

# THE GALERKIN METHOD FOR SOLUTIONS OF NONLINEAR VIBRATION PROBLEMS

VLADIMÍR ĎURIKOVIČ AND JÚLIA GAJDOŠOVÁ

---

## Abstract

In this work we prove the existence, uniqueness and regularity of weak local solutions to the mixed problem associated with the nonlinear equation:

$$u_{tt} = M(\|\nabla u\|^2 + \|u\|^2)(\Delta u - u) - \delta|u|^\alpha u + f.$$

We consider the three basic types of boundary conditions. Thus, our results extend [3] to the Neumann and Newton case with the operator  $-\Delta + I$ .

**Mathematics Subject Classification 2000:** 35A07, 35I70

**Additional Key Words and Phrases:** Boundary condition, existence, regularity and uniqueness of solution, method of Galerkin.

---

## 1. INTRODUCTION

This article is concerned with the study of the following nonlinear vibration equation with the generalized boundary condition

$$u_{tt} + M\left(\|u\|_{H^1(\Omega)}^2\right)(-\Delta u + u) + \delta|u|^\alpha u = f, \quad x \in \Omega, 0 < t < T, \quad (1.1)$$

$$\gamma_1 \frac{\partial u}{\partial \nu} + \gamma_2 u = 0, \quad x \in \partial\Omega, 0 \leq t \leq T, \quad (1.2)$$

$$u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x), \quad x \in \Omega, \quad (1.3)$$

where  $\|u\|_{H^1(\Omega)}^2 = \int_{\Omega} (|\nabla u|^2 + |u|^2) dx$ ,  $\Omega \subset R^n$  is of class  $C^{0,1}$ ,  $\nu$  is the outer normal vector to  $\partial\Omega$  and  $\delta > 0$ ,  $\alpha \geq 0$ ,  $T > 0$  are given constants,  $M(r)$  is a positive  $C^1$ -function on  $[0, \infty)$  and  $\gamma_1, \gamma_2 \in L^\infty(\partial\Omega)$  are non-negative real functions with  $\gamma_1^2 + \gamma_2^2 \neq 0$  on  $\partial\Omega$ . In what follows, we will deal with the Dirichlet, Neumann and Newton boundary condition for equation (1.1) and take  $\delta = 1$  without loss of generality.

Equation (1.1) arises in the study of the nonlinear vibration of particles with mass and with self-interaction assured by the restoring term  $|u|^\alpha u$ . Our model was inspired in

the work of M. Hosoya and Y. Yamada [3], where the operator  $-\Delta + I$  is replaced with Laplacian and the Dirichlet boundary condition is considered. In such a case the equation appears in mathematical physics as a special case of the Kirchhoff-Carrier equation, when modeling planar vibrations. For a background and physical properties of the above mentioned model, we refer the reader to the work of the Japanese authors and their references. Other useful results for the nonlinear operators the reader can find in [5].

A weak solution of the problem (1.1)-(1.3) is a function  $u(t) \in Y_k$  ( $k = 1, 2$ ) satisfying (1.3) and

$$\begin{aligned} (u''(t), w) + M \left( \|u(t)\|_{H^1(\Omega)}^2 \right) \left( (\nabla u(t), \nabla w) + (u(t), w) + \beta (\gamma u(t), w)_{L^2(\partial\Omega)} \right) + \\ + \left( |u(t)|^\alpha u(t), w \right) = (f(t), w) \end{aligned} \quad (1.4)$$

for all  $w \in Y_k$ , where  $Y_1 = H^1(\Omega)$ ,  $Y_2 = H_0^1(\Omega)$ , and  $\beta \in \{0, 1\}$ .  $Y_1$  holds for the Neumann ( $\beta = 0$ ) and Newton ( $\beta = 1$ ) boundary condition and  $Y_2$  for the Dirichlet ( $\beta = 0$ ) one. Here  $\gamma = \gamma_2/\gamma_1 \in L^\infty(\partial\Omega)$  for  $\gamma_1 \neq 0$  on  $\partial\Omega$ .

To obtain the existence of weak solution we employ Galerkin's method and make use of a "special basis".

In this work taking into account all the three types of the boundary conditions, we have to use basis  $\{w_j\}_{j \in N}$  satisfying

$$-\Delta w_j = (\lambda_j - 1)w_j.$$

Note that the Laplacian in the Neumann case is non-coercive. Hence, the Lax-Milgram theorem assuring the existence of eigenfunctions cannot be applied but the operator  $-\Delta + I$  avoids this difficulty.

Our paper is organized as follows. In section 2 we give the notations and state the main result. In Section 3 we introduce some lemmas and hint at the main steps of their proofs. In Section 4 we study the existence and uniqueness of weak solutions by making use of the lemmas given in Section 3.

## 2. NOTATIONS AND MAIN RESULT

In this section we present some notations that will be used throughout this paper and state the main result.

