Journal of the Applied Mathematics, Statistics and Informatics (JAMSI), **5** (2009), No. 2

SOME PROPERTIES OF CERTAIN MULTIVALENT ANALYTIC FUNCTIONS USING A DIFFERENTIAL OPERATOR

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Abstract

Here we introduce a new subclass of multivalent analytic functions of complex order by means of a differential operator. Certain interesting properties such as inclusion relationship and radius problems are investigated for this function class.

Mathematics Subject Classification 2000: 30C45; 30C50.
Key Words and Phrases: Analytic functions, Convex set, Multivalent functions, Starlike functions.

1. INTRODUCTION

Let $\mathrm{A}_n(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n, p \in \mathbf{N} = \{1,2,3,...\}), \tag{1}$$

which are analytic and multivalent in the open unit disc $\mathrm{U} := \{z \in \mathrm{C} : |z| < 1\}$. Suppose that $P(n,\beta)$ be the class of functions $h(z)$ of the form

$$h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + ..., \tag{2}$$

which are analytic in U and satisfy $\mathrm{Re}\{h(z)\} > \beta, 0 \le \beta < 1, z \in \mathrm{U}$. We note that $P(1,0) \equiv P$ is the class of functions with positive real part, $P(1,\beta) \equiv P(\beta)$ and $P(n,0) \equiv P(n)$.

Let $P_k(n,\beta), k \ge 2, 0 \le \beta < 1$, be the class of functions p, analytic in U, such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \tag{3}$$

if and only if $p_1, p_2 \in P(n,\beta)$ for $z \in \mathrm{U}$. It is easy to verify that $P_2(1,\beta) \equiv P(\beta)$ and $P(1,0) \equiv P$. The class $P_k(n,\beta)$ was introduced and studied recently by Noor [4]. Again $P_k(1,0) \equiv P_k$, the class introduced by Pinchuk [5], where he has generalized the concept of functions of bounded boundary rotation. It is worth mentioning that, for $\mathrm{k} > 2$, functions in P_k need not be with the positive real part. It is easy to see that $p \in P_k(n,\beta)$ if and only if there exists $h \in P_k(n,0)$ such that

$$p(z) = (1-\beta)\,h(z) + \beta\,. \qquad (4)$$

Let $f^{(q)}$ denotes the q^{th}-order ordinary differential operator for the function $f \in \mathrm{A}_n(p)$ such that

$$f^{(q)}(z) = \eta(p;q)\,z^{p-q} + \sum_{k=n+p}^{\infty} \eta(k;q)\,a_k\,z^{k-q}\,, \qquad (5)$$

where

$$\eta(j;q) = \frac{j!}{(j-q)!} \quad (j > q,\, j \in N; q \in N_0 = N \cup \{0\}, z \in \mathrm{U})\,. \qquad (6)$$

Following Frasin [2], we define the differential operator $D^m f^{(q)}(z)$ by

$$D^m f^{(q)}(z) = (p-q)^m\,\eta(p;q)\,z^{p-q} + \sum_{k=n+p}^{\infty} (k-q)^m \eta(k;q)\,a_k z^{k-q} \quad (m \in N_0; z \in \mathrm{U}). \qquad (7)$$

Obviously $D^0 f^{(0)}(z) = f(z),\ D^0 f^{(1)}(z) = f'(z)$ and $D^1 f^{(0)}(z) = zf'(z)$.

Also $\quad D^{m+1} f^{(q)}(z) = z(D^m f^{(q)}(z))'.$ (8)

Now, for $g \in \mathrm{A}_n(p)$ defined by

$$g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k, \qquad (9)$$

we introduce a subclass of multivalent analytic functions in the open unit disc U as follows:

DEFINITION 1.1. Let $\mu > 0, k \geq 2, 0 \leq \beta < 1, p > q,\ n, p \in N; m, q \in N_0$, $z \in \mathrm{U}$ and $f \in \mathrm{A}_n(p)$. Then $f \in \Sigma_{n,p}^m(k,\lambda,q,\beta,\delta,\mu)$, if it satisfies

$$(1-\lambda)\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu}+\lambda\frac{D^{m+1}f^{(q)}(z)}{D^{m+1}g^{(q)}(z)}\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu-1}\in P_k(n,\beta), \qquad (10)$$

where $g\in A_n(p)$ satisfies the condition

$$\frac{D^{m+1}g^{(q)}(z)}{(p-q)D^m g^{(q)}(z)}\in P(n,\delta) \qquad (0\le\delta<1, z\in U). \qquad (11)$$

We note that if $q=0,\ p=1, m=0,$ then g is starlike univalent in U.

