SOME PROPERTIES OF CERTAIN MULTIVALENT ANALYTIC FUNCTIONS USING A DIFFERENTIAL OPERATOR

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Abstract

Here we introduce a new subclass of multivalent analytic functions of complex order by means of a differential operator. Certain interesting properties such as inclusion relationship and radius problems are investigated for this function class.

Mathematics Subject Classification 2000: 30C45; 30C50.
Key Words and Phrases: Analytic functions, Convex set, Multivalent functions, Starlike functions.

1. INTRODUCTION

Let \( A_n(p) \) denote the class of functions of the form

\[
 f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n, p \in \mathbb{N} = \{1, 2, 3, \ldots\}),
\]

which are analytic and multivalent in the open unit disc \( U := \{z \in \mathbb{C} : |z| < 1\} \). Suppose that \( P(n, \beta) \) be the class of functions \( h(z) \) of the form

\[
 h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots,
\]

which are analytic in \( U \) and satisfy \( \Re\{h(z)\} > \beta, 0 \leq \beta < 1, z \in U \). We note that \( P(1, 0) \equiv P \) is the class of functions with positive real part, \( P(1, \beta) \equiv P(\beta) \) and \( P(n, 0) \equiv P(n) \).

Let \( P_k(n, \beta), k \geq 2, 0 \leq \beta < 1, \) be the class of functions \( p, \) analytic in \( U \), such that

\[
p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z)
\]

if and only if \( p_1, p_2 \in P(n, \beta) \) for \( z \in U \). It is easy to verify that \( P_2(1, \beta) \equiv P(\beta) \) and \( P(1, 0) \equiv P \). The class \( P_k(n, \beta) \) was introduced and studied.
recently by Noor [4]. Again $P_k(1, 0) \equiv P$, the class introduced by Pinchuk [5], where he has generalized the concept of functions of bounded boundary rotation. It is worth mentioning that, for $k > 2$, functions in $P_k$ need not be with the positive real part. It is easy to see that $p \in P_k(n, \beta)$ if and only if there exists $h \in P_k(n, 0)$ such that

$$p(z) = (1 - \beta)h(z) + \beta.$$  

Let $f^{(q)}$ denotes the $q^{th}$-order ordinary differential operator for the function $f \in A_n(p)$ such that

$$f^{(q)}(z) = \eta(p; q) z^{p-q} + \sum_{k=p}^{\infty} \eta(k; q) a_k z^k,$$  

where

$$\eta(j; q) = \frac{j^1}{(j-q)!} \quad (j > q, j \in N; q \in N_0 = N \cup \{0\}, z \in U).$$  

Following Frasin [2], we define the differential operator $D^m f^{(q)}(z)$ by

$$D^m f^{(q)}(z) = (p-q)^m \eta(p; q) z^{p-q} + \sum_{k=p}^{\infty} (k-q)^m \eta(k; q) a_k z^{k-q} \quad (m \in N_0; z \in U).$$  

Obviously $D^0 f^{(0)}(z) = f(z)$, $D^0 f^{(1)}(z) = f'(z)$ and $D^1 f^{(0)}(z) = zf'(z)$. Also

$$D^{m+1} f^{(q)}(z) = z(D^m f^{(q)}(z))'.$$  

Now, for $g \in A_n(p)$ defined by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k,$$

we introduce a subclass of multivalent analytic functions in the open unit disc $U$ as follows:

**DEFINITION 1.1.** Let $\mu > 0, k \geq 2, 0 \leq \beta < 1, p > q, n, p \in N; m, q \in N_0,$ $z \in U$ and $f \in A_n(p)$. Then $f \in \Sigma_{n,k}^{m}(k, \lambda, q, \beta, \delta, \mu)$, if it satisfies
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\begin{equation}
(1 - \lambda) \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^{\mu} + \lambda \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^{\mu - 1} \in P_1(n, \beta),
\end{equation}

where \( g \in A_n(p) \) satisfies the condition

\begin{equation}
\frac{D^{m+1} g^{(q)}(z)}{(p - q)D^m g^{(q)}(z)} \in P(n, \delta) \quad (0 \leq \delta < 1, z \in \mathbb{U}).
\end{equation}

We note that if \( q = 0, p = 1, m = 0 \), then \( g \) is starlike univalent in \( \mathbb{U} \).