THE GALERKIN METHOD FOR SOLUTIONS OF NONLINEAR VIBRATION  
PROBLEMS

For any Banach space  $X$ , its norm is denoted by  $\|\cdot\|_X$ . In particular, for  $X = L^2(\Omega)$  its norm and inner product are simply denoted by  $\|\cdot\|$  and  $(\cdot, \cdot)$ , respectively. A bilinear form  $a(\cdot, \cdot)$  on  $H^1(\Omega)$  is defined by

$$a(u, v) = (\nabla u, \nabla v) + (u, v) \equiv \sum_{i=1}^n \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_{\Omega} uv dx$$

and  $a(u, u)$  is denoted by  $a(u)$ . We often write

$$a(u, v) + \beta (\gamma u, v)_{L^2(\partial\Omega)},$$

where  $\beta \in \{0, 1\}$  and  $\gamma \equiv \frac{\gamma_2}{\gamma_1} \in L^\infty(\partial\Omega)$  for  $\gamma_1 \neq 0$  on  $\partial\Omega$ . Hereafter,  $\beta = 0$  will always hold for the Dirichlet and Neumann case and  $\beta = 1$  for the Newton case.

For simplicity we will often denote some of the Hilbert spaces by  $Y_k, Z_k$  ( $k = 1, 2$ ) as follows:

$$Y_1 = H^1(\Omega), \quad Y_2 = H_0^1(\Omega)$$

and

$$Z_1 = H^2(\Omega), \quad Z_2 = H_0^1 \cap H^2(\Omega)$$

Just as before,  $Y_k, Z_k$  with  $k = 1$  will always hold for the Neumann and Newton case and  $k = 2$  for the Dirichlet case.

Finally, as usual

$$u' = \frac{\partial u}{\partial t} \quad \text{and} \quad u'' = \frac{\partial^2 u}{\partial t^2}.$$

Now we are in condition to state our main result. Let  $M, \alpha$  and  $f$  satisfy the following assumptions:

$$M \in C^1[0, \infty] \quad \text{and} \quad M(r) \geq m_0 > 0 \quad \text{for} \quad r \geq 0 \quad (\text{A.1})$$

$$0 \leq \alpha \leq \frac{2}{n-4} \quad \text{if } n \geq 5 \quad \text{and} \quad 0 \leq \alpha < \infty \quad \text{if } n = 1, 2, 3, 4 \quad (\text{A.2})$$

$$f \in L^2([0, T]; Y_k) \quad (k = 1, 2) \quad (\text{A.3})$$

Conditions for the existence, regularity and uniqueness are given in the following theorem:

THEOREM 2.1. Assume (A.1)-(A.3). Then, for any initial data

$$u^0 \in Z_k \quad \text{and} \quad u^1 \in Y_k \quad (k = 1, 2),$$

there exists a positive constant  $T_0 < T$  such that (1.1)-(1.3) possesses a unique weak solution  $u$  on  $[0, T_0]$  satisfying

$$u \in C([0, T_0]; Y_k) \cap C_w([0, T_0]; H^2(\Omega)), \quad (\text{B.1})$$

$$u_t \in C([0, T_0]; L^2(\Omega)) \cap C_w([0, T_0]; Y_k), \quad (\text{B.2})$$

$$u_{tt} \in L^2([0, T_0]; L^2(\Omega)), \quad (\text{B.3})$$

where the subscript “ $w$ ” means the weak continuity with respect to  $t$ . Moreover, if  $f \in C_w([0, T_0]; L^2(\Omega))$ , then  $u$  satisfies

$$u_{tt} \in C_w([0, T_0]; L^2(\Omega)). \quad (\text{B.4})$$

### 3. LEMMAS

LEMMA 3.1. Let  $\{\lambda_j\}$  be a sequence of eigenvalues of the weak problem to

$$-\Delta w + w = \lambda w \text{ in } \Omega \in C^{0,1} \quad \text{and} \quad \gamma_1 \frac{\partial w}{\partial \nu} + \gamma_2 w = 0 \quad \text{on } \partial\Omega,$$

and  $w_j \in Z_k$  ( $k = 1, 2$ ) are the corresponding eigenfunctions to  $\lambda_j$ . Then  $\{w_j\}$  is a complete orthonormal system in  $L^2(\Omega)$ .

PROOF. It is sufficient to show that the operator  $(-\Delta + I)^{-1}: Y_k \rightarrow Y_k$  is non-negative, compact and self-adjoint and use the Hilbert-Schmidt theorem.

LEMMA 3.2. Denote the approximate solutions  $u_m$  for a weak solution of (1.1)-(1.3) in the form

$$u_m(t) = \sum_{j=1}^m g_{jm}(t) w_j, \quad m = 1, 2, \dots$$

where  $\{w_j\}$  is a complete orthonormal system in  $L^2(\Omega)$  from Lemma 3.1. and  $g_{jm}$  are sufficiently smooth and determined by the following system of ordinary differential equations

$$\begin{aligned} & (u_m''(t), w_j) + M(a(u_m(t))) (a(u_m(t), w_j) + \beta(\gamma u_m(t), w_j)_{L^2(\partial\Omega)}) + \\ & + (|u_m(t)|^\alpha u_m(t), w_j) = (f(t), w_j) \end{aligned} \quad (\text{3.1})$$

