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SOME NEW WEIGHTED INFORMATION GENERATING FUNCTIONS OF DISCRETE PROBABILITY DISTRIBUTIONS

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Abstract

The aim of this paper is to introduce some new information generating functions with utilities whose derivatives give some well known measures of information. Some properties and particular cases of the proposed functions have also been studied.

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1. INTRODUCTION

It is well – known that the successive derivatives of the moment generating function at point 0 give the successive moments of a probability distribution if these moments exist. In a correspondence, S. Golomb [2] introduced the so-called information generating function of a probability distribution, given by

$$I(t) = \sum_{i\in N} p_i^t \, , \, t \geq 1 \tag{1}$$

Where $\{p_i\}$ is a complete probability distribution with $i \in$ N, N being a discrete sample space and t is a real or a complex variable. The first derivative of the above function, at point $t = 1$, gives Shannon's entropy (in fact the negative entropy) of the corresponding probability distribution i.e. we have

$$-\left(\frac{\partial}{\partial t} I(t)\right)_{t\,=\,1} = -\sum_{i\in N} p_i \ln \ p_i = H\left(P\right) \tag{2}$$

where $H\left(P\right)$ is the Shannon's entropy[8].

The quantity (2) measures the average information but does not take into account the effectiveness or importance of the events. Belis and Guiasu [1] raised the very important issue of integrating the quantitative, objective and probabilistic concept of information with the qualitative, subjective and non – stochastic concept of utility, and proposed the following weighted measure of information

$$-\sum_{i=1}^{n} u_i p_i \ln\ p_i = H(P,\ U) \tag{3}$$

The measure (3) is associated with the following utility information scheme

$$\begin{pmatrix} E_1 & E_2 & \ldots & E_n \\ p_1 & p_2 & \ldots & p_n \\ u_1 & u_2 & \ldots & u_n \end{pmatrix} \tag{4}$$
$$0 \le p_i \le 1,\ \sum_{i=1}^{n} p_i = 1,\quad u_i > 0$$

Here $(E_1, E_2,\ ...,\ E_n)$ denote a family of events with respect to some random experiment and u_i denotes the utility of an event E_i with probability p_i. In general, the utlity u_i of an event is independent of its probability of occurrence p_i.
Analogous to (1), Hooda and Bhaker [3] defined the following weighted information generating function

$$M(P,\ U,\ t) = \sum_{i=1}^{n} u_i p_i^t\ ,\quad t \ge 1 \tag{5}$$

Here also $(p_1, p_2,\ ...,\ p_n)$ and $(u_1, u_2,\ ...,\ u_n)$ are the probability and utility distributions respectively as defined in (4) and t is a real or a complex variable. Further, we have

$$-\left(\frac{\partial}{\partial t} M(P,\ U,\ t)\right)_{t\,=\,1} = -\sum_{i=1}^{n} u_i\ p_i \ln\ p_i = H(P, U) \tag{6}$$

where $H(P, U)$ is the weighted measure of information given by (3).
Longo [6] studied the measure (3) in detail and raised objections about its applicability in various coding procedures. He further suggested that the concept of utility should be introduced in a different way in any information scheme and that a utility measure should exhibit a relative character rather than only a non-negative character. Kapur [4, 5] further studied the measure (3) and asserted that (3) is not a measure of information since it depends on the units in which utility is measured and as such (3) can be expressed in dollar – bits or hour – bits and not in terms of bits only. Keeping in mind the above objections, Kapur [5] considered the following probability distribution

$$\frac{u_i\ p_i}{\sum_{i=1}^{n} u_i\ p_i}, i = 1,\ 2,\ ...,\ n \tag{7}$$

and used this distribution in obtaining the following information – theoretic measures

$$\tilde{H}(P, U) = -\sum_{i=1}^{n} \frac{u_i\ p_i}{\sum_{i=1}^{n} u_i\ p_i} \ln\ \frac{u_i\ p_i}{\sum_{i=1}^{n} u_i\ p_i} \tag{8}$$

$$\tilde{H}_{\alpha}(P,U) = \frac{1}{1-\alpha}\ln\sum_{i=1}^{n}\left(\frac{u_i\,p_i}{\sum_{i=1}^{n}u_i\,p_i}\right)^{\alpha},\ \alpha > 0 \tag{9}$$

Here again $(p_1, p_2, \ldots, p_n)$ and $(u_1, u_2, \ldots, u_n)$ are the probability and utility distributions respectively as defined in (4).