2. PRELIMINARY RESULTS

To establish our main results, we shall require the following known results:

**LEMMA 2.1 ([3]).** Let \( r = r_1 + i r_2 \) and \( s = s_1 + i s_2 \) and let \( \psi(r, s) \) be a complex-valued function satisfying the conditions:

(i) \( \psi(r, s) \) is continuous in a domain \( D \subset \mathbb{C}^2 \),

(ii) \((1, 0) \in D \) and \( \psi(1, 0) > 0 \),

(iii) \( \text{Re} \psi(i r_2, s_1) \leq 0 \) whenever \((i r_2, s_1) \in D \) and \( s_1 \leq -\frac{1}{2}(1 + r_2^2) \).

If \( h(z) \) is analytic in \( \mathbb{U} \) with \( h(0) = 1 \) such that \( (h(z), zh'(z)) \in D \) and \( \text{Re}(h(z), zh'(z)) > 0 \) for \( z \in \mathbb{U} \), then \( \text{Re} \ h(z) > 0 \).

**LEMMA 2.2.** Let \( h \in P(n) \). Then, for \( z \in \mathbb{U} \),

(i) \( \frac{1 - r^n}{1 + r^n} \leq \text{Re} \ h(z) \leq \frac{1 + r^n}{1 - r^n} \).

(ii) \( \text{Re} \ h(z) \leq \frac{2nr^n \text{Re} \ h(z)}{1 - r^{2n}} \).

For part (i), see [7] and for part (ii), see [1].

**LEMMA 2.3 ([6]).** If \( h(z) \) is analytic in \( \mathbb{U} \) with \( h(0) = 1 \) and \( \lambda \) is a complex number satisfying \( \text{Re} \lambda \geq 0, (\lambda \neq 0) \), then \( \text{Re}(h(z) + \lambda zh'(z)) > \alpha \) \( (0 \leq \alpha < 1) \) implies

\( \text{Re} \ h(z) > \alpha + (1 - \alpha)(2\gamma_1 - 1) \)

where \( \gamma_1 \) is given by
which is an increasing function of $\Re \lambda$ and $\frac{1}{2} \leq \gamma_1 < 1$.

This estimate is sharp in the sense that the bound cannot be improved.

3. MAIN RESULTS

**THEOREM 3.1.** The class $P_k(n, \beta)$ is a convex set.

**PROOF.** Let $H_1, H_2 \in P_k(n, \beta)$. We shall show that for $\alpha_1, \alpha_2 > 0$

$$H(z) = \frac{1}{\alpha_1 + \alpha_2} \left[ \alpha_1 H_1(z) + \alpha_2 H_2(z) \right] \in P_k(n, \beta).$$

By definition of $P_k(n, \beta)$, we can write

$$H(z) = \frac{1}{\alpha_1 + \alpha_2} \left\{ \alpha_1 \left[ \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z) \right] + \alpha_2 \left[ \left( \frac{k}{4} + \frac{1}{2} \right) p_3(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_4(z) \right] \right\},$$

where $p_j(z) \in P(n, \beta)$ ($j = 1, 2, 3, 4$).

Now, writing $p_j(z) = (1 - \beta) h_j(z) + \beta$, $h_j \in P(n)$, $j = 1, 2, 3, 4$.

We have

$$\frac{H(z) - \beta}{1 - \beta} = \left( \frac{k}{4} + \frac{1}{2} \right) \left[ \frac{1}{\alpha_1 + \alpha_2} \left[ \alpha_1 h_1(z) + \alpha_2 h_2(z) \right] \right] - \left( \frac{k}{4} - \frac{1}{2} \right) \left[ \frac{1}{\alpha_1 + \alpha_2} \left[ \alpha_1 h_2(z) + \alpha_2 h_1(z) \right] \right]$$

$$= \left( \frac{k}{4} + \frac{1}{2} \right) r_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) r_2(z) = r(z),$$

where $r_1(z), r_2(z) \in P(n)$, since $P(n)$ is a convex set. Thus $r(z) \in P_k(n, 0)$.

Now $H(z) = (1 - \beta) r(z) + \beta$, and $r(z) \in P_k(n, 0)$. 

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Therefore by definition \( H(z) \in P_k(n, \beta) \), and proof of the THEOREM 3.1 is complete.