THE GALERKIN METHOD FOR SOLUTIONS OF NONLINEAR VIBRATION  
PROBLEMS

with  $\beta \in \{0, 1\}$  and for  $j = 1, \dots, m$ . The initial conditions are given by

$$\begin{aligned} u_m(0) &= u_m^0 \equiv \sum_{j=1}^m b_{jm} w_j \rightarrow u^0 \text{ in } Z_k, \\ u'_m(0) &= u_m^1 \equiv \sum_{j=1}^m c_{jm} w_j \rightarrow u^1 \text{ in } Y_k \end{aligned} \quad (3.2)$$

for  $m \rightarrow \infty$ , where  $b_{jm}$  and  $c_{jm}$  are constants, which can be taken as Fourier coefficients of  $u^0$  or  $u^1$ , respectively in  $L^2(\Omega)$  for the complete orthonormal system  $\{w_j\}$  from Lemma 3.1. Then the previous system determines a unique solution  $u_m$  on some interval  $[0, t_m]$ .

PROOF. Denoting  $g_j(t) =: g_{jm}(t)$ ,  $b_j =: b_{jm}$ ,  $c_j =: c_{jm}$  and  $y_j(t) =: g'_j(t)$ , (3.1)-(3.2) can be transformed to a system of ordinary differential equations of the first order with  $2m$  variables

$$\begin{aligned} y'_j(t) &= -\lambda_j \bar{M}(g_1 \dots g_m)(t) g_j(t) - F(g_1 \dots g_m)(t) + (f(t), w_j) \\ g'_j(t) &= y_j(t) \end{aligned}$$

with initial data  $y_j(0) = c_j$ ,  $g_j(0) = b_j$

for  $j = 1 \dots m$ , where

$$\bar{M}(g_1 \dots g_m)(t) = M \left( \left\| \sum_{i=1}^m g_i(t) w_i \right\|_{H^1(\Omega)} \right)$$

and

$$F(g_1 \dots g_m)(t) = \left( \left| \sum_{i=1}^m g_i(t) w_i \right|^\alpha \sum_{i=1}^m g_i(t) w_i, w_j \right).$$

In what follows, the Carathéodory theorem (see [1]) of the existence and uniqueness of local solutions should be applied.

LEMMA 3.3. Consider a linear operator  $A: D(A) \rightarrow L^2[0, T]$  defined as  $Au = u''$ , where  $D(A) = \{u \in C^1[0, T] : u, u' \text{ are absolutely continuous and } u'' \in L^2[0, T]\}$  is the domain of  $A$ . Then the operator  $A$  is closed in  $L^2[0, T]$ .

PROOF. It is sufficient to show that any  $\{u_m\} \subset D(A)$ ,  $u_m \rightarrow u$  in  $C^1[0, T]$  and  $Au_m \rightarrow u_1$  in  $L^2[0, T]$  implies  $u \in D(A)$  and  $Au = u_1$ .

LEMMA 3.4. Let  $R^n \supset \Omega \in C^{0,1}$  be a domain with smooth boundary and let  $u \in H^2(\Omega)$ . Then there exist positive constants  $c_1, c_2, c_3$  independent of  $u$  such that

$$\|u\|_{H^2(\Omega)}^2 \leq c_1 \|\Delta u\|^2 + c_2 \|u\|^2 + c_3 \|\varphi\|_{H^2(\Omega)}^2,$$

where  $\varphi(x)$  is an arbitrary function satisfying  $u(x) - \varphi(x) \in H^2(\Omega) \cap H_0^1(\Omega)$ .

(For the proof of the previous lemma see [4] pg.217.)

#### 4. PROOF OF THE EXISTENCE AND UNIQUENESS OF A WEAK LOCAL SOLUTION

In order to obtain a unique weak solution we employ Galerkin's procedure. Let  $\{\lambda_j\}_{j=1}^\infty$  be a sequence of eigenvalues as defined in Lemma 3.1. Denote by  $w_j \in Z_k$  ( $k = 1, 2$ ) the corresponding eigenfunctions to  $\lambda_j$ . Clearly, we may take  $\{w_j\}$  as a complete orthonormal system in  $L^2(\Omega)$ . First we construct approximate solutions  $u_m$  in the form

$$u_m(t) = \sum_{j=1}^m g_{jm}(t) w_j,$$

where  $g_{jm}$  are determined by the Cauchy problem (3.1)-(3.2). The approximate system (3.1)-(3.2) is a system of  $m$  ordinary differential equations and on account of Lemma 3.2, it has a unique local solution in  $[0, t_m]$ . The restriction of the solution to some interval  $[0, T_0]$  is a consequence of the second estimate we are going to obtain later.

### A PRIORI ESTIMATES

#### 4.1. THE FIRST ESTIMATE

The equation (3.1) is equivalent to

$$\begin{aligned} (u_m''(t), w) + M(a(u_m(t))) (a(u_m(t), w) + \beta(\gamma_m(t), w)_{L^2(\partial\Omega)}) + \\ + (|u_m(t)|^\alpha u_m(t), w) = (f(t), w) \end{aligned} \quad (4.1)$$

for all  $w \in V_m$ , where  $V_m$  is an  $m$ -dimensional vector space spanned by  $\{w_1, w_2, \dots, w_m\}$ .

