)=u_0(x,y)+\int_0^x \left(\frac{1}{2}e^{(\tau-x)}-\frac{1}{2}e^{(x-\tau)}\right)\left(\ddot{u}_0(\tau,y)+u_0''(\tau,y)-u_0(\tau,y)\right)d\tau, \quad (15)$$

Substituting Eq. (14) into Eq. (15) and after simplifications, we have:

$$u_1(x,y)= x\cosh(y)+ye^{(x)}, \quad (16)$$

Which is exactly the same as obtained by Adomian's decomposition method [23], and HPM [26].
In the same way, other iterations can be calculated.

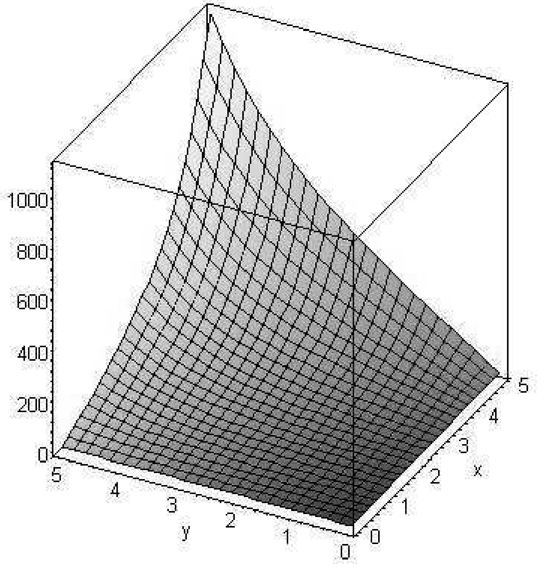

Fig.1. Numerical results obtain by VIM.

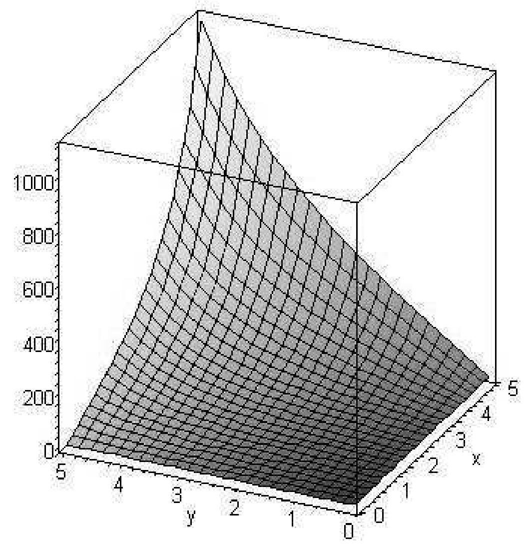

Fig.2. Numerical results obtain by ADM [23], and HPM [26].

Example.2.

We consider the second Helmholtz equation, as follows:

$$\frac{\partial^2 u(x,y)}{\partial x^2}+\frac{\partial^2 u(x,y)}{\partial y^2}+5u(x,y)=0, \tag{17}$$

With the initial conditions of:

$$u(0,y)=0,\ \ u_x(0,y)=3\sinh(2y), \tag{18}$$

Now we start with an arbitrary initial approximation: $u_0(x,y)=A+Bx,$ where A and B are constants in x to be determined using the initial conditions (8), thus we have:

$$u_0(x,y)=3x\sinh(2y), \tag{19}$$

According to the VIM, we can construct the correction functional of (17) as follows:

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^x \lambda(\ddot{u}_n(\tau,y)+u_n''(\tau,y)+5u_n(\tau,y))d\tau, \tag{20}$$

where “dot” denotes differentiation with respect to x and “prime” denotes differentiation with respect to y and λ is general Lagrange multiplier.
Making the above correction functional stationary, we can obtain following stationary condition:

$$\begin{aligned} &\lambda''(\tau)+5\lambda(\tau)=0,\\ &1-\lambda'(\tau)\big|_{\tau=x}=0,\\ &\lambda(\tau)\big|_{\tau=x}=0. \end{aligned} \tag{21}$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda=\frac{\sqrt{5}}{5}(\cos(x\sqrt{5})\sin(\tau\sqrt{5})-\sin(x\sqrt{5})\cos(\tau\sqrt{5})), \tag{22}$$

