STRUCTURE OF LORENTZ-INARIANT DISTRIBUTIONS WITH POINT SUPPORT

V.V. MALYI, V.S. SHCHOLOKOV

Abstract

Univesal representation for generalized functions from $D'(\mathbb{R}^n)$ generalized functions with a point support invariant relative to Lorentz group, is got.

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1. INTRODUCTION

Let's examine the generalized functions $f(x) \in D'(\mathbb{R}^n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ with the point support $\text{supp } f = \{0\}$, invariant relative to Lorentz group: if $A$ is a transform from Lorentz group, which maintain quadric quantic $k(x) \overset{\text{def}}{=} x_1^2 - x_2^2 - \ldots - x_n^2$, $k(Ax) = k(x)$, $A^{-1} = A^T$, then $f(Ax) = f(x)$. The indicated general functions belong to the class of general functions with a compact support, therefore they might be regarded also as space of general temperate functions elements $J'(\mathbb{R}^n)$, meaning extension uniqueness of $f(x)$ with $D'(\mathbb{R}^n)$ on $J'(\mathbb{R}^n)$ by the instrumentality of equality $$(f(x), \varphi(x)) = (f(x), \eta(x) \varphi(x)),$$

where $\varphi(x) \in J(\mathbb{R}^n)$, and $\eta(x) \in D(\mathbb{R}^n)$ and $\eta(x) = 1$ in the neighborhood of the support $f(x)$, as $D(\mathbb{R}^n)$ is compact in $J(\mathbb{R}^n)$.

In the sequel we will carry out all the mathematical constructions, using works [1].

2. PRIMARY HEAD

THEOREM. Let $f(x)$ be a generalized function of $D'(\mathbb{R}^n)$ class with a point support $\text{supp } f = \{0\}$ and $f(Ax) = f(x)$ for all transforms $A$ from Lorentz rotation groups in $\mathbb{R}^n$. Then there exists such a unique polynomial $P(z)$, $z \in \mathbb{C}$ that $f(x) = P(\Box) \delta(x)$, where $\Box = n$-dimensional wave operator.
PROVING. As the generalized function $f(x)$ has a point support $\text{supp } f = \{0\}$, it is uniquely defined as [1, p. 153]:

$$f(x) = \sum_{|\alpha|=0}^{m} c_\alpha D^\alpha \delta(x), \quad (1)$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is multiindex, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $c_\alpha \in \mathbb{C}$ and $D^n = \frac{\partial |\alpha|}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$, $m \geq 0$.

Let’s show how formula (1) is modified subject to Lorentz invariance of the function $f(x) \in D'(\mathbb{R}^n)$, $f(Ax) = f(x)$; and find Fourier transform $F[f(x)](\xi)$ and $F[f(Ax)](\xi)$ for the same purpose. Carrying out Fourier transform in equality (1), we get

$$F[f(x)](\xi) = \sum_{|\alpha|=0}^{m} c_\alpha F[D^\alpha \delta(x)](\xi) =$$

$$= \sum_{|\alpha|=0}^{m} c_\alpha (-i\xi)^\alpha F[\delta(x)](\xi) = \sum_{|\alpha|=0}^{m} c_\alpha (-i\xi)^\alpha. \quad (2)$$

Let’s use the following notation

$$M(\xi) \stackrel{\text{def}}{=} \sum_{|\alpha|=0}^{m} c_\alpha (-i\xi)^\alpha. \quad (3)$$

It’s apparent that $M(\xi)$ is a polynomial of the $m$ power with respect to variable $\xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n$, $|\sum_{|\alpha|=m} c_\alpha| \neq 0$.

Then let’s work out $F[f(Ax)](\xi)$. For any principle function $\varphi(x) \in J(\mathbb{R}^n)$ the following equalities obtain:

$$(F[f(Ax)](\xi), \varphi(x)) = (f(Ax), F[\varphi(\xi)](x)) =$$

$$= (f(x), F[\varphi(\xi)](A^T x)) = \left(f(x), \int \varphi(\xi) e^{i(\xi, A^T x)} d\xi \right) =$$

$$= \left(f(x), \int \varphi(\xi) e^{i(\xi, (A^T x))} d\xi \right) = \left(f(x), \int \varphi(A^T \xi) e^{i(\xi, x)} d\xi \right) =$$

$$= \left(f(x), F[\varphi(A^T \xi)](x) \right) = (F[f(x)](\xi), \varphi(A^T \xi)) =$$

$$= (F[f(x)](A \xi), \varphi(\xi)). \quad (4)$$
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From (4) it follows that

\[ F[f(Ax)](\xi) = F[f(x)](A\xi). \] (5)

Thus, on the basis of (2), (3), (5), we get

\[ F[f(Ax)](\xi) = M(A\xi). \] (6)

Fourier transform is bijective mapping \( \mathcal{F}'(\mathbb{R}^n) \) on \( \mathcal{F}'(\mathbb{R}^n) \), therefore from the equality \( f(Ax) = f(x) \) such relation follows:

\[ M(\xi) = M(A\xi). \] (7)

As \( M(\xi) \) is an integer function and \( k(\xi) = k(A\xi) \) for all \( \xi \in \mathbb{R}^n \), the following equality

\[ M(\xi) = M_o(k(\xi)) = \sum_{p=0}^{s} c_p^' k^p(\xi), \quad s = \frac{m}{2}, \] (8)

where \( M_o(k(\xi)) \) – polynomial with respect to variable \( k(\xi) \), is necessary and sufficient condition of identity (7) fulfillment.

Using inverse Fourier transform characteristics, on the basis of (2), (8), we get

\[ F^{-1}[F[f(x)](\xi)](x) = f(x) = F^{-1}[M_o(k(\xi))](x) = \frac{1}{(2\pi)^n} F[M_o(k(\xi))](x). \] (9)

Let’s define explicit expression for the quantity \( F[M_o(k(\xi))](x) \), meaning structure (8) of the polynomial \( M_o(k(\xi)) \):

\[ F[M_o(k(\xi))](x) = \sum_{p=0}^{s} c_p^' F[k^p(\xi)](x) = \sum_{p=0}^{s} c_p^' \left( -F\left[ (i\xi)_1^2 - \sum_{l=2}^{n} (i\xi)_l^2 \right] k^{p-1}(\xi) \cdot 1 \right)(x) = \sum_{p=0}^{s} c_p^' \left( \text{with } F[k^{p-1}(\xi) \cdot 1](x) \right) = \] (10)
where the wave operator $\Box = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$.

Substituting (10) into (9) we finally get

$$f(x) = P(\Box)\delta(x), \quad P(\Box) = P(z) \mid_{z=\delta},$$

$$P(z) = \sum_{p=0}^{\infty} (-1)^p c_p^\prime z^p, \quad z \in C.$$