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EXTENSION OF DISTRIBUTIONS FROM $D'(R^{n+1})$

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Abstract

The variety of functionals, defined in the basic space $D(R^n)$ of finitary infinitely differentiable functions and assuming extension on $\varphi(x)\cdot 1(t)$, $\varphi(x)\in D(R^n)$ functions, is examined in the work. This operation correctness condition and its role in the generalized interpretation of equations of mathematical physics solutions is defined, especially in that case, when an elementary solution of some functional operator is known.

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1. INTRODUCTION

Linear operations continuity properties in topological vector spaces, which are the universal mathematical environment of distributions theory mechanism modelling, were examined in the work [1]. Possibility of such universalization is conditional on the fact that in the basic space $D(R^n)$ (here and in the sequel we will adhere to works [1], [2] notations) topology is generated by the system of zero neighborhoods, which are defined by arbitrary finite continuous positive functions packages $\{\gamma_i(x)\}_{i=0}^{m}$ in R^n and consist of all functions belonging to $D(R^n)$ that fit the inequalities:

$$|\varphi(x)| < \gamma_i(x),\ x\in R^n,\ i=\overline{1,m}.$$

Here and in the subsequent works on distributions theory we will substantially use aforesaid approaches equivalence to convergence fundamental notions interpretation. And we will also use continuities of operations of differentiation, nonsingular change of variables, $C^\infty(R^n)$ class functions multiplication, etc.

2. PRIMARY HEAD

THEOREM. Let L^* be linear set of distributions $u(x,t)\in D'(R^{n+1})$, for which there exists convolution $u(x,t)*[\delta(x)\cdot 1(t)]$. Then $u(x,t)\in L^*$ assumes $\varphi(x)\cdot 1(t)$, $\varphi(x)\in D(R^n)$ functions extension, and for the functional $u_0(x)\in D'(R^n)$, which is defined by the formula $(u_0(x),\varphi(x))=(u(x,t),\varphi(x)\cdot 1(t))$, the equality $u(x,t)*[\delta(x)\cdot 1(t)]=u_0(x)\cdot 1(t)$ is true.

PROVING. Let $u(x,t)$ be arbitrary set L^* member and $u(x,t)*[\delta(x)\cdot 1(t)]$ – its convolution with direct product $\delta(x)\cdot 1(t)$. Then there is a limit for any principal function $\varphi(x,t)\in D(R^{n+1})$

$$
\begin{aligned}
&(u(x,t)*[\delta(x)\cdot 1(t)],\ \varphi(x,t))= \\
&=\lim_{k\to\infty}(u(x,t)\cdot[\delta(y)\cdot 1(\tau)],\ \eta_k(x,t;\ y,\tau)\varphi(x+y,\ t+\tau)),
\end{aligned}
\tag{1}
$$

where $\{\eta_k(x,t;\ y,\tau)\}_{k=1}^{\infty}$ is principle $D(R^{n+1})$ functions sequence, converging to 1 in R^{2n+2}, that means:

1) for any compact $G\subset R^{2n+2}$ there will be such a number $K(G)$, that $\eta_k(x,t;\ y,\tau)=1$ for all $(x,t;\ y,\tau)\in G$ and $k\geq K(G)$;

2) functions $\{\eta_k(x,t;\ y,\tau)\}_{k=1}^{\infty}$ are equibounded in R^{2n+2} together with their derivatives, $\left|D^{\alpha}\eta_k(x,t;\ y,\tau)\right|<\gamma_{\alpha}$, $\alpha=(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{2n+2})$ it is arbitrary multiindex.

The limit in formula (1) exists and does not depend on the choice of the sequence $\{\eta_k(x,t;\ y,\tau)\}_{k=1}^{\infty}$. Subject to this fact we will examine the sequence of such a type

$$
\eta_k(x,t;\ y,\tau)=\eta_k^{(1)}(x)\eta_k^{(2)}(t)\eta_k^{(3)}(y)\eta_k^{(4)}(t+\tau),
\tag{2}
$$

where $\eta_k^{(1)}(x)$ and $\eta_k^{(3)}(y)$ are principle functions from $D(R^n)$, converging to 1 in R^n, and $\eta_k^{(2)}(t)$ and $\eta_k^{(4)}(\tau)$ are principle functions from (R^1), converging to 1 in R^1.

Substituting (2) in (1), we will get

$$
\begin{aligned}
&(u(x,t)*[\delta(x)\cdot 1(t)],\varphi(x,t))= \\
&=\lim_{k\to\infty}\left(u(x,t)\cdot[\delta(y)\cdot 1(\tau)],\ \eta_k^{(1)}(x)\eta_k^{(2)}(t)\eta_k^{(3)}(y)\eta_k^{(4)}(t+\tau)\cdot\varphi(x+y,\ t+\tau)\right)= \\
&=\lim_{k\to\infty}\left(u(x,t),\left(\delta(y)\cdot 1(\tau)],\ \eta_k^{(1)}(x)\eta_k^{(2)}(t)\eta_k^{(3)}(y)\eta_k^{(4)}(t+\tau)\cdot\varphi(x+y,\ t+\tau)\right)\right)= \\
&=\lim_{k\to\infty}\left(u(x,t),\ \eta_k^{(1)}(x)\eta_k^{(2)}(t)\eta_k^{(3)}(0)\int\eta_k^{(4)}(t+\tau)\cdot\varphi(x,\ t+\tau)d\tau\right)= \\
&=\lim_{k\to\infty}\left(u(x,t),\ \eta_k^{(1)}(x)\eta_k^{(2)}(t)\eta_k^{(3)}(0)\int\eta_k^{(4)}(\tau)\cdot\varphi(x,\tau)d\tau\right).
\end{aligned}
\tag{3}
$$