THEOREM 3.2. Let \( f \in \Sigma_{n, \mu}(k, \lambda, q, \beta, \delta, \mu), \lambda \geq 0. \) Then
\[
\left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^\mu
\in P_k(n, \gamma), \text{ where}
\]
\[
\gamma = \frac{2\mu(p-q)\beta + \lambda \xi}{2\mu(p-q) + \lambda \xi},
\]
and \( g \in A_s(p) \) satisfies the condition (11) and
\[
\xi = \frac{\operatorname{Re} q_0(z)}{|q_0(z)|^2},
\quad q_0(z) = \frac{D^{n+1} g^{(q)}(z)}{(p-q)D^m g^{(q)}(z)}
\]

PROOF. Set
\[
h(z) = \frac{1}{1 - \gamma} \left( \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^\mu - \gamma \right),
\]
\( h(0) = 1 \) and \( h(z) \) is analytic in \( U \) and we write
\[
h(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z).
\]

By simple calculations, we get
\[
(1 - \Lambda) \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^\mu + \Lambda \left( \frac{D^{n+1} f^{(q)}(z)}{D^{n+1} g^{(q)}(z)} \right) \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^{\mu-1} - \beta
\]
\[
= \left( \frac{k}{4} + \frac{1}{2} \right) \left( 1 - \gamma \right) h_1(z) + \gamma - \beta + \frac{\lambda(1 - \gamma) z h_1'(z)(z)}{\mu(p-q) q_0(z)}
\]
\[
- \left( \frac{k}{4} - \frac{1}{2} \right) \left( 1 - \gamma \right) h_2(z) + \gamma - \beta + \frac{\lambda(1 - \gamma) z h_2'(z)}{\mu(p-q) q_0(z)}.
\]

Now we form the functional \( \psi(r, s) \) by choosing \( r = h_j(z) = r_1 + ir_2 \) and \( s = zh_j'(z) \)
\[
= s_1 + is_2 \). Thus
\[ \psi(r,s) = (1-\gamma) r + \gamma - \beta + \frac{\lambda(1-\gamma)s}{\mu(p-q) q_0(z)}, \quad (15) \]

The first two conditions of LEMMA 2.1 are clearly satisfied. We verify the condition (iii) as follows.

\[ \Re \psi(ir_2,s_1) = \gamma - \beta + \frac{\lambda(1-\gamma)s_1 \Re q_0(z)}{\mu(p-q) |q_0(z)|^2} \]

\[ = \gamma - \beta + \frac{\lambda(1-\gamma)\xi}{\mu(p-q)} s_1, \]

where

\[ \xi = \frac{\Re q_0(z)}{|q_0(z)|^2}. \quad (16) \]

Now, for \( s_1 \leq -\frac{1}{2}(1+r_2^2) \), we have

\[ \Re \psi(ir_2,s_1) \leq (\gamma - \beta) - \frac{\lambda(1-\gamma)(1+r_2^2)\xi}{2\mu(p-q)} \]

\[ = \frac{2\mu(p-q)(\gamma - \beta) - \lambda \xi(1-\gamma) - \lambda(1-\gamma)\xi r_2^2}{2\mu(p-q)} \]

\[ = \frac{A + Br_2^2}{2C}, \quad C > 0, \]

where \( A = 2\mu(p-q)(\gamma - \beta) - \lambda \xi(1-\gamma), \)

\( B = -\lambda(1-\gamma)\xi r_2^2 \leq 0. \)

Now \( \Re \psi(ir_2,s_1) \leq 0 \) if \( A \leq 0 \), which gives \( \gamma \) as defined by (12). Now, we apply Lemma 2.1 to conclude that \( h_j \in P(n,0), z \in U \) and thus \( h \in P_k(n,0) \) and hence by definition \( p \in P_k(n,\beta) \), which gives the required result.

We note that for \( \delta = 0, \gamma = \beta \). For \( \mu = 1 \) we thus get

**THEOREM 3.3.** For \( \lambda \geq 1 \), let \( f \in \Sigma_{n,p}^m(k,\lambda,q,\beta,0,1) \). Then

\[ \frac{D_{m+1} f^{(q)}(z)}{D_{m+1} g^{(q)}(z)} \]

\[ \in P_k(n,\beta). \]
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PROOF. We can write \( \hat{\lambda} \geq 1, \)

\[
\hat{\lambda} \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} = \left[ (1 - \hat{\lambda}) \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} + \hat{\lambda} \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} \right] + (\hat{\lambda} - 1) \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)}.
\]

(17)

This implies that

\[
\frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} = \frac{1}{\hat{\lambda}} \left[ (1 - \hat{\lambda}) \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} + \hat{\lambda} \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} \right] + \left( 1 - \frac{1}{\hat{\lambda}} \right) \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} = \frac{1}{\hat{\lambda}} H_1 + (1 - \frac{1}{\hat{\lambda}}) H_2.
\]

Since \( H_1, H_2 \in P_k(n, \beta) \), by THEOREM 3.2, DEFINITION 1.1 and \( P_k(n, \beta) \) is a convex set (by THEOREM 3.1), we get the required result.

