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ASYMPTOTIC STABILITY OF TIME VARYING DELAY-DIFFERENCE CONTROL SYSTEM WITH TIME-VARYING DELAY OF HOPFIELD NEURAL NETWORKS VIA MATRIX INEQUALITIES*

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Abstract

In this paper, we derive a sufficient condition for asymptotic stability of the zero solution of time varying delay-difference control system with time-varying delay of Hopfield neural networks in terms of certain matrix inequalities by using a discrete version of the Lyapunov second method.

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General Terms: Asymptotic Stability, Lyapunov Second Method, Hopfield Neural Networks
Additional Key Words and Phrases: Asymptotic stability, Hopfield neural networks, Lyapunov function, Time varying delay-difference control system with time-varying delay, Matrix inequalities.

1. INTRODUCTION

In this paper, we consider time-varying delay-difference control system with time-varying delay of Hopfield neural networks of the form

$$v(k+1) = -A(k)v(k) + B(k)S(v(k-h(k))) + C(k)u(k) + f\ , \qquad (1)$$

where $v(k) \in \Omega \subseteq \mathbf{R}^n$ is the neuron state vector, $h(k)$ is the continuous function describing the time-varying transmission delay in the network system and satisfies $0 \le h(k) \le h,\quad A(k) = diag\{a_1(k),\ldots,a_n(k)\}$, $a_i(k) \ge 0$, $i = 1,2,...,n$ is the $n \times n$ relaxation matrix functions, $B(k)$ is the $n \times n$ weight matrix functions, $C(k)$ is the $n \times m$ matrix functions, $u(k) \in \mathbf{R}^m$ is the control vector, $f = (f_1,\ldots,f_n) \in \mathbf{R}^n$ is the constant external input vector and $S(z) = [s_1(z_1),\ldots,s_n(z_n)]^T$ with $s_i \in C^1[\mathbf{R},(-1,1)]$ where s_i is the neuron activations and monotonically increasing for each $i = 1,2,...,n$.

The asymptotic stability of the zero solution of the delay-differential system of Hopfield neural networks has been developed during the past several years. We refer to monographs by Burton [3] and Arik [2] and the references cited therein. Much less is known regarding the asymptotic stability of the zero solution of delay-difference system of Hopfield neural networks. Therefore, the purpose of this paper is to establish sufficient condition for the asymptotic stability of the zero solution of (1) in terms of certain matrix inequalities.

2. PRELIMINARIES

We assume that the n-vector function nonlinear perturbations are bounded and satisfy the following hypotheses, respectively:

$$0 \le \frac{f_i(r_1) - f_i(r_2)}{r_1 - r_2} \le l_i, \quad \forall r_1, r_2 \in \mathbf{R}, \text{ and } r_1 \ne r_2, \tag{2}$$

where $l_i > 0$ are constants for $i = 1, 2, ..., n$.

By assumption (2) we know that the functions $f_i(\cdot)$ satisfy

$$|f_i(x_i)| \le l_i |x_i|, \; i = 1, 2, ..., n,$$

and

$$f^2{}_i(x_i) \le l_i x_i f_i(x_i), \quad i = 1, 2, ..., n. \tag{3}$$

Fact 2.1 For any positive scalar ε and vectors x and y, the following inequality holds:

$$x^T y + y^T x \le \varepsilon x^T x + \varepsilon^{-1} y^T y.$$

Lemma 2.1 [2] The zero solution of difference system is asymptotic stability if there exists a positive definite function $V(x): \mathbf{R}^n \to \mathbf{R}^+$ such that

$$\exists \beta > 0 : \Delta V(x(k)) = V(x(k+1)) - V(x(k)) \le -\beta \|x(k)\|^2,$$

along the solution of the system. In the case the above condition holds for all $x(k) \in V_\delta$, we say that the zero solution is locally asymptotically stable.

Lemma 2.2 [3] For any constant symmetric matrix $M \in \mathbf{R}^{n \times n}$, $M = M^T > 0$, scalar $s \in \mathbf{Z}^+ / \{0\}$, vector function $W : [0, s] \to \mathbf{R}^n$, we have

$$s \sum_{i=0}^{s-1} (w^T(i) M w(i)) \ge \left(\sum_{i=0}^{s-1} w(i) \right)^T M \left(\sum_{i=0}^{s-1} w(i) \right).$$

3. MAIN RESULTS

In this section, we consider the sufficient condition for asymptotic stability of the zero solution v^* of (1) in terms of certain matrix inequalities. Without loss of generality, we can assume that $v^* = 0, S(0) = 0$ and f =0 (for otherwise, we let $x = v - v^*$ and define

$$S(x) = S(x + v^*) - S(v^*)).$$

The new form of (1) is now given by

$$x(k+1) = -A(k)x(k) + B(k)S(x(k-h(k))) + C(k)u(k). \quad (4)$$

This is a basic requirement for controller design. Now, we are interested designing a feedback controller for the system (4) as

$$u(k) = Kx(k),$$

where K is $n \times m$ constant control gain matrix.

