Journal of Applied Mathematics, Statistics and Informatics (JAMSI), **5** (2009), No. 1

OSCILLATION OF A CLASS OF HIGHER ORDER FORCED FUNCTIONAL DIFFERENTIAL EQUATIONS. II

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Abstract

Oscillatory behavior of certain higher order forced functional differential equations was investigated by Gera and Hamedani (2005). It was shown there that the oscillation criteria presented for odd order equations can be applied to a class of both forced and unforced differential equations. The results of the present note, which are established for any order (even or odd), will complete the work mentioned above.

Mathematics Subject Classification 2000: 34K11, 34K15
Additional Key Words and Phrases: Forced and unforced nonlinear differential equations; oscillatory and nonoscillatory solutions; n^{th} order equations.

1. INTRODUCTION

Gera and Hamedani, [1], study the oscillatory behavior of the n^{th} order forced functional differential equations

$$(x(t) - c(t)g(x(t-\tau)))^{(n)} + a(t)f(x(q(t))) = e(t), \quad t \geq t_0. \tag{1}$$

They established, among other things, an oscillation criterion for 1 when $c(t) \geq 0$ and $n \geq 3$ is an odd integer [1, Theorem 2.7] which can be applied to a class of both forced and unforced differential equations. The proof of Theorem 2.7 of [1] is based on a lemma [1, Lemma 2.5] whose conclusion holds only for n odd. In the present work we establish an oscillation criterion with $n \geq 2$ any integer, whose proof is much simpler than that of [1, Theorem 2.7] and does not require results similar to [1, Lemma 2.5]. For further details of the importance and the domain of applicability of our results, we refer the reader to [1] and the references therein.

2. MAIN RESULTS

For the sake of completeness, we shall state the following well-known lemmas which are basic in our discussion.

LEMMA 2.1. *Let $y \in C^n([T,\infty),\mathbb{R})$ be such that $y^{(n)}(t) \leq 0$ for $t \geq T$ and such that $y^{(n)}(t)$ is not identically zero in any interval $[t_1,\infty)$. Then there exist signs $s_\ell \in \{-1,1\}$, $\ell = 0,1,\dots,n-1$ and $T_1 \geq T$ such that $s_\ell y^{(\ell)}(t) > 0$ for $t \geq T_1$, $\ell = 0,1,\dots,n-1$. There is $j \in \{0,1,\dots,n\}$ such that $s_0 = s_1 = \ldots = s_j$ and $s_\ell = (-1)^{n+\ell-1}$ for $j \leq \ell \leq n$, where $s_n = -1$.*

Lemma 2.2. *Let $q \in C([T,\infty),\mathbb{R})$ be such that $q(t) \le t$, $\lim_{t\to\infty} q(t) = \infty$, and let $y \in C^2([T,\infty),\mathbb{R})$ be such that $y(t) > 0$, $y'(t) > 0$, and $y''(t) \le 0$ for $t \ge T$. Then for each $k_1 \in (0,1)$ there exists a $T_{k_1} \ge T$ such that*

$$y(q(t)) \ge k_1 \frac{q(t)}{t} y(t), \quad t \ge T_{k_1}.$$

Lemma 2.3. *Let $y \in C^2([T,\infty),\mathbb{R})$ be such that $y(t) > 0$, $y'(t) > 0$, and $y''(t) \le 0$ for $t \ge T$. Then for each $k_2 \in (0,1)$ there is a $T_{k_2} \ge T$ such that*

$$y(t) \ge k_2 t y'(t), \quad t \ge T_{k_2}.$$

We consider equation 1 under the following assumptions:

(i) $n \ge 2$ is an integer;
(ii) $\tau > 0$;
(iii) $a : [t_0,\infty) \to [0,\infty)$ is continuous;
(iv) $q : [t_0,\infty) \to \mathbb{R}$ is continuous, $q(t) \le t$ for all $t \ge t_0$, and $\lim_{t\to\infty} q(t) = \infty$;
(v) there is $\eta \in C^n([t_0,\infty),\mathbb{R})$ such that $\eta^{(n)}(t) = e(t)$ and $\lim_{t\to\infty} \eta(t) = 0$;
(vi) $f : \mathbb{R} \to \mathbb{R}$ is continuous and $uf(u) > 0$ for all $u \ne 0$;
(vii) there is a constant $\xi > 0$ such that

