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STRUCTURE OF LORENTZ-INVARIANT DISTRIBUTIONS WITH POINT SUPPORT

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Abstract

Univesal representation for generalized functions from $D'(R^n)$ generalized functions with a point support invariant relative to Lorentz group, is got.

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1. INTRODUCTION

Let's examine the generalized functions $f(x) \in D'(R^n)$, $x = (x_1, x_2, \ldots, x_n) \in R^n$ with the point support $\text{supp } f = \{0\}$, invariant relative to Lorentz group: if A is a transform from Lorentz group, which maintain quadric quantic $k(x) \overset{\text{def}}{=} x_1^2 - x_2^2 - \ldots - x_n^2$, $k(Ax) = k(x)$, $A^{-1} = A^T$, then $f(Ax) = f(x)$. The indicated general functions belong to the class of general functions with a compact support, therefore they might be regarded also as space of general temperate functions elements $J'(R^n)$, meaning extension uniqueness of $f(x)$ with $D'(R^n)$ on $J'(R^n)$ by the instrumentality of equalityhe

$$(f(x), \varphi(x)) = (f(x), \eta(x)\varphi(x)),$$

where $\varphi(x) \in J(R^n)$, and $\eta(x) \in D(R^n)$ and $\eta(x) = 1$ in the neighborhood of the support $f(x)$, as $D(R^n)$ is compact in $J(R^n)$.

In the sequel we will carry out all the mathematical constructions, using works [1].

2. PRIMARY HEAD

THEOREM. Let $f(x)$ be a generalized function of $D'(R^n)$ class with a point support $\text{supp } f = \{0\}$ and $f(Ax) = f(x)$ for all transforms A from Lorentz rotation groups in R^n. Then there exists such a unique polynomial $P(z)$, $z \in C$ that $f(x) = P(\Box)\,\delta(x)$, where $\Box$ – n-dimensional wave operator.

PROVING. As the generalized function $f(x)$ has a point support $\text{supp } f = \{0\}$, it is uniquely defined as [1, p. 153]:

$$f(x) = \sum_{|\alpha|=0}^{m} c_\alpha D^\alpha \delta(x), \tag{1}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is multiindex, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $c_\alpha \in C$ and $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$, $m \geq 0$.

Let's show how formula (1) is modified subject to Lorentz invariance of the function $f(x) \in D'(R^n)$, $f(Ax) = f(x)$; and find Fourier transform $F[f(x)](\xi)$ and $F[f(Ax)](\xi)$ for the same purpose. Carrying out Fourier transform in equality (1), we get

$$\begin{aligned} F[f(x)](\xi) &= \sum_{|\alpha|=0}^{m} c_\alpha F[D^\alpha \delta(x)](\xi) = \\ &= \sum_{|\alpha|=0}^{m} c_\alpha (-i\xi)^\alpha F[\delta(x)](\xi) = \sum_{|\alpha|=0}^{m} c_\alpha (-i\xi)^\alpha . \end{aligned} \tag{2}$$

Let's use the following notation

$$M(\xi) \overset{\text{def}}{=} \sum_{|\alpha|=0}^{m} c_\alpha (-i\xi)^\alpha . \tag{3}$$

It's apparent that $M(\xi)$ is a polynomial of the m power with respect to variable $\xi = (\xi_1, \xi_2, ..., \xi_n) \in R^n$, $\sum_{|\alpha|=m} |c_\alpha| \neq 0$.

Then let's work out $F[f(Ax)](\xi)$. For any principle function $\varphi(x) \in J(R^n)$ the following equalities obtain:

$$\begin{aligned} &(F[f(Ax)](\xi), \varphi(x)) = (f(Ax), F[\varphi(\xi)](x)) = \\ &= (f(x), F[\varphi(\xi)](A^T x)) = \left(f(x), \int \varphi(\xi) e^{i(\xi, A^T x)} d\xi\right) = \\ &= \left(f(x), \int \varphi(\xi) e^{i(A\xi, x)} d\xi\right) = \left(f(x), \int \varphi(A^T \xi) e^{i(\xi, x)} d\xi\right) = \\ &= (f(x), F[\varphi(A^T \xi)](x)) = (F[f(x)](\xi), \varphi(A^T \xi)) = \\ &= (F[f(x)](A\xi), \varphi(\xi)). \end{aligned} \tag{4}$$

From (4) it follows that

$$F[f(Ax)](\xi) = F[f(x)](A\xi). \tag{5}$$

Thus, on the basis of (2), (3), (5), we get

$$F[f(Ax)](\xi) = M(A\xi). \tag{6}$$

Fourier transform is bijective mapping $J'(R^n)$ on $J'(R^n)$, therefore from the equality $f(Ax) = f(x)$ such relation follows:

$$M(\xi) = M(A\xi). \tag{7}$$

As $M(\xi)$ is an integer function and $k(\xi) = k(A\xi)$ for all $\xi \in R^n$, the following equality

$$M(\xi) = M_0(k(\xi)) = \sum_{p=0}^{s} c'_p k^p(\xi), \ s = \frac{m}{2}, \tag{8}$$

where $M_0(k(\xi))$ – polynomial with respect to variable $k(\xi)$, is necessary and sufficient condition of identity (7) fulfillment.

Using inverse Fourier transform characteristics, on the basis of (2), (8), we get

$$\begin{aligned} F^{-1}[F[f(x)](\xi)](x) = f(x) = F^{-1}[M_0(k(\xi))](x) = \\ = \frac{1}{(2\pi)^n} F[M_0(k(\xi))](x). \end{aligned} \tag{9}$$

Let's define explicit expression for the quantity $F[M_0(k(\xi))](x)$, meaning structure (8) of the polynomial $M_0(k(\xi))$:

$$\begin{aligned} F[M_0(k(\xi))](x) = \sum_{p=0}^{s} c'_p F[k^p(\xi)](x) = \\ = \sum_{p=0}^{s} c'_p \left(-F\left[\left((i\xi_1)^2 - \sum_{l=2}^{n} (i\xi_l)^2 \right) k^{p-1}(\xi) \cdot 1 \right](x) \right) = \\ = \sum_{p=0}^{s} c'_p \ (\text{with } F[k^{p-1}(\xi) \cdot 1](x)) = \end{aligned} \tag{10}$$

$$= \sum_{p=0}^{s} (-1)^p c'_p \Box^p F[1](x) =$$

$$= (2\pi)^n \sum_{p=0}^{s} (-1)^p c'_p \Box^p \delta(x),$$

where the wave operator $\Box = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \ldots - \frac{\partial^2}{\partial x_n^2}$.

Substituting (10) into (9) we finally get

$$f(x) = P(\Box)\delta(x), \; P(\Box) = P(z)\big|_{z=\Box},$$

$$P(z) = \sum_{p=0}^{s} (-1)^p c'_p z, \; z \in C.$$

Uniqueness of the polynomial $P(z)$ follows from uniqueness of representation (1). The theorem is proved.

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