2. PRELIMINARY RESULTS

To establish our main results, we shall require the following known results:

LEMMA 2.1 ([3]). Let $r=r_1+ir_2$ and $s=s_1+is_2$ and let $\psi(r,s)$ be a complex-valued function satisfying the conditions:

(i) $\psi(r,s)$ is continuous in a domain $D\subset C^2$,

(ii) $(1,0)\in D$ and $\psi(1,0)>0,$

(iii) $\operatorname{Re}\psi(ir_2,s_1)\le 0$ whenever $(ir_2,s_1)\in D$ *and* $s_1\le-\frac{1}{2}(1+r_2^2)$.

If $h(z)$ is analytic in U with $h(0)=1$ such that $(h(z),zh'(z))\in D$ and $\operatorname{Re}(h(z),zh'(z))>0$ for $z\in U$, then $\operatorname{Re}h(z)>0$.

LEMMA 2.2. Let $h\in P(n)$. Then, for $z\in U$,

(i) $\frac{1-r^n}{1+r^n}\le\operatorname{Re}h(z)\le|h(z)|\le\frac{1+r^n}{1-r^n}$.

(ii) $|zh'(z)|\le\frac{2nr^n\operatorname{Re}h(z)}{1-r^{2n}}$.

For part (i), see [7] and for part (ii), see [1].

LEMMA 2.3 ([6]). If $h(z)$ is analytic in U with $h(0)=1$ and λ is a complex number satisfying $\operatorname{Re}\lambda\ge 0,\ (\lambda\ne 0)$, then $\operatorname{Re}\{h(z)+\lambda zh'(z)\}>\alpha\quad(0\le\alpha<1)$ implies

$$\operatorname{Re}h(z)>\alpha+(1-\alpha)(2\gamma_1-1)$$

where γ_1 is given by

$$\gamma_1 = \int_0^1 (1+t^{Re\,\lambda})^{-1}\, dt,$$

which is an increasing function of $\operatorname{Re}\lambda$ and $\frac{1}{2} \le \gamma_1 < 1.$

This estimate is sharp in the sense that the bound cannot be improved.

3. MAIN RESULTS

THEOREM 3.1. The class $P_k(n,\beta)$ is a convex set.

PROOF. Let $H_1, H_2 \in P_k(n,\beta).$ We shall show that for $\alpha_1, \alpha_2 > 0$

$$H(z) = \frac{1}{\alpha_1+\alpha_2}\left[\alpha_1 H_1(z) + \alpha_2 H_2(z)\right] \in P_k(n,\beta).$$

By definition of $P_k(n,\beta)$, we can write

$$H(z) = \frac{1}{\alpha_1+\alpha_2}\left\{\alpha_1\left[\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z)\right]\right.$$

$$\left. +\alpha_2\left[\left(\frac{k}{4}+\frac{1}{2}\right)p_3(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_4(z)\right]\right\},$$

where $p_j(z) \in P(n,\beta) \quad (j = 1,2,3,4).$

Now, writing $p_j(z) = (1-\beta)\, h_j(z) + \beta, \quad h_j \in P(n), \quad j = 1,2,3,4.$

We have

$$\frac{H(z)-\beta}{1-\beta} = \left(\frac{k}{4}+\frac{1}{2}\right)\left\{\frac{1}{\alpha_1+\alpha_2}\left[\alpha_1 h_1(z) + \alpha_2 h_3(z)\right]\right\}$$

$$-\left(\frac{k}{4}-\frac{1}{2}\right)\left\{\frac{1}{\alpha_1+\alpha_2}\left[\alpha_1 h_2(z) + \alpha_2 h_4(z)\right]\right\}$$

$$= \left(\frac{k}{4}+\frac{1}{2}\right) r_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) r_2(z) = r(z),$$

where $r_1(z), r_2(z) \in P(n),$ since $P(n)$ is a convex set. Thus $r(z) \in P_k(n,0).$

Now $H(z) = (1-\beta)\, r(z) + \beta,$ and $r(z) \in P_k(n,0).$

Therefore by definition $H(z) \in P_k(n,\beta)$, and proof of the THEOREM 3.1 is complete.