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} \ge \xi;$$

and

$$\limsup_{t\to\infty} t \int_t^\infty a(s) \frac{q(s)}{s} ds > \xi^{-1};$$

(viii) $g : \mathbb{R} \to \mathbb{R}$ is continuous and $ug(u) > 0$ for all $u \ne 0$;
(ix) $c : [t_0,\infty) \to [0,\infty)$ is continuous;
(x) there is a constant $\omega > 0$ and $t^* \ge t_0$ such that $c(t^* + j\tau) \le \omega$ for all $j = 0, 1, 2, \dots$ and $\frac{g(u)}{u} \le \omega^{-1}$ for all $u \ne 0$.

Theorem 2.4. *Under the assumptions (i)—(x) every solution $x(t)$, $t \ge t_1$, of 1 is either oscillatory or nonoscillatory and* $\liminf_{t\to\infty} |x(t)| = 0$.

Proof. The proof is by contradiction. If the conclusion of the theorem is false, then there is a solution $x(t)$ of 1, $t_1 \ge t_0$ and $\epsilon > 0$ such that $|x(t)| \ge \epsilon$ for all $t \ge t_1$. We assume that

$$x(t) \ge \epsilon, \quad t \ge t_1; \tag{2}$$

the proof under the assumption $x(t) \le -\epsilon < 0$ is similar. We define

$$y(t) = x(t) - c(t)g(x(t-\tau)) - \eta(t). \tag{3}$$

Then, by (v),

$$y^{(n)}(t) + a(t) f(x(q(t))) = 0. \tag{4}$$

It follows from 3, 4 and (iii), (iv), (vi) that $y^{(n)}(t) \le 0$ for $t \ge t_2 \ge t_1$, and by (vii), for every T there is $t \ge T$ such that $y^{(n)}(t) < 0$. By Lemma 2.1, there is $t_3 \ge t_2$ and signs $s_\ell \in \{-1,1\}$ such that $s_\ell y^{(\ell)}(t) > 0$ for $t \ge t_3$ and $\ell = 0, 1, \dots, n-1$.

Suppose $s_{n-1} = -1$. By Lemma 2, $s_\ell = -1$ for $\ell = 0, 1, \ldots, n-1$. Set $\delta = -y(t_3) > 0$. Since $s_1 = -1$, $y(t) \le -\delta$ for $t \ge t_3$. Using 3, (v) and (x) we see that, for large j,

$$x(t^* + j\tau) \le c(t^* + j\tau)g(x(t^* + (j-1)\tau)) - \frac{\delta}{2} \le x(t^* + (j-1)\tau) - \frac{\delta}{2}.$$

By (ii) this implies that $\liminf_{t\to\infty} x(t) = -\infty$ which, in view of 2, is impossible. Therefore, $s_{n-1} = 1$.

Integrating 4 from t to ∞ we have

$$y^{(n-1)}(t) = y^{(n-1)}(\infty) + \int_t^\infty a(s)f(x(q(s)))\,ds \ge \int_t^\infty a(s)f(x(q(s)))\,ds. \qquad (5)$$

Since $n \ge 2$ we can integrate again from α to t and obtain

$$y^{(n-2)}(t) \ge y^{(n-2)}(\alpha) + \int_\alpha^t \int_u^\infty a(s)f(x(q(s)))\,ds\,du.$$

By 2 and (iv), $x(q(s)) \ge \epsilon$ for large s. By (vi), (vii) there is $\gamma > 0$ such that $f(x(q(s))) \ge \gamma$ for large s. Therefore for large $\alpha \ge t_3$ and $t \ge \alpha$,

$$y^{(n-2)}(t) \ge y^{(n-2)}(\alpha) + \gamma \int_\alpha^t \int_u^\infty a(s)\,ds\,du. \qquad (6)$$

From (iv) and (vii) we obtain that

$$\limsup_{t\to\infty} t \int_t^\infty a(s)\,ds > 0. \qquad (7)$$

Since

$$\int_t^\infty \int_u^\infty a(s)\,ds\,du = \int_t^\infty (s-t)a(s)\,ds \ge t\int_{2t}^\infty a(s)\,ds,$$