Putting  $w = u_m'(t) \in V_m$  in (4.1) we get with  $p = \alpha + 2$

THE GALERKIN METHOD FOR SOLUTIONS OF NONLINEAR VIBRATION  
PROBLEMS

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \|u'_m(t)\|^2 + \int_0^{a(u_m(t))} M(s) ds \right\} + \frac{\beta}{2} M(a(u_m(t))) \frac{d}{dt} \left\| \gamma^{\frac{1}{2}} u_m(t) \right\|_{L^2(\partial\Omega)}^2 + \\ + \frac{1}{p} \frac{d}{dt} \|u_m(t)\|_{L^p(\Omega)}^p = (f(t), u'_m(t)), \end{aligned} \quad (4.2)$$

which can be verified directly by the derivation of (4.2).

Denote a functional by

$$E(u_m(t)) = \frac{1}{2} \left\{ \|u'_m(t)\|^2 + \int_0^{a(u_m(t))} M(s) ds + \frac{2}{p} \|u_m(t)\|_{L^p(\Omega)}^p \right\}$$

Hence,

$$\frac{d}{dt} E(u_m(t)) + \frac{\beta}{2} M(a(u_m(t))) \frac{d}{dt} \left\| \gamma^{\frac{1}{2}} u_m(t) \right\|_{L^2(\partial\Omega)}^2 = (f(t), u'_m(t)).$$

Integrating (4.2) over  $(0, t)$  and employing Young's inequality yields

$$\begin{aligned} \frac{1}{2} \left\{ \|u'_m(t)\|^2 + m_0 a(u_m(t)) + \beta m_0 \left\| \gamma^{\frac{1}{2}} u_m(t) \right\|_{L^2(\partial\Omega)}^2 \right\} + \frac{1}{p} \|u_m(t)\|_{L^p(\Omega)}^p \leq \\ \leq E(u_m(0)) + \frac{1}{2} \int_0^t \left\{ \|f(s)\|^2 + \|u'_m(s)\|^2 + m_0 a(u_m(s)) + \beta m_0 \left\| \gamma^{\frac{1}{2}} u_m(s) \right\|_{L^2(\partial\Omega)}^2 \right\} + \\ + \frac{1}{p} \int_0^t \|u_m(s)\|_{L^p(\Omega)}^p ds. \end{aligned} \quad (4.3)$$

Applying Gronwall's lemma to (4.3) we obtain

$$\begin{aligned} \left\{ \|u'_m(t)\|^2 + m_0 a(u_m(t)) + \beta m_0 \left\| \gamma^{\frac{1}{2}} u_m(t) \right\|_{L^2(\partial\Omega)}^2 \right\} + \frac{2}{p} \|u_m(t)\|_{L^p(\Omega)}^p \leq \\ \leq \left\{ 2E(u_m^0) + \int_0^T \|f(s)\|^2 ds \right\} e^T. \end{aligned}$$

(4.4)

Using embedding  $H^2(\Omega)$  do  $L^p(\Omega)$  and (3.2) one can easily see the  $E(u_m(0))$  is bounded by a positive constant independent of  $m$  which enables us to take  $t_m = T_0$  for all  $m$  and so

$$\|u'_m(t)\| \leq C_1 \quad \text{for } 0 \leq t \leq T_0, \quad (4.5)$$

$$\|\nabla u_m(t)\| \leq C_2 \quad \text{for } 0 \leq t \leq T_0, \quad (4.6)$$

$$\|u_m(t)\| \leq C \|u_m(t)\|_{L^p(\Omega)} \leq C_3 \quad \text{for } 0 \leq t \leq T_0, \quad (4.7)$$

$$\|u_m(t)\|_{L^2(\partial\Omega)} \leq C_4 \quad \text{for } 0 \leq t \leq T_0, \quad (4.8)$$

with some positive constants  $C_i, i = 1, \dots, 4$  independent of  $m$ .

#### 4.2. THE SECOND ESTIMATE

In the following steps we will gradually put  $w = \Delta u'_m(t) - u'_m(t)$  in (4.2). Then we have

$$(u''_m(t), \Delta u'_m(t) - u'_m(t)) = -\frac{1}{2} \frac{d}{dt} \left\{ a(u'_m(t)) + \beta \left\| \gamma^{\frac{1}{2}} u'_m(t) \right\|_{L^2(\partial\Omega)}^2 \right\}. \quad (4.9)$$

Define a scalar product in  $H^1(\Omega) \times H^1(\Omega)$  in the following way:

$$a_1(x, y) = (\nabla x, \nabla y) + (x, y) + \beta (\gamma x, y)_{L^2(\Omega)}$$

with  $\beta \in \{0, 1\}$ . Putting  $x = u_m(t), y = \Delta u'_m(t) - u'_m(t)$  into  $a_1(x, y)$  one obtains

$$\begin{aligned} & a_1(u_m(t), \Delta u'_m(t) - u'_m(t)) = \\ & = -\frac{1}{2} \frac{d}{dt} \left\{ \|\Delta u_m(t)\|^2 + 2 \|\nabla u_m(t)\|^2 + \|u_m(t)\|^2 + 2\beta \left\| \gamma^{\frac{1}{2}} u_m(t) \right\|_{L^2(\partial\Omega)}^2 \right\} \end{aligned} \quad (4.10)$$