Substituting Eq. (22) into the correction functional Eq. (20) results in the following iteration formula:

$$u_{n+1}(x,y)=u_n(x,y)+\int_0^x\left(\frac{\sqrt{5}}{5}(\cos(x\sqrt{5})\sin(\tau\sqrt{5})-\sin(x\sqrt{5})\cos(\tau\sqrt{5}))\right)(\ddot{u}_n(\tau,y)+u''_n(\tau,y)-u_n(\tau,y))d\tau, \tag{23}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,y)=3x\sinh(2y), \tag{24}$$

Using the above variational formula (23), we have:

$$u_1(x,y)=u_0(x,y)+\int_0^x\left(\frac{\sqrt{5}}{5}(\cos(x\sqrt{5})\sin(\tau\sqrt{5})-\sin(x\sqrt{5})\cos(\tau\sqrt{5}))\right)(\ddot{u}_0(\tau,y)+u''_0(\tau,y)-u_0(\tau,y))d\tau, \tag{25}$$

Substituting Eq. (24) into Eq. (25) and after simplifications, we have:

$$\begin{aligned} u_1(x,y) &= 3x\sinh(2y)+\frac{54\sqrt{5}}{25}\sin(x\sqrt{5})\sinh(y)\cosh(y)\\ &-\frac{54}{5}\cos^2(x\sqrt{5})x\sinh(y)\cosh(y)-\frac{54}{5}\sin^2(x\sqrt{5})\,x\sinh(y)\cosh(y) \end{aligned} \tag{26}$$

In the same way, other iterations can be obtained.

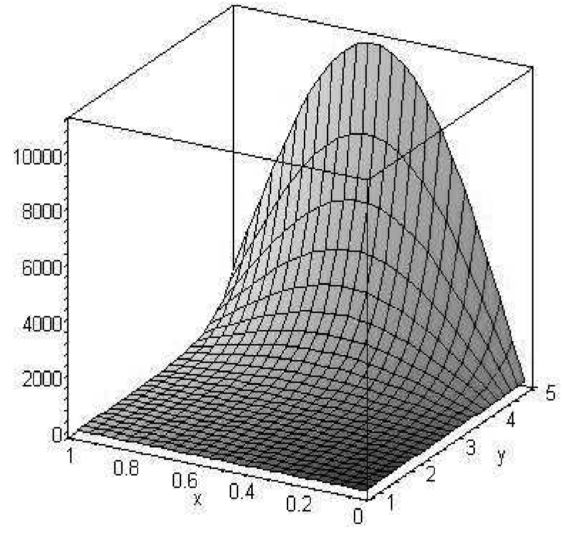

Fig.3. Numerical results obtain by VIM.

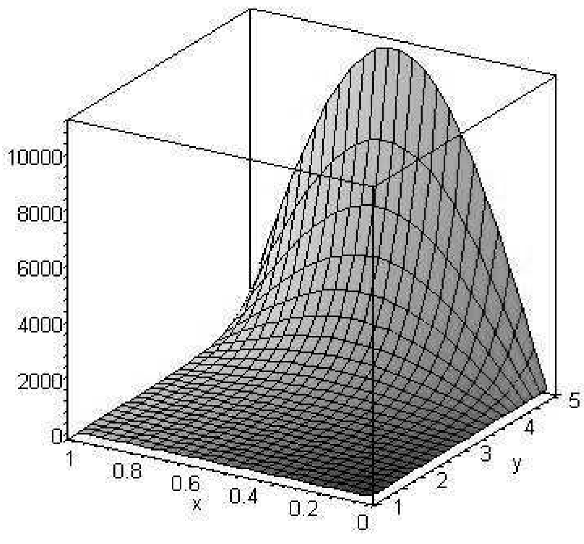

Fig.4. Numerical results obtain by ADM [23], and HPM [26].