As the function $\varphi(x,\tau)$ in (3) is finitary, it is not difficult to see that

$$
\eta_k^{(1)}(x)\eta_k^{(2)}(t)\eta_k^{(3)}(0)\eta_k^{(4)}(\tau)=\eta_k^{(2)}(t),\ k\geq k_0,
\tag{4}
$$

where k_0 is a sufficiently great number.

Thus, subject to (4), the last of the equalities in (3) will take the form:

$$\begin{aligned}&\lim_{k\to\infty}\left(u(x,t),\ \eta_k^{(1)}(x)\eta_k^{(2)}(t)\eta_k^{(3)}(0)\int\eta_k^{(4)}(\tau)\cdot\varphi(x,\tau)d\tau\right)=\\&=\lim_{k\to\infty}\left(u(x,t),\ \eta_k^{(2)}(t)\int\varphi(x,\tau)d\tau\right).\end{aligned}\tag{5}$$

Relation (5) is performed for all principle functions $\varphi(x,t)\in D\left(R^{n+1}\right)$ and, particularly, for the functions of such a type

$$\varphi(x,t)=\varphi(x)\omega_\varepsilon(t),\tag{6}$$

where $\varphi(x,t)\in D\left(R^n\right)$, and an averaging kernel $\omega_\varepsilon(t)$ [2, p. 86] it is defined by a formula

$$\omega_\varepsilon(t)=\begin{cases}c_\varepsilon\exp\left(-\dfrac{\varepsilon^2}{\varepsilon^2-t^2}\right), & |t|<\varepsilon,\\0, & |t|\geq\varepsilon.\end{cases}\tag{7}$$

$$\int\omega_\varepsilon(t)dt=1.$$

Using (6) and (7), on the basis of (5) we will get

$$\lim_{k\to\infty}\left(u(x,t),\eta_k^{(2)}(t)\varphi(x)\right)=\left(u_0(x),\ \varphi(x)\right),\tag{8}$$

where $u_0(x)\in D'\left(R^n\right)$.

The equality (8) might be interpreted as the functional $u(x,t)\in D'\left(R^{n+1}\right)$ extension on function of the type $\varphi(x)\cdot 1(t)$, because the limit (8) exists for any sequences $\eta_k^{(2)}(t)\in D\left(R^1\right)$, converging to 1 in R^1 [2, p.196]. So,

$$\left(u_0(x),\varphi(x)\right)=\left(u(x,t),\varphi(x)\cdot 1(t)\right),\tag{9}$$

$\varphi(x)\in D\left(R^n\right)$.Let's return to the equality (5) and mark that for all $\varphi(x,t)\in D\left(R^{n+1}\right)$

$$\int\varphi(x,\tau)d\tau\in D\left(R^n\right).\tag{10}$$

Relation (10) results from the fact that the function $\varphi(x,t)$ is finitary and $\varphi(x,t)\in C^\infty\left(R^{n+1}\right)$. It results in the equality

$$D_x^{\alpha} \int \varphi(x,\tau) d\tau = \int D_x^{\alpha} \varphi(x,\tau) d\tau, \tag{11}$$

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n+2})$ is arbitrary.

Considering (8), (10), (5), (3), it is not difficult to arrive at

$$\begin{aligned}(u(x,t) * [\delta(x)\cdot 1(t)], \varphi(x,t)) &= \lim_{k\to\infty}\left(u(x,t),\ \eta_k^{(2)}(t) \int \varphi(x,\tau) d\tau\right) = \\ &= \left(u_0(x),\ \int \varphi(x,\tau) d\tau\right) = (u_0(x)\cdot 1(t), \varphi(x,t)).\end{aligned} \tag{12}$$

Under arbitrariness of $\varphi(x,t) \in D(R^{n+1})$, the identity

$$u(x,t) * [\delta(x)\cdot 1(t)] = u_0(x)\cdot 1(t) \tag{13}$$

results from (12).

The theorem is proved.

3. COMMENTS

1) Equality (13) plays a major role in the generalized interpretation of some types of equations of mathematical physics solutions, especially in that case, when $u(x,t)$ is an elementary solution of some differential operator. Then with the help of (13) it is possible to arrive at elementary solutions of truncated differential operators with respect to variable t, avoiding sufficiently tedious method of descent constructions with respect to t.

2) If in (13) $\int |u(x,t)| dt$ is locally integrable function in R^n according to Lebegue, it will take the form $u(x,t) * [\delta(x)\cdot 1(t)] = 1(t)\cdot \int u(x,\tau) d\tau$.

4. REFERENCES

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