The new form of (4) is now given by

$$x(k+1) = -A(k)x(k) + B(k)S(x(k-h(k))) + C(k)Kx(k). \quad (5)$$

Theorem 3.1 The zero solution of the delay-difference system (5) is asymptotically stable if there exist symmetric positive definite matrices $P,\ G,\ W,\ L = diag[l_1, \ldots, l_n] > 0$ and $\varepsilon > 0$ satisfying the following matrix inequalities of the form

$$\psi = \begin{pmatrix} (1,1) & 0 & 0 \\ 0 & (2,2) & 0 \\ 0 & 0 & (3,3) \end{pmatrix} < 0, \quad (6)$$

where

$$\begin{aligned}(1,1) = {} & A(k)PA(k) - A(k)PC(k)K - K^T C^T(k)PA(k) \\ & -C^T(k)K^T PC(k) - P + h(k)G + W \\ & +\varepsilon A(k)PB(k)B^T(k)PA(k) \\ & +\varepsilon_1 K^T C^T(k)PB(k)B^T(k)PC(k)K\end{aligned},$$

$$(2,2) = \varepsilon^{-1}LL + \varepsilon_1^{-1}LL + LB^T(k)PB(k)L - W,$$

$$(3,3) = -h(k)G.$$

Proof Consider the Lyapunov function $V(y(k)) = V_1(y(k)) + V_2(y(k)) + V_3(y(k))$, where

$$V_1 = x^T(k)Px(k),$$

$$V_2 = \sum_{i=k-h(k)}^{k-1} (h(k) - k + i)x^T(i)Gx(i),$$

$$V_3 = \sum_{i=k-h(k)}^{k-1} x^T(i)Wx(i),$$

P, G, and W being symmetric positive definite solutions of (6) and $y(k) = [x(k), x(k-h)]$.

Then difference of $V(y(k))$ along trajectory of solution of (5) is given by

$$\Delta V(y(k)) = \Delta V_1(y(k)) + \Delta V_2(y(k)) + \Delta V_3(y(k)),$$

where

$$\begin{aligned}
\Delta V_1(y(k)) = V_1(x(k+1)) - V_1(x(k)) \\
&= [-A(k)x(k) + B(k)S(x(k-h(k))) + C(k)Kx(k)]^T \\
&\times P[-A(k)x(k) + B(k)S(x(k-h(k))) + C(k)Kx(k)] \\
&-x^T(k)Px(k) \\
&= x^T(k)[A(k)PA(k) - A(k)PC(k)K - K^T C^T(k)PA(k) \\
&-C^T(k)K^T PC(k) - P]x(k) \\
&-x^T(k)A(k)PB(k)S(x(k-h(k))) \\
&-S^T(x(k-h(k)))B^T(k)PA(k)x(k) \\
&+x^T(k)K^T C^T(k)PB(k)S(x(k-h(k))) \\
&+S^T(x(k-h(k)))B^T(k)PC(k)Kx(k) \\
&+S^T(x(k-h(k)))B^T(k)PB(k)S(x(k-h(k))),
\end{aligned}$$

$$\Delta V_2 = \Delta\left(\sum_{i=k-h(k)}^{k-1} (h(k) - k + i)x^T(i)Gx(i)\right) = h(k)x^T(k)Gx(k) - \sum_{i=k-h(k)}^{k-1} x^T(i)Gx(i),$$

and

$$\Delta V_3 = \Delta\left(\sum_{i=k-h(k)}^{k-1} x^T(i)Wx(i)\right) = x^T(k)Wx(k) - x^T(k-h(k))Wx(k-h(k)), \quad (7)$$

where (3) and **Fact 2.1** are utilized in (7), respectively.