In the present work, we have defined new weighted information generating functions. It is interesting to note that differentiation of these information generating functions at point 1 yields the measures given by (8) and (9). Some particular and limiting cases of these functions have also been studied. Further we have also defined a new generating function for Shannon entropy given by (2).

The whole paper is organized as follows. In section 2, we have defined new weighted information generating functions. In section 3, we have discussed the limiting cases of the generating function defined in section 2. Section 4 concludes this paper.

2. NEW INFORMATION GENERATING FUNCTIONS

Consider the following functions

$$I(P,\,U,\,t) = \sum_{i=1}^{n}\left(\frac{u_i\,p_i}{\sum_{i=1}^{n}u_i\,p_i}\right)^{t},\ t \geq 1 \tag{10}$$

$$\tilde{I}_{\alpha}(P,\,U,\,t) = \left(\sum_{i=1}^{n}\left(\frac{u_i\,p_i}{\sum_{i=1}^{n}u_i\,p_i}\right)^{\alpha}\right)^{\frac{t-1}{1-\alpha}} = \left(\frac{\sum_{i=1}^{n}u_i^{\alpha}\,p_i^{\alpha}}{\left(\sum_{i=1}^{n}u_i\,p_i\right)^{\alpha}}\right)^{\frac{t-1}{1-\alpha}} \quad t \geq 1,\ \alpha > 0,\ \alpha \neq 1 \tag{11}$$

Here $P = (p_1, p_2, \ldots, p_n)$ and $U = (u_1, u_2, \ldots, u_n)$ are the probability and utility distributions respectively as defined in (4) and t is a real or a complex variable. Also we have $I(P,\,U,\,1) = 1 = \tilde{I}_{\alpha}(P,\,U,\,1)$. Also it is very much clear that the functions given by (10) and (11) are convergent for $t \geq 1$. Similar definitions will hold if the information scheme defined by (4) contains a countable number of events.

It further follows from (10) and (11) that

$$-\left(\frac{\partial}{\partial t}I(P,\,U,\,t)\right)_{t=1} = -\sum_{i=1}^{n}\frac{u_i\,p_i}{\sum_{i=1}^{n}u_i\,p_i}\ln\frac{u_i\,p_i}{\sum_{i=1}^{n}u_i\,p_i} = \tilde{H}(P,U) \tag{12}$$

and

$$\left(\frac{\partial}{\partial t}\tilde{I}_{\alpha}(P,\,U,\,t)\right)_{t=1} = \frac{1}{1-\alpha}\ln\sum_{i=1}^{n}\left(\frac{u_i\,p_i}{\sum_{i=1}^{n}u_i\,p_i}\right)^{\alpha} = \tilde{H}_{\alpha}(P,U) \tag{13}$$

where $\tilde{H}(P,U)$ and $\tilde{H}_{\alpha}(P,U)$ are the information – theoretic measures given by (8) and (9) respectively. Therefore (10) and (11) serve as the information generating functions for the information – theoretic measures given by (8) and (9) respectively. Also we have

$$\left(\frac{\partial^r}{\partial t^r} I\,(P,\ U,\ t)\right)_{t\,=\,1} = \sum_{i=1}^{n} \frac{u_i p_i}{\sum_{i=1}^{n} u_i\, p_i} \left(\ln\ \frac{u_i\, p_i}{\sum_{i=1}^{n} u_i\, p_i}\right)^r$$