Example.3.
Let's consider a special case of the FKDV equation [14], as follows:

$$u_t + u_x + u^2 u_{xx} + u_x u_{xx} - 20u^2 u_{xxx} + u_{xxxxx} = 0,$$
$$u(x,0) = \frac{1}{x}, \tag{27}$$

According to the VIM, we can construct the correction functional of (27) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^\tau \lambda(\dot{u}_n + u'_n + u_n^2 u''_n + u'_n u''_n - 20u_n^2 u'''_n + u'''''_n)d\tau, \tag{28}$$

where "dot" denotes differentiation with respect to t and "prime" denotes differentiation with respect to x and λ is general Lagrange multiplier.
Making the above correction functional stationary, we can obtain following stationary condition:

$$\lambda'(\tau) = 0,$$
$$1 + \lambda(\tau)\big|_{\tau=t} = 0. \tag{29}$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda = -1, \tag{30}$$

Substituting Eq. (30) into the correction functional Eq. (28) results in the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^\tau (\dot{u}_n + u'_n + u_n^2 u''_n + u'_n u''_n - 20u_n^2 u'''_n + u'''''_n)d\tau, \tag{31}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$\mathrm{u}_0(\mathrm{x,t}) = \frac{1}{\mathrm{x}}, \tag{32}$$

Using the above variational formula (31), we have:

$$u_1(x,t) = u_0(x,t) - \int_0^{\tau} (\dot{u}_0 + u_0' + u_0^2 u_0'' + u_0' u_0'' - 20 u_0^2 u_0''' + u_0''''')d\tau, \tag{33}$$

Substituting Eq. (32) into Eq. (33) and after simplifications, we have:

$$u_1(x,t) = \frac{1}{x} + \frac{t}{x^2} \tag{34}$$

Which is exactly the same as obtained by Adomian's decomposition method [24], and HPM [26].
In the same way, other iterations can be obtained.

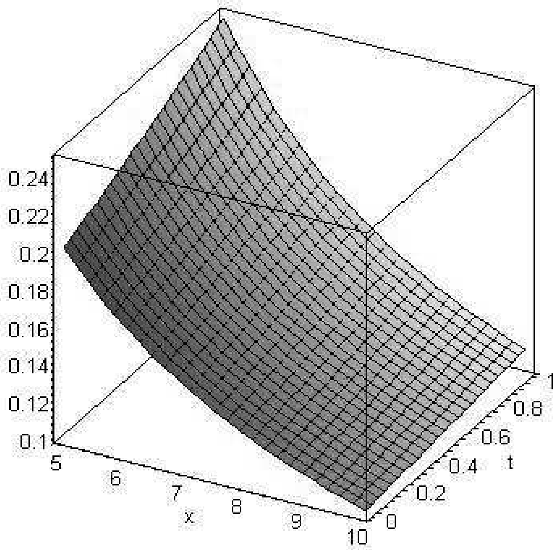

Fig.5. Numerical results obtain by VIM.

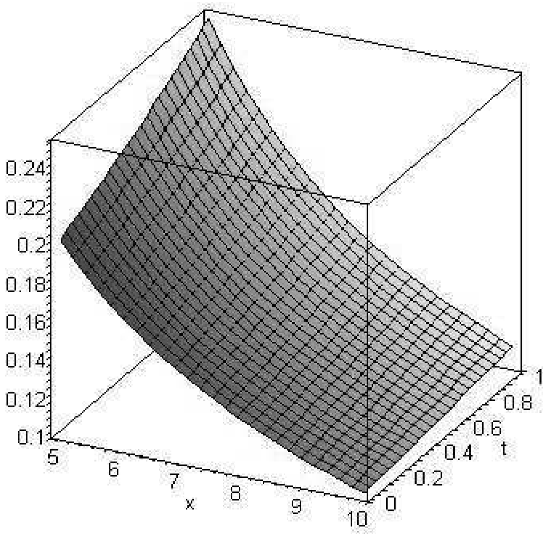

Fig.6. Numerical results obtain by ADM [23], and HPM [26].