7 yields

$$\limsup_{t\to\infty} \int_t^\infty \int_u^\infty a(s)\,ds\,du > 0.$$

We conclude that

$$\int_\alpha^\infty \int_u^\infty a(s)\,ds\,du = \infty.$$

Therefore, from 6 we obtain $\lim_{t\to\infty} y^{(n-2)}(t) = \infty$. It follows that $\lim_{t\to\infty} y^{(\ell)}(t) = \infty$ and $s_\ell = 1$ for every $\ell = 0, 1, \ldots, n-2$. In particular,

$$\lim_{t\to\infty} y(t) = \infty. \qquad (8)$$

From 2, 3, (viii) and (ix) we obtain

$$x(t) \ge y(t) + \eta(t), \quad t \ge t_1 + \tau. \qquad (9)$$

Then 8, 9 and (v) give

$$\lim_{t\to\infty} x(t) = \infty. \qquad (10)$$

By Lemma 2.3, for $k_2 \in (0,1)$, there exists $t_4 \ge t_3$ such that

$$y^{(n-2)}(t) \ge k_2 t\, y^{(n-1)}(t), \quad t \ge t_4. \tag{11}$$

It follows from 8, 9 and (v) that, for $k_3 \in (0,1)$, there is $t_5 \ge t_4$ such that $x(q(t)) \ge k_3 y(q(t))$. Therefore, from 5 and 11 we have

$$y^{(n-2)}(t) \ge k_2 k_3 \left(t \int_t^\infty a(s) y(q(s))\, ds \right) \inf_{s \ge t} \frac{f(x(q(s)))}{x(q(s))}. \tag{12}$$

It follows from 11 that

$$y^{(n-2)}(t) \ge k_2 (t - t_4) y^{(n-1)}(t), \quad t \ge t_4.$$

If $n > 2$ we integrate both sides from t_4 to t integrating by parts on the right-hand side. We obtain

$$y^{(n-3)}(t) - y^{(n-3)}(t_4) \ge k_2 (t - t_4) y^{(n-2)}(t) - k_2 y^{(n-3)}(t) + k_2 y^{(n-3)}(t_4)$$

which gives

$$y^{(n-3)}(t) \ge \frac{k_2}{1 + k_2} (t - t_4) y^{(n-2)}(t), \quad t \ge t_4.$$

If $n > 3$ we repeat this procedure until we obtain

$$y(t) \ge \frac{k_2}{1 + (n-2) k_2} (t - t_4) y'(t), \quad t \ge t_4.$$

Therefore, there is $\delta > 0$ such that

$$y(t) \ge \delta (t - t_4)^{n-2} y^{(n-2)}(t), \quad t \ge t_4.$$

Using again (iv), we find $t_6 \ge t_5$ such that

$$y(q(t)) \ge y^{(n-2)}(q(t)), \quad t \ge t_6. \tag{13}$$

Of course, 13 is also true if $n = 2$.

By Lemma 2.2, for $k_1 \in (0,1)$, there exists $t_7 \ge t_6$ such that

$$y^{(n-2)}(q(t)) \ge k_1 \frac{q(t)}{t} y^{(n-2)}(t), \quad t \ge t_7. \tag{14}$$

Combining 12, 13 and 14 yields

$$y^{(n-2)}(t) \ge y^{(n-2)}(t) k_1 k_2 k_3 \left(t \int_t^\infty a(s) \frac{q(s)}{s}\, ds \right) \inf_{s \ge t} \frac{f(x(q(s)))}{x(q(s))}, \quad t \ge t_7.$$

From this we obtain

$$1 \ge k_1 k_2 k_3 \left(t \int_t^\infty a(s) \frac{q(s)}{s}\, ds \right) \inf_{s \ge t} \frac{f(x(q(s)))}{x(q(s))}, \quad t \ge t_7, \tag{15}$$

and by 10, (vii) and the fact that $k_1, k_2, k_3 \in (0,1)$ are arbitrary, we obtain

$$\xi^{-1} < \limsup_{t \to \infty} t \int_t^\infty a(s) \frac{q(s)}{s}\, ds \le \xi^{-1},$$

which is a contradiction. $\square$

THEOREM 2.5. *If $n \geq 3$, the conclusion of Theorem 2.4 holds with (vii) replaced by a weaker assumption*