Note that

$$-x^T(k)A(k)PB(k)S(x(k-h(k))) - S^T(x(k-h(k)))B^T(k)PA(k)x(k) \le$$
$$\varepsilon x^T(k)A(k)PB(k)B^T(k)PA(k)x(k) + \varepsilon^{-1}S^T(x(k-h(k)))S(x(k-h(k))),$$
$$x^T(k)K^TC^T(k)PB(k)S(x(k-h(k))) + S^T(x(k-h(k)))B^T(k)PC(k)Kx(k) \le$$
$$\varepsilon_1 x^T(k)K^TC^T(k)PB(k)B^T(k)PC(k)Kx(k)$$
$$+\varepsilon_1^{-1}S^T(x(k-h(k)))S(x(k-h(k))),$$
$$S^T(x(k-h(k)))B^T(k)PB(k)S(x(k-h(k))) \le$$
$$x^T(k-h(k))LB^T(k)PB(k)Lx(k-h(k)),$$
$$\varepsilon^{-1}S^T(x(k-h(k)))S(x(k-h(k))) \le \varepsilon^{-1}x^T(k-h(k))LLx(k-h(k)),$$
$$\varepsilon_1^{-1}S^T(x(k-h(k)))S(x(k-h(k))) \le \varepsilon_1^{-1}x^T(k-h(k))LLx(k-h(k)),$$

Hence

$$\begin{aligned}\Delta V_1 \le\ & x^T(k)[A(k)PA(k) - A(k)PC(k)K - K^TC^T(k)PA(k)\\ & -C^T(k)K^TPC(k) - P]x(k) + \varepsilon x^T(k)A(k)PB(k)B^T(k)PA(k)x(k)\\ & +\varepsilon_1 x^T(k)K^TC^T(k)PB(k)B^T(k)PC(k)Kx(k)\\ & +\varepsilon^{-1}x^T(k-h(k))LLx(k-h(k)) + \varepsilon_1^{-1}x^T(k-h(k))LLx(k-h(k))\\ & +x^T(k-h(k))LB^T(k)PB(k)Lx(k-h(k)).\end{aligned}$$

Then we have

$$\begin{aligned}\Delta V \le\ & x^T(k)[A(k)PA(k) - A(k)PC(k)K\\ & -K^TC^T(k)PA(k) - C^T(k)K^TPC(k) - P\\ & +h(k)G + W + \varepsilon A(k)PB(k)B^T(k)PA(k)\\ & +\varepsilon_1 K^TC^T(k)PB(k)B^T(k)PC(k)K]x(k)\\ & +x^T(k-h(k))[\varepsilon^{-1}LL + \varepsilon_1^{-1}LL + LB^T(k)PB(k)L\\ & -W]x(k-h(k))\\ & -\sum_{i=k-h(k)}^{k-1} x^T(i)Gx(i).\end{aligned}$$

Using **Lemma 2.2**, we obtain

$$\sum_{i=k-h(k)}^{k-1} x^T(i)Gx(i) \geq \left(\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right)^T (h(k)G)\left(\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right).$$

From the above inequality it follows that:

$$\begin{aligned}\Delta V \leq\ & x^T(k)[A(k)PA(k) - A(k)PC(k)K - K^T C^T(k)PA(k) - C^T(k)K^T PC(k)\\ & -P + h(k)G + W + \varepsilon A(k)PB(k)B^T(k)PA(k)\\ & +\varepsilon_1 K^T C^T(k)PB(k)B^T(k)PC(k)K]x(k)\\ & +x^T(k-h(k))[\varepsilon^{-1}LL + \varepsilon_1^{-1}LL + LB^T(k)PB(k)L - W]x(k-h(k))\\ & -\left(\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right)^T (h(k)G)\left(\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right)\end{aligned}$$

$$= \left(x^T(k), x^T(k-h(k)), (\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i))^T\right)\begin{pmatrix}(1,1) & 0 & 0\\ 0 & (2,2) & 0\\ 0 & 0 & (3,3)\end{pmatrix}$$

$$\times\begin{pmatrix} x(k)\\ x(k-h(k))\\ (\frac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i))\end{pmatrix}$$

$$= y^T(k)\psi y(k),$$

where

$$\begin{aligned}(1,1) =\ & A(k)PA(k) - A(k)PC(k)K - K^T C^T(k)PA(k)\\ & -C^T(k)K^T PC(k) - P + h(k)G + W\\ & +\varepsilon A(k)PB(k)B^T(k)PA(k)\\ & +\varepsilon_1 K^T C^T(k)PB(k)B^T(k)PC(k)K,\\ (2,2) =\ & \varepsilon^{-1}LL + \varepsilon_1^{-1}LL + LB^T(k)PB(k)L - W,\\ (3,3) =\ & -h(k)G,\end{aligned}$$

$$y(k)=\begin{pmatrix} x(k) \\ x(k-h(k)) \\ \left(\dfrac{1}{h(k)}\sum_{i=k-h(k)}^{k-1} x(i)\right) \end{pmatrix}.$$

By the condition (6), $\Delta V(y(k))$ is negative definite, namely there is a number $\beta > 0$ such that $\Delta V(y(k)) \le -\beta \|y(k)\|^2$, and hence, the asymptotic stability of the system immediately follows from **Lemma 2.1**. This completes the proof.