provided that this sum converges. Except for a factor of$(-\,1)^{\,r}$, this the r^{th} moment of the generalized self information of the probability distribution given by (7).
If we consider $u_1 = u_2 = \ ... \ = u_n = u(\text{say})$, then (10) reduces to (1), the information – generating function introduced by Golomb [2] and (11) reduces to

$$\left(\sum_{i=1}^{n} p_i^{\alpha}\right)^{\frac{t\,-\,1}{1\,-\,\alpha}} = \tilde{I}_\alpha\,(P,\ t)\ \ (\text{say}) \tag{14}$$

which can be considered as the information generating function for Renyi's entropy of order α, since

$$\left(\frac{\partial}{\partial t} \tilde{I}_\alpha\,(P,\ t)\right)_{t\,=\,1} = \frac{1}{1-\alpha} \ln \left(\sum_{i=1}^{n} p_i^{\alpha}\right) = \tilde{H}_\alpha\,(P)\ \ (\text{say}) \tag{15}$$

where $\tilde{H}_\alpha\,(P)$ is the well - known Renyi's entropy of order α proposed by Renyi [7].

Particular Cases

(a) Uniform Distribution

If we consider

$$u_1 = u_2 = \ ... \ = u_n = u(\text{say})\ \ \text{and}\ \ p_i = \frac{1}{n},\ i = 1,\ 2,\ ...,\ n$$

then the information generating functions given by (10) and (11) reduces to

$$I\,(P,\ U,\ t) = n^{1\,-\,t} \qquad \text{and} \qquad \tilde{I}_\alpha\,(P,\ U,\ t) = n^{t\,-\,1}$$

Therefore we have

$$-\left(\frac{\partial}{\partial t} I\,(P,\ U,\ t)\right)_{t\,=\,1} = \ln n = \left(\frac{\partial}{\partial t} \tilde{I}_\alpha\,(P,\ U,\ t)\right)_{t\,=\,1}$$

which is expected since $\ln\, n$ is the Shannon entropy for uniform distribution.

(b) Geometric Distribution

If we consider

$$u_1 = u_2 = \ ... \ = u_n = u(\text{say}) \qquad \text{and}$$

$$p_i = q\, p^{\,i},\ p + q = 1\ ,\ i = 0,1,\ 2,\ ...,\ \infty$$

then the information generating functions given by (10) and (11) reduces to

$$I\,(P,\ U,\ t) = \frac{q^{\,t}}{1\ -\ p^t} \qquad \text{and}$$

$$\tilde{I}_\alpha(P,\ U,\ t) = \left(\frac{q^{\alpha}}{1 - p^{\alpha}}\right)^{\frac{t-1}{1-\alpha}}$$

Further we have

$$-\left(\frac{\partial}{\partial t} I(P,\ U,\ t)\right)_{t=1} = -\left(\frac{p \ln p + q \ln q}{q}\right)$$

which is exactly the Shannon entropy for this distribution and

$$\left(\frac{\partial}{\partial t}\tilde{I}_\alpha(P,\ U,\ t)\right)_{t=1} = \frac{1}{1-\alpha} \ln \left(\frac{q^{\alpha}}{1 - p^{\alpha}}\right),\ \alpha \neq 1$$

which is exactly the Renyi's entropy for this distribution.
Also in this case, we have

$$\lim_{\alpha \to 1} \frac{1}{1-\alpha} \ln \left(\frac{q^{\alpha}}{1 - p^{\alpha}}\right) = -\left(\frac{p \ln p + q \ln q}{q}\right)$$

$$i.\ e. \qquad \lim_{\alpha \to 1} \left(\frac{\partial}{\partial t}\tilde{I}_\alpha(P,\ U,\ t)\right)_{t=1} = -\left(\frac{\partial}{\partial t} I(P,\ U,\ t)\right)_{t=1} \tag{16}$$

which is a well – known result in information theory.
(c) β - Power Distribution
If we consider