Define a functional

$$F(u) = \|\Delta u\|^2 + 2 \|\nabla u\|^2 + \|u\|^2 + 2\beta \left\| \gamma^{\frac{1}{2}} u \right\|_{L^2(\partial\Omega)}^2 \quad \text{for } u \in Z_k.$$

Also,



THE GALERKIN METHOD FOR SOLUTIONS OF NONLINEAR VIBRATION  
PROBLEMS

$$(f(t), \Delta u'_m(t) - u'_m(t)) = - \left\{ a(f(t), u'_m(t)) + \beta (f(t), \gamma u'_m(t))_{L^2(\partial\Omega)} \right\}. \quad (4.11)$$

Finally,

$$\begin{aligned} & \frac{d}{dt} \left\{ M(a(u_m(t))) F(u_m(t)) \right\} = \\ & = 2M'(a(u_m(t))) a(u_m(t), u'_m(t)) F(u_m(t)) + M \left( a(u_m(t)) \frac{d}{dt} F(u_m(t)) \right). \end{aligned} \quad (4.12)$$

Making use of (4.7)-(4.12) one gets

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left\{ a(u'_m(t)) + \beta \left\| \gamma^{\frac{1}{2}} u'_m(t) \right\|_{L^2(\partial\Omega)} + M(a(u_m(t))) F(u_m(t)) \right\} = \\ & = \left( |u_m(t)|^\alpha u_m(t), \Delta u'_m(t) - u'_m(t) \right) + a(f(t), u'_m(t)) + \\ & + \beta (f(t), \gamma u'_m(t))_{L^2(\partial\Omega)} + M'(a(u_m(t))) a(u_m(t), u'_m(t)) F(u_m(t)). \end{aligned} \quad (4.13)$$

The first term in the right-hand side of (4.13) is estimated as follows:

$$\begin{aligned} & \left| \left( |u_m(t)|^\alpha u_m(t), \Delta u'_m(t) - u'_m(t) \right) \right| \leq \left| \left( |u_m(t)|^\alpha u_m(t), \Delta u'_m(t) \right) \right| + \\ & + \left| \left( |u_m(t)|^\alpha u_m(t), u'_m(t) \right) \right|. \end{aligned} \quad (4.14)$$

The first term in the right-hand side of (4.14) is further estimated as

$$\begin{aligned} & \left| \left( |u_m(t)|^\alpha u_m(t), \Delta u'_m(t) \right) \right| \leq \left| \left( \nabla \left( |u_m(t)|^\alpha u_m(t) \right), \nabla u'_m(t) \right) \right| + \\ & + \beta \int_{\partial\Omega} \gamma |u_m(t)|^\alpha u_m(t) u'_m(t) ds \leq (\alpha + 1) \left| \left( |u_m(t)|^\alpha \nabla u_m(t), \nabla u'_m(t) \right) \right| + \\ & + \beta C \int_{\partial\Omega} |u_m(t)|^{\alpha+1} u'_m(t) ds \leq (\alpha + 1) \|u_m(t)\|_{L^{q\alpha}(\Omega)}^\alpha \|\nabla u_m(t)\|_{L^r(\Omega)} \|\nabla u'_m(t)\| + \\ & + \beta C \|u_m(t)\|_{L^{2(\alpha+1)}(\partial\Omega)}^{\alpha+1} \|u'_m(t)\|_{L^2(\partial\Omega)}. \end{aligned}$$

We take  $q$  and  $r$  such that

$$\frac{1}{q} + \frac{1}{r} = \frac{1}{2}, \quad \frac{1}{q\alpha} \geq \frac{1}{2} - \frac{2}{n} \quad \text{and} \quad \frac{1}{r} \geq \frac{1}{2} - \frac{1}{n}.$$

It is easy to check that  $H^2(\Omega) \hookrightarrow L^{q\alpha}(\Omega)$  and  $H^1(\Omega) \hookrightarrow L^r(\Omega)$  for the parameters  $q$  and  $r$  defined above. Also,  $H^2(\Omega) \hookrightarrow L^{2(\alpha+1)}(\partial\Omega)$  and  $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ . Hence,

$$\begin{aligned} & \left| \left( |u_m(t)|^\alpha u_m(t), \Delta u'_m(t) \right) \right| \leq (\alpha + 1) \|u_m(t)\|_{H^2(\Omega)}^{\alpha+1} + \\ & + \beta C \|u_m(t)\|_{H^2(\Omega)}^{\alpha+1} \|u'_m(t)\|_{H^1(\Omega)}. \end{aligned}$$

From Lemma (3.4) and (4.7) follows that

$$\|u_m(t)\|_{H^2(\Omega)}^2 \leq C \left( \|\Delta u_m(t)\|^2 + 1 \right). \quad (4.15)$$