Remark 3.1 Theorem 3.1 gives a sufficient condition for the asymptotic stability of delay-difference system (5) via matrix inequalities. These conditions are described in terms of certain diagonal matrix inequalities, which can be realized by using the linear matrix inequality algorithm proposed in [4]. But Hu and Wang [7] these conditions are described asymptotic stability of delay-difference system via matrix inequalities in terms of certain symmetric matrix inequalities, which can be realized by using the Schur complement lemma and linear matrix inequality algorithm proposed in [4].

Example 3.1 Let us consider the delay-difference system (4), given by the system

$$x(k+1) = -A(k)x(k) + B(k)S(x(k-h(k))) + C(k)u(k),$$

where the matrices are

$$A(k)=\begin{pmatrix} 1.9-0.5e^{-5.8t}-e^{-5.8t} & 0 \\ 0 & 0.5e^{-t}-2 \end{pmatrix}, B(k)=\begin{pmatrix} 3-e^{6t} & -1 \\ 0 & -0.5e^{-2t} \end{pmatrix},$$

$$C(k)=\begin{pmatrix} e^{t} & -1 \\ 1 & e^{-t} \end{pmatrix},$$

$s_i(x_i)=\dfrac{2}{\pi}\tan^{-1}(x_i), i=1,2$, $\quad K=\begin{pmatrix}1.2369 & 3.9865\end{pmatrix}, \varepsilon = 0.5, \varepsilon_1 = 0.5,$ and $0 \le h(k) = e^{-k} \le 1$.

Using the LMI Toolbox in MATLAB, we found that the LMIs in **Theorem 3.1** are feasible and

$$P=\begin{pmatrix}0.2795 & 0.0722\\ 0.0722 & 0.5725\end{pmatrix},\quad G=\begin{pmatrix}3.4259 & 0.0183\\ 0.0183 & 3.6663\end{pmatrix},\quad W=\begin{pmatrix}0.9018 & 0.3625\\ 0.3625 & 1.0628\end{pmatrix}$$

are the set of solutions to

the LMIs (6).

By a straightforward, we have

$$\psi=\begin{pmatrix}-0.4890 & 0\\ 0 & -0.8251\end{pmatrix}.$$

The eigenvalues are -0.4890 and -0.8251, respectively. This implies the matrix $\psi<0$. It follows from Lemma 2.1 that the zero solution of time varying delay-difference control system with time-varying delay of Hopfield neural networks is asymptotically stable.

Therefore, the system is asymptotically stable.

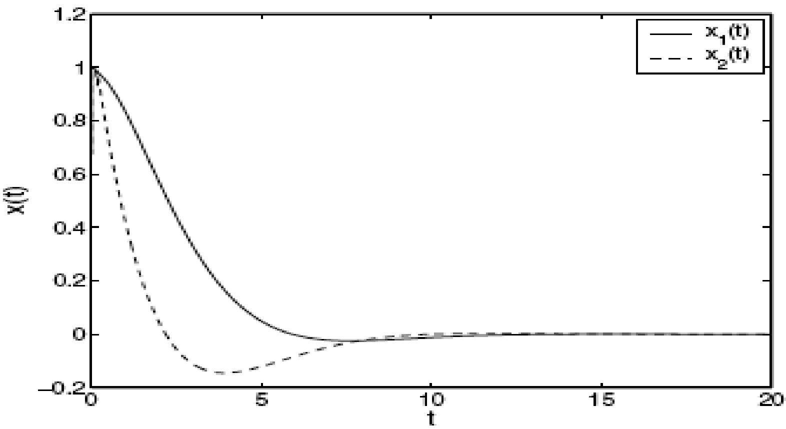

Fig. 3.1. Numerical simulation of a solution for the **example 3.1.**

For a given initial condition $x(\theta)=[1,1]^T$, convergence behavior of is shown in Fig. 3.1. As we can see from this figure, the steady state of delay-difference system is indeed asymptotically stable.

4. CONCLUSIONS

In this paper, based on a discrete analog of the Lyapunov second method, we have established a sufficient condition for the asymptotic stability of time varying delay-difference control system with time-varying delay of Hopfield neural networks in terms of certain matrix inequalities.

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