Taking \( b_k = 0 \) \( \forall \ k \geq n + p, \ n, p \in N \), we get the result contained in the following theorem:

THEOREM 3.4. Let \( \mu > 0 \), and \( \hat{\lambda} \in C \) such that \( \text{Re} \hat{\lambda} > 0 \). Let \( f \in A\), and satisfy the condition

\[
(1 - \hat{\lambda}) \left( \frac{D^m f^{(q)}(z)}{(p-q)^m \eta(p; q) z^{p-q}} \right)^\mu + \hat{\lambda} \left( \frac{D^{m+1} f^{(q)}(z)}{(p-q)^m \eta(p; q) z^{p-q}} \right)^\mu + \left( 1 - \frac{1}{\hat{\lambda}} \right) \left( \frac{D^m f^{(q)}(z)}{(p-q)^m \eta(p; q) z^{p-q}} \right)^\mu 
\]

\( \in P_k(n, \beta) \).

Then

\[
\left( \frac{D^m f^{(q)}(z)}{(p-q)^m \eta(p; q) z^{p-q}} \right)^\mu \in P_k(n, \sigma) \ \forall \ z \in U,
\]

where \( \sigma = \beta + (1 - \beta)(2\rho - 1) \)

\[
\rho = \frac{1}{2} {}_2F_1 \left( 1, 1; 1 + \frac{\mu(p-q) - 1}{\text{Re} \{\hat{\lambda}\}} \right).
\]

The value of \( \sigma \) is best possible and cannot be improved.
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PROOF. We set
\[
\left( \frac{D^m f^{(q)}(z)}{(p-q)^m \eta(p;q) z^{p-q}} \right)^\mu = h(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z),
\]
where \( h(0) = 1 \) and \( h \) is analytic in \( U \). Then
\[
(1 - \lambda) \left( \frac{D^m f^{(q)}(z)}{(p-q)^m \eta(p;q) z^{p-q}} \right) + \lambda \frac{D^{m+1} f^{(q)}(z)}{(p-q)^{m+1} \eta(p;q) z^{p-q}} = h(z) + \frac{\lambda z h'(z)}{\mu(p-q)} \in P_\beta(n, \beta), \quad z \in U.
\]

Using Lemma 2.3, we note that \( h_j \in P(n, \sigma) \), where \( \sigma = \beta + (1 - \beta)(2 \rho - 1) \),
\[
\text{and} \quad \rho = \int_0^1 t^{\frac{\text{Re} \{ \lambda \}}{\mu(p-q)} - 1} \, dt.
\]

Putting \( \text{Re} \{ \lambda \} = \lambda > 0 \), we have
\[
\rho = \int_0^1 t^{\frac{\text{Re} \{ \lambda \}}{\mu(p-q)} - 1} \, dt = \frac{\mu(p-q)}{\lambda} \int_0^1 u^{\frac{\mu(p-q)}{\lambda} - 1} (1 + u)^{-1} \, du
\]
\[
= \text{$_2$F$_1$} \left( 1, \frac{\mu(p-q)}{\lambda}; 1 + \frac{\mu(p-q)}{\lambda}; -1 \right)
\]
\[
= \frac{1}{2} \text{$_2$F$_1$} \left( 1, 1; 1 + \frac{\mu(p-q)}{\lambda}; \frac{1}{2} \right). \quad (20)
\]

On substituting \( \mu = 1 \) and \( \lambda \in R \) s.t. \( \lambda \geq 1 \) in THEOREM 3.4, we get the following result:

**COROLLARY 3.5.** If \( f \in A_\lambda(p) \) satisfies
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Using a differential operator

\[
(1 - \lambda) \frac{D^m f^{(q)}(z)}{(p - q)^m \eta(p; q) z^{p-q}} + \lambda \frac{D^{m+1} f^{(q)}(z)}{(p - q)^{m+1} \eta(p; q) z^{p-q}} \in P_k(n, \beta).
\]

Then

\[
\frac{D^{m+1} f^{(q)}(z)}{(p - q)^{m+1} \eta(p; q) z^{p-q}} \in P_k(n, \sigma^*) \quad \forall \ z \in U,
\]

where \( \sigma^* = \beta + (1 - \beta)(2 \rho^* - 1)(1 - \lambda^{-1}) \)

and \( \rho^* = \frac{1}{2} \; _2F_1 \left( 1, 1; 1 + \frac{p - q}{\lambda}; 1 - \frac{1}{2} \right) \).