$$u_1 = u_2 = \ldots = u_n = u \text{ (say)} \qquad \text{and}$$

$$p_i = \frac{i^{-\beta}}{\zeta(\beta)},\ \zeta(\beta) = \sum_{i=1}^{\infty} i^{-\beta},\ i = 1,\ 2,\ \ldots$$

then the information generating functions given by (10) and (11) reduces to

$$I(P,\ U,\ t) = \frac{\zeta(\beta t)}{\zeta(\beta)} \qquad \text{and}$$

$$\tilde{I}_\alpha(P,\ U,\ t) = \left(\frac{\zeta(\beta\alpha)}{(\zeta(\beta))^{\alpha}}\right)^{\frac{t-1}{1-\alpha}}$$

Further we have

$$-\left(\frac{\partial}{\partial t} I(P,\ U,\ t)\right)_{t=1} = \ln \zeta(\beta) - \frac{\beta\, \zeta'(\beta)}{\zeta(\beta)}$$

which is exactly the Shannon entropy for this distribution and

$$\left(\frac{\partial}{\partial t}\tilde{I}_\alpha(P,\ U,\ t)\right)_{t=1} = \frac{1}{1-\alpha} \ln \left(\frac{\zeta(\beta\alpha)}{(\zeta(\beta))^{\alpha}}\right)$$

which is exactly the Renyi's entropy for this distribution. The relationship given by (16) is again satisfied in this case.The above results are similar to those obtained by Golomb [2].

(d) Geometric Probability and Utility Distributions (Non constant utility distribution)

If we consider

$$u_i = v\,u^i,\ i = 0, 1, 2, ..., \infty \quad \text{and} \qquad p_i = q\,p^i,\ p + q = 1,\ i = 0, 1, 2, ..., \infty$$

then the information generating functions given by (10) and (11) reduces to

$$I\,(P,\ U,\ t) = \frac{(1 - pu)^t}{1 - p^t u^t} \qquad \text{and}$$

$$\tilde{I}_\alpha\,(P,\ U,\ t) = \left(\frac{(1 - pu)^\alpha}{1 - p^\alpha u^\alpha}\right)^{\frac{t-1}{1-\alpha}}$$

Further we have

$$-\left(\frac{\partial}{\partial t} I\,(P,\ U,\ t)\right)_{t=1} = -\ln\,(1 - pu) - \frac{pu}{1 - pu}\ln\,(pu)$$

which is exactly the Shannon entropy for this distribution and

$$\left(\frac{\partial}{\partial t}\tilde{I}_\alpha\,(P,\ U,\ t)\right)_{t=1} = \frac{1}{1-\alpha}\ln\left(\frac{(1 - pu)^\alpha}{1 - p^\alpha u^\alpha}\right),\ \alpha \neq 1$$

which is exactly the Renyi's entropy for this distribution. The relationship given by (16) is again satisfied in this case.

(e) Geometric Utility Distribution and Uniform Probability Distribution (Non constant utility distribution)

If we consider

$$u_i = v\,u^i,\ i = 0, 1, 2, ..., \infty \quad \text{and}$$

$$p_i = 1/n,\ \ i = 0, 1, 2, ..., n$$

then the information generating functions given by (10) and (11) reduces to

$$I\,(P,\ U,\ t) = \frac{v^t}{1 - u^t} \qquad \text{and}$$

$$\tilde{I}_\alpha\,(P,\ U,\ t) = \left(\frac{v^\alpha}{1 - u^\alpha}\right)^{\frac{t-1}{1-\alpha}}$$

Further we have

$$-\left(\frac{\partial}{\partial t} I\,(P,\ U,\ t)\right)_{t=1} = -\left(\frac{u\ln u + v\ln v}{v}\right)$$

which is exactly the Shannon entropy for this distribution and

$$\left(\frac{\partial}{\partial t}\tilde{I}_\alpha\,(P,\ U,\ t)\right)_{t\,=\,1} = \frac{1}{1-\alpha}\ \ln\ \left(\frac{v^{\alpha}}{1\ -\ u^{\alpha}}\right),\ \alpha \neq 1$$

which is exactly the Renyi's entropy for this distribution. The relationship given by (16) is again satisfied in this case.