Making use of (4.15) and (4.5) one obtains that

$$\begin{aligned} & \left| \left( |u_m(t)|^\alpha u_m(t), \Delta u'_m(t) \right) \right| \leq \\ & \leq C \left\{ \left( \|\Delta u_m(t)\|^{\alpha+1} + 1 \right) \|\nabla u'_m(t)\| + \beta \left( \|\Delta u_m(t)\|^{\alpha+1} + 1 \right) \left( \|\nabla u'_m(t)\| + 1 \right) \right\} \leq \\ & \leq C \left( \|\Delta u_m(t)\|^{\alpha+1} + 1 \right) \left( \|\nabla u'_m(t)\| + 1 \right) \end{aligned} \quad (4.16)$$

The second term in the right-hand side of (4.14), using (4.5), is estimated as

$$\left| \left( |u_m(t)|^\alpha u_m(t), u'_m(t) \right) \right| \leq C \|u_m(t)\|_{L^{2(\alpha+1)}(\Omega)}^{\alpha+1}.$$

From (4.15) and  $H^2(\Omega) \hookrightarrow L^{2(\alpha+1)}(\Omega)$  it follows that

$$\left| \left( |u_m(t)|^\alpha u_m(t), u'_m(t) \right) \right| \leq C \left( \|\Delta u_m(t)\|^{\alpha+1} + 1 \right). \quad (4.17)$$

Combining (4.16) and (4.17) we obtain

$$\left| \left( |u_m(t)|^\alpha u_m(t), \Delta u'_m(t) - u'_m(t) \right) \right| \leq C \left( \|\Delta u_m(t)\|^{\alpha+1} + 1 \right) \left( \|\nabla u'_m(t)\| + 1 \right). \quad (4.18)$$

By virtue of (4.5) the sum of the second and third term in the right-hand side of (4.14) is bounded by

$$\begin{aligned} & |a(f(t), u'_m(t))| + \beta |f(t), \mathcal{M}'_m(t)| \leq \\ & \leq C \left\{ \|\nabla f(t)\|^2 + \|\nabla u'_m(t)\|^2 + \|f(t)\|^2 + 1 \right\}. \end{aligned} \quad (4.19)$$

THE GALERKIN METHOD FOR SOLUTIONS OF NONLINEAR VIBRATION  
PROBLEMS

Employing (4.5)-(4.8) we obtain the following estimate for the fourth term in the right-hand side for (4.14);

$$\begin{aligned} & |M'(a(u_m(t))) a(u_m(t), u'_m(t)) F(u_m(t))| \leq \\ & \leq M_1 C (\|\nabla u'_m(t)\| + 1) (\|\Delta u_m(t)\|^2 + 1), \end{aligned} \quad (4.20)$$

where  $M_1 = \max\{M'(r): 0 \leq r \leq C_2^2 + C_3^2\}$  (use (4.6) and (4.9)).

Define a functional  $G(u)$  by

$$G(u(t)) = a(u'(t)) + \beta \left\| \gamma^{\frac{1}{2}} u'(t) \right\|_{L^2(\partial\Omega)} + M(a(u(t))) F(u(t))$$

for  $u(t) \in Y_k \cap H_2(\Omega)$  and  $u'(t) \in Y_k$ ,  $t \in [0, T]$ .

Then, again, by virtue of (4.6) and (4.7) we have

$$\begin{aligned} & \|\nabla u'_m(t)\|^2 + \|u'_m(t)\|^2 + \beta \left\| \gamma^{\frac{1}{2}} u'_m(t) \right\|_{L^2(\partial\Omega)}^2 + m_0 F(u_m(t)) \leq G(u_m(t)) \leq \\ & \leq \|\nabla u'_m(t)\|^2 + \|u'_m(t)\|^2 + \beta \left\| \gamma^{\frac{1}{2}} u'_m(t) \right\|_{L^2(\partial\Omega)} + M_2 F(u_m(t)), \end{aligned} \quad (4.21)$$

where  $M_2 = \max\{M(r): 0 \leq r \leq C_2^2 + C_3^2\}$ . Making use of (4.18), (4.19) and (4.20)

we rearrange (4.13);

$$\begin{aligned} & \frac{d}{dt} G_k(u_m(t)) \leq C \{ (\|\Delta u_m(t)\|^{\alpha+1} + 1) (\|\nabla u'_m(t)\| + 1) + \|\nabla f(t)\|^2 + \\ & + \|\nabla u'_m(t)\|^2 + \|f(t)\|^2 + 1 + (\|\nabla u'_m(t)\| + 1) (\|\Delta u_m(t)\|^2 + 1) \} \leq \\ & \leq C \{ G^{\frac{\alpha+2}{2}}(u_m(t)) + G^{\frac{\alpha+1}{2}}(u_m(t)) + G^{\frac{1}{2}}(u_m(t)) + G(u_m(t)) + \\ & + G^{\frac{3}{2}}(u_m(t)) + \|\nabla f(t)\|^2 + \|f(t)\|^2 + 1 \}, \end{aligned}$$

where (4.21) is used in the last inequality. Taking  $\omega = \max\left\{\frac{\alpha + 2}{2}, \frac{3}{2}\right\}$  if  $G(u_m(t)) \geq 1$

and  $\omega = \frac{1}{2}$  if  $G(u_m(t)) < 1$  we get  $\frac{d}{dt}G(u_m(t)) \leq C\{G^\omega(u_m(t)) + 1\}$ .