Example.4.
We consider an equation with initial condition is given by:

$$u_t + uu_x - uu_{xxx} + u_{xxxxx} = 0, \qquad u(x,0) = e^x, \qquad (35)$$

According to the VIM, we can construct the correction functional of (35) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^{\tau} \lambda(\dot{u}_n + u_n u_n' - u_n u_n''' + u_n''''')d\tau, \quad (36)$$

where "dot" denotes differentiation with respect to t and "prime" denotes differentiation with respect to x and λ is general Lagrange multiplier.
Making the above correction functional stationary, we can obtain following stationary condition:

$$\lambda'(\tau) = 0,$$
$$1 + \lambda(\tau)\big|_{\tau=t} = 0. \qquad (37)$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda = -1, \qquad (38)$$

Substituting Eq. (38) into the correction functional Eq. (36) results in the following iteration formula:

$$u_{n+1}(x,t)=u_n(x,t)-\int_0^{\tau}(\dot{u}_n+u_n u_n'-u_n u_n'''+u_n''''')d\tau, \tag{39}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,t)=e^{(x)}, \tag{40}$$

Using the above variational formula (39), we have:

$$u_1(x,t)=u_0(x,t)-\int_0^{\tau}(\dot{u}_0+u_0 u_0'-u_0 u_0'''+u_0''''')\,d\tau, \tag{41}$$

Substituting Eq. (40) into Eq. (41) and after simplifications, we have:

$$u_1(x,t)=e^{(x)}-t\,e^{(x)}, \tag{42}$$

Which is exactly the same as obtained by Adomian's decomposition method [24], and HPM [26].
In the same way other iterationas can be obtained.

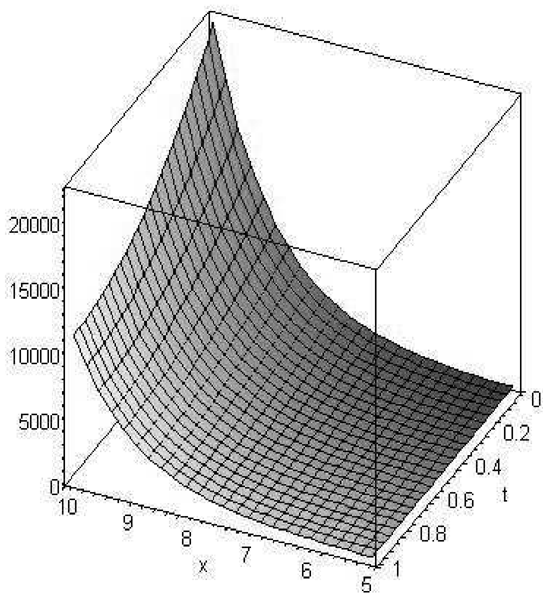

Fig.7. Numerical results obtain by VIM.

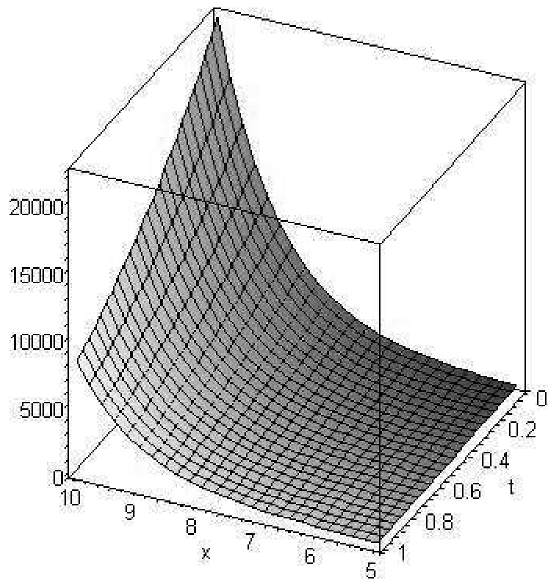

Fig.8. Numerical results obtain by ADM [23], and HPM [26].

Example.5.

As an example of the application of the self-canceling phenomena [25], let's seek the explicit solution of the inhomogeneous FKdV equation, as follows:

$$u_t - uu_x + u_{xxx} - u_{xxxxx} = \cos(x) + 2t\sin(x) + \frac{t^2}{2}\sin(2x),$$
$$u(x,0) = 0, \tag{43}$$

According to the VIM, we can construct the correction functional of (43) as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^{\tau} \lambda(\dot{u}_n - u_n u'_n + u'''_n - u'''''_n$$
$$-\cos(x) - 2t\sin(x) - \frac{t^2}{2}\sin(2x))d\tau, \tag{44}$$

where “dot” denotes differentiation with respect to t and “prime” denotes differentiation with respect to x and λ is general Lagrange multiplier.