THEOREM 3.2. Let $f \in \Sigma_{n,p}^m(k,\lambda,q,\beta,\delta,\mu)$, $\lambda \geq 0$. Then $\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^\mu$ $\in P_k(n,\gamma)$, where

$$\gamma = \frac{2\mu(p-q)\beta + \lambda\xi}{2\mu(p-q) + \lambda\xi}, \tag{12}$$

and $g \in \mathrm{A}_n(p)$ satisfies the condition (11) and

$$\xi = \frac{\operatorname{Re} q_0(z)}{|q_0(z)|^2}, \qquad q_0(z) = \frac{D^{m+1} g^{(q)}(z)}{(p-q) D^m g^{(q)}(z)}$$

PROOF. Set

$$h(z) = \frac{1}{1-\gamma}\left\{\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^\mu - \gamma\right\}, \tag{13}$$

$h(0) = 1$ and $h(z)$ is analytic in U and we write

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z). \tag{14}$$

By simple calculations, we get

$$(1-\lambda)\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^\mu + \lambda \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)}\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu-1} - \beta$$

$$= \left(\frac{k}{4} + \frac{1}{2}\right)\left\{(1-\gamma) h_1(z) + \gamma - \beta + \frac{\lambda(1-\gamma)}{\mu(p-q)} \frac{z h_1'(z)}{q_0(z)}\right\}$$

$$- \left(\frac{k}{4} - \frac{1}{2}\right)\left\{(1-\gamma) h_2(z) + \gamma - \beta + \frac{\lambda(1-\gamma)}{\mu(p-q)} \frac{z h_2'(z)}{q_0(z)}\right\}.$$

Now we form the functional $\psi(r,s)$ by choosing $r = h_j(z) = r_1 + i r_2$ and $s = z h_j'(z)$ $= s_1 + i s_2$. Thus

$$\psi(r,s)=(1-\gamma)r+\gamma-\beta+\frac{\lambda(1-\gamma)s}{\mu(p-q)\,q_0(z)}. \tag{15}$$

The first two conditions of LEMMA 2.1 are clearly satisfied. We verify the condition (iii) as follows.

$$\operatorname{Re}\psi(ir_2,s_1)=\gamma-\beta+\frac{\lambda(1-\gamma)\,s_1}{\mu(p-q)}\frac{\operatorname{Re}q_0(z)}{|\,q_0(z)|^2}$$

$$=\gamma-\beta+\frac{\lambda(1-\gamma)\xi}{\mu(p-q)}s_1,$$

where

$$\xi=\frac{\operatorname{Re}q_0(z)}{|\,q_0(z)|^2}\;. \tag{16}$$

Now, for $s_1\le-\frac{1}{2}(1+r_2^{\,2})$, we have

$$\operatorname{Re}\psi(ir_2,s_1)\le(\gamma-\beta)-\frac{\lambda(1-\gamma)(1+r_2^{\,2})\xi}{2\mu(p-q)}$$

$$=\frac{2\mu\,(p-q)(\gamma-\beta)-\lambda\,\xi\,(1-\gamma)-\lambda\,(1-\gamma)\,\xi\,r_2^{\,2}}{2\mu\,(p-q)}$$

$$=\frac{A+Br_2^{\,2}}{2C},\quad C>0,$$

where $A=2\mu\,(p-q)(\gamma-\beta)-\lambda\,\xi\,(1-\gamma)$,

$$B=-\lambda\,(1-\gamma)\,\xi\,r_2^{\,2}\le 0.$$

Now $\operatorname{Re}\psi(ir_2,s_1)\le 0$ if $A\le 0,$ which gives γ as defined by (12). Now, we apply Lemma 2.1 to conclude that $h_j\in P(n,0), z\in \mathrm{U}$ and thus $h\in P_k(n,0)$ and hence by definition $p\in P_k(n,\beta),$ which gives the required result.

We note that for $\delta=0, \gamma=\beta.$ For $\mu=1$ we thus get

THEOREM. 3.3. For $\lambda\ge 1,$ let $f\in\Sigma_{n,p}^{m}(k,\lambda,q,\beta,0,1).$ Then $\dfrac{D^{m+1}f^{(q)}(z)}{D^{m+1}g^{(q)}(z)}$ $\in P_k(n,\beta).$

PROOF. We can write $\lambda \geq 1,$

$$\lambda \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} = \left[(1-\lambda) \frac{D^{m} f^{(q)}(z)}{D^{m} g^{(q)}(z)} + \lambda \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} \right] + (\lambda - 1) \frac{D^{m} f^{(q)}(z)}{D^{m} g^{(q)}(z)}. \tag{17}$$

This implies that

$$\frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} = \frac{1}{\lambda} \left[(1-\lambda) \frac{D^{m} f^{(q)}(z)}{D^{m} g^{(q)}(z)} + \lambda \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} \right] + \left(1 - \frac{1}{\lambda} \right) \frac{D^{m} f^{(q)}(z)}{D^{m} g^{(q)}(z)}$$
$$= \frac{1}{\lambda} H_1 + (1 - \frac{1}{\lambda}) H_2.$$

Since $H_1, H_2 \in P_k(n, \beta)$, by THEOREM 3.2, DEFINITION 1.1 and $P_k(n, \beta)$ is a convex set (by THEOREM 3.1), we get the required result.