The value of \( \sigma^* \) is best possible and cannot be improved.

Taking \( q = 0 \) in Theorem 3.4, we get the following result:

**Corollary 3.6.** Let \( \mu > 0 \) and \( \lambda \in C \) such that \( \Re \lambda > 0 \). Let \( f \in A_n(p) \) and satisfy the condition

\[
(1 - \lambda) \left( \frac{D^m f(z)}{p^m z^p} \right)^\mu + \lambda \frac{D^{m+1} f(z)}{p^{m+1} z^p} \left( \frac{D^m f(z)}{p^m z^p} \right)^{\mu-1} \in P_k(n, \beta).
\]

Then

\[
\left( \frac{D^m f(z)}{p^m z^p} \right)^\mu \in P_k(n, \sigma_1) \quad \forall \ z \in U,
\]

where \( \sigma_1 = \beta + (1 - \beta)(2 \rho_1 - 1) \)

and \( \rho_1 = \frac{1}{2} \; _2F_1 \left( 1, 1; 1 + \frac{\mu p}{\Re \lambda}; 1 - \frac{1}{2} \right) \).

The value of \( \sigma_1 \) is best possible and cannot be improved.

Further taking \( m = 0 \) in Corollary 3.6, we get the following result:

**Corollary 3.7.** Let \( \lambda \in C \) such that \( \Re \lambda > 0 \). Let \( f \in A_n(p) \) and satisfy

the condition

\[
(1 - \lambda) \left( \frac{f(z)}{z^p} \right)^\mu + \lambda \frac{z f'(z)}{p f(z)} \left( \frac{f(z)}{z^p} \right)^\mu \in P_k(n, \beta).
\]
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Then
\[
\left( \frac{f(z)}{z^p} \right)^\mu P_k(n, \sigma) \quad \forall z \in U,
\]
where \( \sigma \) is defined with COROLLARY 3.6. Also the value of \( \sigma \) is best possible and cannot be improved.

THEOREM 3.8. For \( 0 \leq \lambda_2 < \lambda_1 \),
\[
\sum_{n,p}^m (k, \lambda_2, q, \beta, 0, \mu) \subset \sum_{n,p}^m (k, \lambda_1, q, \beta, 0, \mu).
\]

PROOF. If \( \lambda_2 = 0 \), then the proof is immediate from THEOREM 3.2. So we let \( \lambda_2 > 0 \) and \( f \in \sum_{n,p}^m (k, \lambda_1, q, \beta, 0, \mu) \). Then there exist two functions \( H_1, H_2 \in P_k(n, \beta) \) such that
\[
(1 - \lambda_2) \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^\mu = H_1(z),
\]
and
\[
\left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^\mu = H_2(z).
\]

Now
\[
(1 - \lambda_2) \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^\mu + \lambda_2 \left( \frac{D^{m+1} f^{(q)}(z)}{D^{m+1} g^{(q)}(z)} \right)^{\mu-1} = \frac{\lambda_2}{\lambda_1} H_1(z) + \left( 1 - \frac{\lambda_2}{\lambda_1} \right) H_2(z),
\]
and since \( P_k(n, \beta) \) is a convex set, it follows that R.H.S. of (21) belongs to \( P_k(n, \beta) \) and this completes the proof.

We now consider the converse case of THEOREM 3.2 as follows:

THEOREM 3.9. Let
\[
\left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^\mu \in P_k(n, \beta) \quad \text{with} \quad \frac{D^{m+1} g^{(q)}(z)}{(p-q)D^m g^{(q)}(z)}
\]
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\[ e \in P(n, \delta) \text{ for } z \in U. \text{ Then } f \in \sum_{n=0}^m (k, \lambda, q, \beta, \delta, \mu) \text{ for } |z| < R(\mu, \delta, p, q, n, \lambda), \]
where

\[ R(\mu, \delta, p, q, n, \lambda) = \left[ \frac{\mu(p-q)}{[(1-\delta)\mu(p-q)+\lambda n]+\sqrt{(\delta\mu(p-q))^2+\lambda^2n^2+2\lambda n(1-\delta)(p-q)}} \right]^{\frac{1}{n}}. \quad (22) \]