3. NEW GENERATING FUNCTION FOR SHANNON ENTROPY

Consider the function $\tilde{I}_\alpha\,(P,\ U,\ t)$ given by (11). For this function, we have

$$\lim_{\alpha\to 1}\ \tilde{I}_\alpha\,(P,\ U,\ t)\ =$$

$$\exp\left\{(t\ -\ 1)\left(-\frac{\sum_{i=1}^n\ u_i\ p_i \ln\ p_i}{\sum_{i=1}^n\ u_i\ p_i} - \frac{\sum_{i=1}^n\ u_i\ p_i \ln\ u_i}{\sum_{i=1}^n\ u_i\ p_i} + \ln\left(\sum_{i=1}^n\ u_i\ p_i\right)\right)\right\}$$

which on simplification yields

$$= \ \exp\left\{(t\ -\ 1)\left(-\sum_{i=1}^n\ \frac{u_i\ p_i}{\sum_{i=1}^n\ u_i\ p_i}\ln\ \frac{u_i\ p_i}{\sum_{i=1}^n\ u_i\ p_i}\right)\right\}$$

$$= \ \tilde{I}\,(P,\ U,\ t)\ \ \{\text{say}\} \tag{17}$$

The function given by (17) can be considered as a information generating function for the measure of information given by (8) since

$$\left(\frac{\partial}{\partial t}\tilde{I}\,(P,\ U,\ t)\right)_{t\,=\,1} = \ -\ \sum_{i=1}^n\ \frac{u_i\ p_i}{\sum_{i=1}^n\ u_i\ p_i}\ln\ \frac{u_i\ p_i}{\sum_{i=1}^n\ u_i\ p_i} = \tilde{H}\,(P,U)$$

Further in (17), if we consider

$$u_1\ =\ u_2 =\ \ ...\ =\ \ u_n\ =\ u(\text{say}),$$

then we obtain

$$\lim_{\alpha\to 1}\ \tilde{I}_\alpha\,(P,\ U,\ t) = \ \exp\left\{(t\ -\ 1)\left(-\sum_{i=1}^n\ p_i \ln\ p_i\right)\right\} = \ \tilde{I}\,(t)\,(\text{say}) \tag{18}$$

The function $\tilde{I}\,(t)$ can be considered as a generating function for Shannon entropy since

$$\left(\frac{\partial}{\partial t}\tilde{I}\,(t)\right)_{t\,=\,1} = \ -\sum_{i\,=\,1}^n\ p_i \ln\ p_i = H\,(P)$$

For uniform distribution, we have

$$\tilde{I}(t) = n^{1-t} \qquad \text{and} \qquad \left(\frac{\partial}{\partial t}\tilde{I}(t)\right)_{t=1} = \ln n$$

Further for geometric and β - power distributions, we have

$$\tilde{I}(t) = \frac{q^t}{1 - p^t} \qquad \text{and} \qquad \left(\frac{\partial}{\partial t}\tilde{I}(t)\right)_{t=1} = -\left(\frac{p\ln p + q\ln q}{q}\right)$$

$$\tilde{I}(t) = \frac{\zeta(\beta t)}{\zeta(\beta)} \qquad \text{and} \qquad \left(\frac{\partial}{\partial t}\tilde{I}(t)\right)_{t=1} = \ln\zeta(\beta) - \frac{\beta\,\zeta'(\beta)}{\zeta(\beta)}$$

Again, the above results are similar to those obtained by Golomb [2].

4. CONCLUDING REMARKS

In this paper, we have defined new information generating functions with utilities whose derivatives yields some well known measures of information. Work on further generalizations of these functions is in progress and will be reported elsewhere.

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