Taking $b_k = 0 \ \forall \ k \geq n+p, \ n, p \in N,$ we get the result contained in the following theorem:

THEOREM 3.4. Let $\mu > 0,$ and $\lambda \in C$ such that $\mathrm{Re}\lambda > 0$. Let $f \in \mathrm{A}_n(p)$ and satisfy the condition

$$(1-\lambda) \left(\frac{D^{m} f^{(q)}(z)}{(p-q)^{m} \eta(p;q) z^{p-q}} \right)^{\mu} + \lambda \frac{D^{m+1} f^{(q)}(z)}{(p-q)^{m+1} \eta(p;q) z^{p-q}} \left(\frac{D^{m} f^{(q)}(z)}{(p-q)^{m} \eta(p;q) z^{p-q}} \right)^{\mu - 1} \in P_k(n, \beta).$$

Then
$$\left(\frac{D^{m} f^{(q)}(z)}{(p-q)^{m} \eta(p;q) z^{p-q}} \right)^{\mu} \in P_k(n, \sigma) \quad \forall z \in \mathrm{U},$$

where $\sigma = \beta + (1-\beta)(2\rho - 1)$

and $$\rho = \frac{1}{2} \, {}_2F_1\left(1, 1; 1 + \frac{\mu(p-q)}{\mathrm{Re}\{\lambda\}}; \frac{1}{2} \right).$$

The value of σ is best possible and cannot be improved.

PROOF. We set

$$\left(\frac{D^m f^{(q)}(z)}{(p-q)^m\,\eta(p;q)\,z^{p-q}}\right)^{\mu}=h(z)=\left(\frac{k}{4}+\frac{1}{2}\right)h_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)h_2(z),$$

where $h(0)=1$ and h is analytic in U. Then

$$(1-\lambda)\left(\frac{D^m f^{(q)}(z)}{(p-q)^m\,\eta(p;q)\,z^{p-q}}\right)^{\mu}+\lambda\frac{D^{m+1} f^{(q)}(z)}{(p-q)^{m+1}\,\eta(p;q)\,z^{p-q}}\left(\frac{D^m f^{(q)}(z)}{(p-q)^m\,\eta(p;q)\,z^{p-q}}\right)^{\mu-1}$$

$$=h(z)+\frac{\lambda z h'(z)}{\mu(p-q)}\in P_k(n,\beta),\qquad z\in \mathrm{U}.$$

Using Lemma 2.3, we note that $h_j\in P(n,\sigma)$,

where $\sigma=\beta+(1-\beta)(2\rho-1)$, (18)

and $\rho=\int_0^1\left(1+t^{\frac{Re\{\lambda\}}{\mu(p-q)}}\right)^{-1}dt$. (19)

Putting $\operatorname{Re}\{\lambda\}=\lambda_1>0$, we have

$$\rho=\int_0^1\left(1+t^{\frac{Re\{\lambda\}}{\mu(p-q)}}\right)^{-1}dt=\frac{\mu(p-q)}{\lambda_1}\int_0^1 u^{\frac{\mu(p-q)}{\lambda_1}-1}(1+u)^{-1}du$$

$$={}_2F_1\left(1,\frac{\mu(p-q)}{\lambda_1};1+\frac{\mu(p-q)}{\lambda_1};-1\right)$$

$$=\frac{1}{2}\left[{}_2F_1\left(1,1;1+\frac{\mu(p-q)}{\lambda_1};\frac{1}{2}\right)\right]. \tag{20}$$

On substituting $\mu=1$ and $\lambda\in R$ s.t. $\lambda\geq 1$ in THEOREM 3.4, we get the following result:

COROLLARY 3.5. If $f\in \mathrm{A}_n(p)$ satisfies

$$(1-\lambda)\frac{D^m f^{(q)}(z)}{(p-q)^m\,\eta(p;q)\,z^{p-q}}+\lambda\frac{D^{m+1} f^{(q)}(z)}{(p-q)^{m+1}\,\eta(p;q)\,z^{p-q}}\in P_k(n,\beta).$$