PROOF. Let

\[ H = \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^\mu, \quad \text{and} \quad Q_0 = \frac{D^{m+1} g^{(q)}(z)}{(p-q)D^m g^{(q)}(z)}. \]

Then

\[ H \in P_1(n, \beta), \quad Q_0 \in P(n, \delta). \]

Proceeding as in THEOREM 3.2, for \( p > q, \ p \in N; \ q, m \in N_0, \ k \geq 2, \ \Re \lambda \geq 0, \ 0 \leq \beta, \delta < 1, \) and

\[ H = (1-\beta)h + \beta, \quad Q_0 = (1-\delta)q_0 + \delta, \] with \( h \in P_1(n,0) \) and \( q_0 \in P(n), \) we have

\[ \frac{1}{(1-\beta)} \left( 1-\lambda \right)^\mu \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^\mu + \lambda \frac{D^{m+1} f^{(q)}(z)}{D^m g^{(q)}(z)} \left( \frac{D^m f^{(q)}(z)}{D^m g^{(q)}(z)} \right)^{\mu-1} \]

\[ = \left\{ h(z) + \frac{\lambda}{\mu(p-q)(1-\delta)q_0(z)+\delta} z h'(z) \right\} \]

\[ = \left\{ \frac{k}{4} + \frac{1}{2} \right\} \left[ h_1(z) + \frac{\lambda}{\mu(p-q)(1-\delta)q_0(z)+\delta} \frac{z h_1'(z)}{\mu(p-q)(1-\delta)q_0(z)+\delta} \right] \]

\[ - \left\{ \frac{k}{4} - \frac{1}{2} \right\} \left[ h_2(z) + \frac{\lambda}{\mu(p-q)(1-\delta)q_0(z)+\delta} \frac{z h_2'(z)}{\mu(p-q)(1-\delta)q_0(z)+\delta} \right]. \]

Using LEMMA 2.2, for \( h_j \in P(n), \)

\[ \frac{1-r^n}{1+r^n} \leq \Re h_j(z) \leq |h_j(z)| \leq \frac{1+r^n}{1-r^n} \]
and

\[ |zh_j'(z)| \leq \frac{2nr^o \Re h_j(z)}{1-r^{2n}}. \]

We have

\[
\begin{align*}
\Re \left[ h_j(z) + \frac{\lambda}{\mu(p-q)} \frac{zh_j'(z)}{(1-\delta)q_o(z) + \delta} \right] \\
\geq \Re h_j(z) \left[ 1 - \frac{\lambda}{\mu(p-q)} \frac{2nr^o}{1-r^{2n}} \frac{1}{(1-\delta)q_o(z) + \delta} \right] \\
= \Re h_j(z) \left[ 1 - \frac{\lambda}{\mu(p-q)} \frac{2nr^o}{1-r^{2n}} \frac{1+r^o}{1-(1-2\delta)r^o} \right] \\
= \Re h_j(z) \left[ \frac{2\lambda nr^o}{\mu(p-q)(1-r^n)[1-(1-2\delta)r^n]} \right] \\
= \Re h_j(z) \left[ \frac{\mu(p-q)[1-r^n-(1-2\delta)r^o + (1-2\delta)r^{2n}] - 2\lambda nr^n}{\mu(p-q)(1-r^n)[1-(1-2\delta)r^n]} \right] \\
\geq \Re h_j(z) \left[ \frac{\mu(p-q)(1-2\delta)r^{2n} - 2[(1-\delta)\mu(p-q) + \lambda n]r^n + \mu(p-q)}{\mu(p-q)(1-r^n)[1-(1-2\delta)r^n]} \right].
\end{align*}
\]

(23)

R.H.S. of (23) is positive for \( r = |z| < R(\mu, \delta, p, q, n, \lambda) \), where \( R(\mu, \delta, p, q, n, \lambda) \) is given by (22).

We note that, for \( p = 1, q = 0, m = 0, \mu = 1, \delta = 0, f \in P_2(n, \beta) \), for \( z \in U \) implies \( f' \in P_2(n, \beta) \) for \( |z| < R^* = \left( \frac{1}{(n+1) + \sqrt{n^2 + 2n}} \right)^{1/2} \).

4. ACKNOWLEDGMENT

The second author is thankfull to CSIR, India, for providing Junior Research Fellowship under research scheme No. 09/149(0498)/2008-EMR-I.
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5. REFERENCES


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Received June 2009