*(vii)**

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 0,$$

and

$$\limsup_{t\to\infty} t \int_t^\infty a(s)\frac{q(s)}{s}\, ds > 0.$$

PROOF. Follow the proof of Theorem 2.4 and replace (13) with $y(q(t)) \geq \beta q(t) y^{(n-2)}(q(t))$ for some $\beta > 0$. Then obtain 15 but with an additional arbitrarily large factor k_4 on the right-hand side. This leads to a contradiction assuming (vii)*. □

COROLLARY 2.6. *Let $e(t) = 0$ in 1, then the conclusion of Theorem 2.4 (and of Theorem 2.5) holds.*

COROLLARY 2.7. *Let $q(t) = t$ in 1, then the conclusion of Theorem 2.4 (and of Theorem 2.5 holds.*

THEOREM 2.8. *Let (i)—(viii) hold. Assume further that*

(ix) g is a bounded function;*
(x) $c(t) \leq 0$, $t \geq t_0$ and bounded.*

Then every solution $x(t)$, $t \geq t_1$ of 1 is either oscillatory or nonoscillatory and $\liminf_{t\to\infty} |x(t)| = 0$.

COROLLARY 2.9. *Let $n \geq 3$ and assume that (i)—(vi), (vii)*, (viii), (ix)* and (x)* hold. Then every solution $x(t)$, $t \geq t_1$ of 1 is either oscillatory or nonoscillatory and* $\liminf_{t\to\infty} |x(t)| = 0$.

COROLLARY 2.10. *Let $e(t) = 0$ in 1, then the conclusion of Theorem 2.8 (and of Corollary 2.9) holds.*

REMARK 2.11. *It is clear from Theorems 2.4 and 2.5 (as well as Theorem 2.8, Corollary 2.9 and Corollary 2.10) that the oscillation of the unforced equation*

$$(x(t) - c(t)g(x(t-\tau)))^{(n)} + a(t)f(x(q(t))) = 0,$$

is maintained under the effect of certain forcing terms.

REMARK 2.12. *In [2], the present authors study the oscillatory behavior of 1 for the special case of $c(t) = 1$ and $g(x) = x$.*

3. EXAMPLES

A few examples with n odd were presented in [1]. Here we just give some examples for n even.

EXAMPLE 3.1. *Equation*

$$(x(t) - c(t)x(t-1))^{(4)} + \frac{k}{t^2}x(t) = \frac{10}{t^5}(3\sin\ln t - 5\cos\ln t), \quad t \geq 1,$$

satisfies all the assumptions of Theorem 2.5 for $k > 0$ and appropriate $c(t)$. Here $\eta(t) = \frac{1}{t}(\sin\ln t + \cos\ln t)$.

Example 3.2. *The nonlinear equation*

$$(x(t) - c(t)x(t-1))^{(4)} + \frac{k}{t^5} f(x(t)) = \frac{10}{t^5}(3\sin\ln t - 5\cos\ln t), \quad t \geq 1,$$

satisfies all the assumptions of Theorem 2.5 for $k > 0$, where $f(x) = x\ln(e + x^2)$ or $f(x) = xe^{|x|}$ and $c(t)$ is chosen appropriately. Clearly $\eta(t)$ is as in Example 3.1.

Example 3.3. *The nonlinear even order equation*

$$(x(t) - c(t)x(t-1))^{(n)} + \frac{k}{t^2} x(t)e^{|\sin x(t)|} = \left(\frac{\sin\ln t}{t}\right)^{(n)}, \quad t \geq 1,$$

satisfies all the assumptions of Theorem 2.4 for $k > 1$ and appropriate $c(t)$. Here $\eta(t) = \frac{\sin\ln t}{t}$.

REFERENCES

[1] Gera, M., and Hamedani, G.G. : Oscillation of a Class of Higher Order Forced Functional Differential Equations, JAMSI, **1** (2005), No.2, pp. 49–62.

[2] Hamedani, G.G., and Volkmer, H: Oscillation of nth order forced functional differential equations, to appear in JAMSI.

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Received March 2009