Then $\dfrac{D^{m+1} f^{(q)}(z)}{(p-q)^{m+1}\,\eta(p;q)\,z^{p-q}}\in P_k(n,\sigma^*)\ \ \forall z\in \mathrm{U},$

where $\sigma^*=\beta+(1-\beta)(2\rho^*-1)(1-\lambda^{-1})$

and $\rho^*=\dfrac{1}{2}\,{}_2F_1\left(1,1;1+\dfrac{p-q}{\lambda};\dfrac{1}{2}\right).$

The value of σ^* is best possible and cannot be improved.

Taking $q=0$ in THEOREM 3.4, we get the following result:

COROLLARY 3.6. Let $\mu>0,$ and $\lambda\in C$ such that $\operatorname{Re}\lambda>0$. Let $f\in \mathrm{A}_n(p)$ and satisfy the condition

$$(1-\lambda)\left(\frac{D^m f(z)}{p^m z^p}\right)^{\mu}+\lambda\frac{D^{m+1} f(z)}{p^{m+1} z^p}\left(\frac{D^m f(z)}{p^m z^p}\right)^{\mu-1}\in P_k(n,\beta).$$

Then $\left(\dfrac{D^m f(z)}{p^m z^p}\right)^{\mu}\in P_k(n,\sigma_1)\ \ \forall z\in \mathrm{U},$

where $\sigma_1=\beta+(1-\beta)(2\rho_1-1)$

and $\rho_1=\dfrac{1}{2}\,{}_2F_1\left(1,1;1+\dfrac{\mu p}{\operatorname{Re}\{\lambda\}};\dfrac{1}{2}\right).$

The value of σ_1 is best possible and cannot be improved.

Further taking $m=0$ in COROLLARY 3.6, we get the following result:

COROLLARY 3.7. Let $\lambda\in C$ such that $\operatorname{Re}\lambda>0$. Let $f\in \mathrm{A}_n(p)$ and satisfy the condition

$$(1-\lambda)\left(\frac{f(z)}{z^p}\right)^{\mu}+\lambda\frac{z f'(z)}{p\, f(z)}\left(\frac{f(z)}{z^p}\right)^{\mu}\in P_k(n,\beta).$$

Then $$\left(\frac{f(z)}{z^p}\right)^{\mu} \in P_k(n,\sigma_1) \quad \forall z \in \mathrm{U},$$

where σ_1 is defined with COROLLARY 3.6. Also the value of σ_1 is best possible and cannot be improved.

THEOREM 3.8. For $0 \le \lambda_2 < \lambda_1$,

$$\Sigma_{n,p}^{m}(k,\lambda_1,q,\beta,0,\mu) \subset \Sigma_{n,p}^{m}(k,\lambda_2,q,\beta,0,\mu).$$

PROOF. If $\lambda_2 = 0$, then the proof is immediate from THEOREM 3.2. So we let $\lambda_2 > 0$ and $f \in \Sigma_{n,p}^{m}(k,\lambda_1,q,\beta,0,\mu)$. Then there exist two functions $H_1, H_2 \in P_k(n,\beta)$ such that

$$(1-\lambda_1)\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu} + \lambda_1 \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)}\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu-1} = H_1(z),$$

and

$$\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu} = H_2(z).$$

Now

$$(1-\lambda_2)\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu} + \lambda_2 \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)}\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu-1}$$

$$= \frac{\lambda_2}{\lambda_1} H_1(z) + \left(1 - \frac{\lambda_2}{\lambda_1}\right) H_2(z), \tag{21}$$

and since $P_k(n,\beta)$ is a convex set, it follows that R.H.S. of (21) belongs to $P_k(n,\beta)$ and this completes the proof.