One can easily show that there exists a positive number  $T_0$  such that

$$G(u_m(t)) \leq C_5 \quad \text{for } 0 \leq t \leq T_0,$$

which yields by (4.21)

$$\|\nabla u'_m(t)\| \leq C_6 \quad \text{for } 0 \leq t \leq T_0, \quad (4.22)$$

$$\|\Delta u_m(t)\| \leq C_7 \quad \text{for } 0 \leq t \leq T_0, \quad (4.23)$$

$$\|u'_m(t)\|_{L^2(\partial\Omega)} \leq C_8 \quad \text{for } 0 \leq t \leq T_0, \quad (4.24)$$

From (4.15) and (4.23) also follows

$$\|u_m(t)\|_{H^1(\Omega)} \leq C \|u_m(t)\|_{H^2(\Omega)} \leq C_9 \quad \text{for } 0 \leq t \leq T_0 \quad (4.25)$$

### 4.3. THE THIRD ESTIMATE

Put  $w = u''_m(t)$  in (4.4). We estimate

$$\begin{aligned} & \left| a(u_m(t), u''_m(t)) + (\mathcal{A}u_m(t), u''_m(t))_{L^2(\partial\Omega)} \right| \leq \\ & \leq C \left( \|\Delta u_m(t)\| + \|u_m(t)\| \right) \|u''_m(t)\|. \end{aligned}$$

Thus, it is easy to obtain from (4.1)

$$\|u''_m(t)\| \leq M_2 C \left( \|\Delta u_m(t)\| + \|u_m(t)\| \right) + \|u_m(t)\|_{L^{2(\alpha+1)}(\Omega)}^{\alpha+1} + \|f(t)\|.$$

Therefore, using  $H^2(\Omega) \hookrightarrow L^{2(\alpha+1)}(\Omega)$  and (4.8), (4.24), (4.26), it follows from the previous inequality that

$$\|u''_m(t)\| \leq C_{10} \quad \text{for } 0 \leq t \leq T_0 \quad (4.26)$$

### PASSAGE TO THE LIMIT

THE GALERKIN METHOD FOR SOLUTIONS OF NONLINEAR VIBRATION  
PROBLEMS

1. From (4.25) we have that  $\{u_m\}$  is uniformly bounded in  $C([0, T_0], Z_k)$ . Since the embedding from  $H^2(\Omega)$  into  $H^1(\Omega)$  and from  $H_0^1(\Omega) \cap H^2(\Omega)$  into  $H_0^1(\Omega)$  is compact,  $\{u_m\}$  is uniformly bounded in  $C([0, T_0]; Y_k)$ .

Moreover, the sequence  $\{u_m(t)\}$  is relatively compact in  $Y_k$  for every  $t \in [0, T_0]$ . From (4.5) and (4.22) we have the equicontinuity of  $\{u_m(t)\}$  in  $Y_k$ , that is

$$\limsup_{\delta \rightarrow 0^+} \left\{ \left\| u_m(t_1) - u_m(t_2) \right\|_{Y_k}; \left| t_1 - t_2 \right| < \delta \right\} = 0.$$

Generalized Ascoli-Arzelà theorem enables us to choose a subsequence, again denoted by  $\{u_m\}$ , such that

$$\lim_{m \rightarrow \infty} u_m = u \quad \text{in } C([0, T_0]; Y_k) \quad (4.27)$$

with some  $u \in C([0, T_0]; Y_k)$ .

2. We can also study some weak convergence properties of the second derivatives of  $u_m$ . Consider the  $L^2(\Omega)$ - inner product of  $D^\mu u_m$  ( $|\mu| = 2$ ) with the fixed function  $\varphi \in C_0^\infty(\Omega)$ . From (4.6), (4.8), (4.22) and (4.24) follows that  $\left\{ \left( D^\mu u_m, \varphi \right) \right\}$  is a family of uniformly bounded equicontinuous functions on  $[0, T_0]$  for every  $\mu$  ( $|\mu| = 2$ ).

Thus, we may choose a subsequence denoted by  $\left\{ \left( D^\mu u_m, \varphi \right) \right\}$ , such that

$$\lim_{m \rightarrow \infty} \left( D^\mu u_m, \varphi \right) = \left( D^\mu u, \varphi \right) \quad \text{in } C([0, T_0]; \mathbb{R}) \quad (4.28)$$

by (4.27). Since  $C_0^\infty(\Omega)$  is dense in  $L^2(\Omega)$ , (4.28) remains valid for every  $\varphi \in L^2(\Omega)$ ; that

$$\text{is} \quad \lim_{m \rightarrow \infty} D^\mu u_m = D^\mu u \quad \text{in } C_w([0, T_0]; L^2(\Omega)).$$

3. Also,

$$\lim_{m \rightarrow \infty} (u_m, \varphi) = (u, \varphi) \quad \text{and} \quad \lim_{m \rightarrow \infty} \left( \frac{\partial u_m}{\partial x_i}, \varphi \right) = \left( \frac{\partial u}{\partial x_i}, \varphi \right) \quad \text{in } C([0, T_0]; \mathbb{R}), \quad i = 1, \dots, n$$

so that

$$\lim_{m \rightarrow \infty} u_m = u \quad \text{in } C_w([0, T_0]; H^2(\Omega)). \quad (4.29)$$