Making the above correction functional stationary, we can obtain following stationary condition:

$$\lambda'(\tau) = 0,$$
$$1 + \lambda(\tau)\big|_{\tau=t} = 0. \tag{45}$$

The Lagrange multiplier, therefore, can be identified as:

$$\lambda = -1, \tag{46}$$

Substituting Eq. (46) into the correction functional Eq. (44) results in the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^{\tau} (\dot{u}_n - u_n u_n' + u_n''' - u_n'''''$$
$$-\cos(x) - 2t\sin(x) - \frac{t^2}{2}\sin(2x))d\tau, \tag{47}$$

Now we start with an arbitrary initial approximation that satisfies the initial condition:

$$u_0(x,t) = t\cos(x), \tag{48}$$

Using the above variational formula (47), we have:

$$u_1(x,t) = u_0(x,t) - \int_0^{\tau} (\dot{u}_0 - u_0 u_0' + u_0''' - u_0'''''$$
$$-\cos(x) - 2t\sin(x) - \frac{t^2}{2}\sin(2x))d\tau, \tag{49}$$

Substituting Eq. (48) into Eq. (49) and after simplifications, we have:

$$u_1(x,t) = t\cos(x), \tag{50}$$

This is exactly the same as obtained by Adomian's decomposition method [24], and HPM [26].
In the same way, other iterations can be obtained.

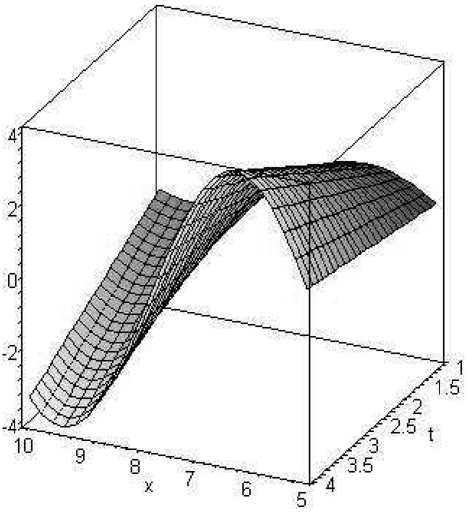

Fig.9. Numerical results obtain by VIM.

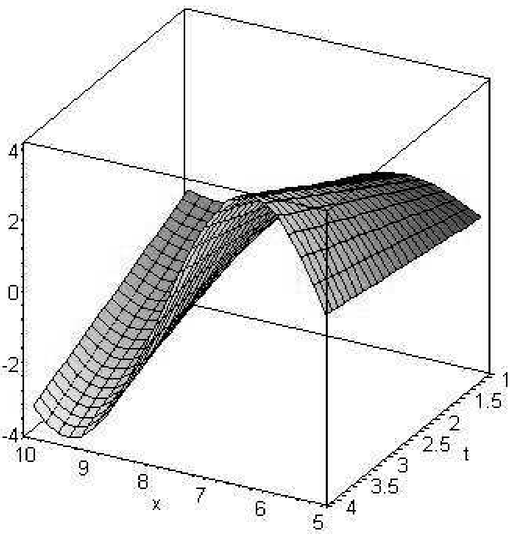

Fig.10. Numerical results obtain by ADM [23], and HPM [26].

4. CONCLUSIONS

In this paper, He's variational iteration method has been successfully applied to finding the exact solutions of the linear Helmholtz partial differential equation and some nonlinear fifth-order Korteweg-de Vries (FKdV) partial differential equations with specified initial conditions. The obtained solutions are compared with the Adomian's decomposition method and Homotopy perturbation Method. All the examples show that the results of the present method are in excellent accordance with those obtained by the Adomian's decomposition method and Homotopy perturbation Method. Some of the advantages of VIM are that the initial solution can be freely chosen with some unknown parameters and that we can easily achieve the unknown parameters in the initial solution. This technique is based on a general weighted residual method. The weighted factor or the general Lagrange multiplier λ can be determined by variational theory; the more exact λ is, the more it leads to rapid convergence to exact solutions. An interesting point about VIM is that with the fewest number of iterations or even in some cases, once, it can converge to correct results. The results show that the VIM is a powerful mathematical tool for solving linear and nonlinear partial differential equations, and therefore can be widely applied in engineering.

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