We now consider the converse case of THEOREM 3.2 as follows:

THEOREM 3.9. Let $\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu} \in P_k(n,\beta)$ with $\frac{D^{m+1} g^{(q)}(z)}{(p-q) D^m g^{(q)}(z)}$

$\in P(n,\delta)$ for $z\in$ U. Then $f\in \Sigma_{n,p}^{m}(k,\lambda,q,\beta,\delta,\mu)$ for $|z|< R(\mu,\delta,p,q,n,\lambda)$,

where

$$R(\mu,\delta,p,q,n,\lambda)$$

$$=\left[\frac{\mu(p-q)}{[(1-\delta)\mu(p-q)+\lambda n]+\sqrt{(\delta\mu(p-q))^2+\lambda^2 n^2+2\lambda n(1-\delta)(p-q)}}\right]^{\frac{1}{n}}. \quad (22)$$

PROOF. Let

$$H=\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu}, \text{ and } Q_0=\frac{D^{m+1}g^{(q)}(z)}{(p-q)D^m g^{(q)}(z)}.$$

Then $\quad H\in P_k(n,\beta),\quad Q_0\in P(n,\delta).$

Proceeding as in THEOREM 3.2, for $p>q,\ p\in N;\ q,m\in N_0,\ k\geq 2,\ \operatorname{Re}\lambda\geq 0$, $0\leq \beta,\delta<1$, and

$$H=(1-\beta)h+\beta,\quad Q_0=(1-\delta)q_0+\delta,\ \text{ with } h\in P_k(n,0) \text{ and } q_0\in P(n),$$

we have

$$\frac{1}{(1-\beta)}\left\{(1-\lambda)\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu}+\lambda\frac{D^{m+1}f^{(q)}(z)}{D^{m+1}g^{(q)}(z)}\left(\frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}\right)^{\mu-1}-\beta\right\}$$

$$=\left\{h(z)+\frac{\lambda}{\mu(p-q)}\frac{z\,h'(z)}{(1-\delta)q_0(z)+\delta}\right\}$$

$$=\left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z)+\frac{\lambda}{\mu(p-q)}\frac{z\,h_1'(z)}{(1-\delta)q_0(z)+\delta}\right]$$

$$-\left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z)+\frac{\lambda}{\mu(p-q)}\frac{z\,h_2'(z)}{(1-\delta)q_0(z)+\delta}\right].$$

Using LEMMA 2.2, for $h_j\in P(n)$,

$$\frac{1-r^n}{1+r^n}\leq \operatorname{Re}h_j(z)\leq |h_j(z)|\leq \frac{1+r^n}{1-r^n}$$

and

$$|zh_j'(z)| \le \frac{2nr^n \operatorname{Re} h_j(z)}{1-r^{2n}}.$$

We have

$$\operatorname{Re}\left[h_j(z) + \frac{\lambda}{\mu(p-q)} \frac{z h_j'(z)}{(1-\delta)q_0(z)+\delta}\right]$$

$$\ge \operatorname{Re} h_j(z)\left[1 - \frac{\lambda}{\mu(p-q)} \frac{2nr^n}{1-r^{2n}} \frac{1}{(1-\delta)q_0(z)+\delta}\right]$$

$$= \operatorname{Re} h_j(z)\left[1 - \frac{\lambda}{\mu(p-q)} \frac{2nr^n}{1-r^{2n}} \frac{1+r^n}{1-(1-2\delta)r^n}\right]$$

$$= \operatorname{Re} h_j(z)\left[1 - \frac{2\lambda n r^n}{\mu(p-q)(1-r^n)[1-(1-2\delta)r^n]}\right]$$

$$= \operatorname{Re} h_j(z)\left[\frac{\mu(p-q)[1-r^n-(1-2\delta)r^n+(1-2\delta)r^{2n}] - 2\lambda n r^n}{\mu(p-q)(1-r^n)[1-(1-2\delta)r^n]}\right]$$

$$\ge \operatorname{Re} h_j(z)\left[\frac{\mu(p-q)(1-2\delta)r^{2n} - 2[(1-\delta)\mu(p-q)+\lambda n]r^n + \mu(p-q)}{\mu(p-q)(1-r^n)[1-(1-2\delta)r^n]}\right]. \tag{23}$$

R.H.S. of (23) is positive for $r=|z|<R(\mu,\delta,p,q,n,\lambda)$, where $R(\mu,\delta,p,q,n,\lambda)$ is given by (22).

We note that, for $p=1,\ q=0,\ m=0,\ \mu=1,$ and $\lambda=1, \delta=0, \frac{f}{g} \in P_k(n,\beta)$, for $z \in \mathrm{U}$ implies $\frac{f'}{g'} \in P_k(n,\beta)$ for $|z| < R^* = \left[\frac{1}{(n+1)+\sqrt{n^2+2n}}\right]^{\frac{1}{n}}$.

4. ACKNOWLEDGMENT

The second author is thankful to CSIR, India, for providing Junior Research Fellowship under research scheme No. 09/149(0498)/2008-EMR-I.

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Received June 2009