4. Now we can study the convergency of  $\{u'_m\}$ . By (4.5) and (4.22) the sequence  $\{u'_m\}$  is uniformly bounded in  $C([0, T_0]; Y_k)$ . The compact embedding from  $Y_k$  into  $L^2(\Omega)$  and (4.26) assure that  $\{u'_m\}$  is a family of uniformly bounded equicontinuous functions in

$C([0, T_0]; L^2(\Omega))$  and that  $\{u'_m(t)\}$  is relatively compact in  $L^2(\Omega)$  for every  $t \in [0, T_0]$ . Hence,

$$\lim_{m \rightarrow \infty} u'_m = u' \quad \text{in } C([0, T_0]; L^2(\Omega)). \quad (4.30)$$

5. Just as before, consider the  $L^2(\Omega)$ -inner product of  $\nabla u'_m$  with any fixed function  $\varphi \in L^2(\Omega)$ . From (4.22) and (4.26) follows that  $\{(\nabla u'_m, \varphi)\}$  is a family of uniformly bounded equicontinuous functions on  $[0, T_0]$ . Again,

$$\lim_{m \rightarrow \infty} (\nabla u'_m, \varphi) = (\nabla u', \varphi) \quad \text{in } C([0, T_0]; \mathbb{R}). \quad (4.31)$$

(4.30) and (4.31) assure that

$$\lim_{m \rightarrow \infty} u'_m = u' \quad \text{in } C_w([0, T_0]; Y_k). \quad (4.32)$$

6. Furthermore, (4.26) implies that  $\{u''_m\}$  is uniformly bounded in  $L^2([0, T_0]; L^2(\Omega))$ . The reflexivity enables us to choose a subsequence denoted by  $\{u''_m\}$ , which converges weakly to some  $u_1 \in L^2([0, T_0]; L^2(\Omega))$ . Lemma 3.3 implies that  $u_1 = u''$ . Thus, we get

$$\lim_{m \rightarrow \infty} u''_m = u'' \quad \text{weakly in } L^2([0, T_0]; L^2(\Omega)). \quad (4.33)$$

7. Similarly, we can show that

$$\lim_{m \rightarrow \infty} u_m = u \quad \text{weakly in } L^2([0, T_0]; L^2(\partial\Omega)). \quad (4.34)$$

The estimate (4.8) implies that  $\{u_m\}$  is uniformly bounded in  $L^2([0, T_0]; L^2(\partial\Omega))$  and on account of the Eberlein-Schmuljan theorem there exists a subsequence denoted by  $\{u_m\}$  converging weakly to some  $u_2 \in L^2([0, T_0]; L^2(\partial\Omega))$ . The continuous embedding from  $H^1(\Omega)$  into  $L^2(\partial\Omega)$  assures that  $u_m \rightarrow u$  in  $L^2([0, T_0]; L^2(\partial\Omega))$  and  $u_2 = u$  by (4.27). Hence, we get (4.35).

8. Finally, we will prove

$$\lim_{m \rightarrow \infty} |u_m|^\alpha u_m = |u|^\alpha u \quad \text{in } C([0, T_0]; L^2(\Omega)). \quad (4.35)$$

Rearranging the following inequality we obtain

$$\begin{aligned} \int_{\Omega} \left| |u_m|^\alpha u_m - |u|^\alpha u \right|^2 dx &\leq C \int_{\Omega} \left( |u|^\alpha + |u_m|^\alpha \right)^2 |u_m - u|^2 dx \leq \\ &\leq C \left\{ \|u\|_{L^{\alpha\omega}(\Omega)}^\alpha + \|u_m\|_{L^{\alpha\omega}(\Omega)}^\alpha \right\}^2 \|\Delta u_m - u\|_{L^r(\Omega)}^2, \end{aligned}$$

THE GALERKIN METHOD FOR SOLUTIONS OF NONLINEAR VIBRATION  
PROBLEMS

where  $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$  and the parametres  $q, r$  are taken such that  $H^2(\Omega) \hookrightarrow L^{aq}(\Omega)$

and  $H^1(\Omega) \hookrightarrow L^r(\Omega)$ , which yields to

$$\left\| |u(t)|^\alpha u(t) - |u_m(t)|^\alpha u_m(t) \right\| \leq C \|u_m - u\|_{C([0, T_0], Y_k)} \quad (4.36)$$

by (4.25). Therefore, (4.35) follows from (4.36) with use of (4.27).

Now we are ready to prove that  $u$  is a solution of (1.1)-(1.3). From (4.27), (4.29), (4.30) and (4.32)-(4.35) one can pass to the limit in (4.1) to get

$$\begin{aligned} & (u''(t), w) + M\left(\|u(t)\|_{H^1(\Omega)}^2\right) (-\Delta u(t) + u(t), w) + \\ & + \left(|u|^\alpha u(t), w\right) = (f(t), w) \text{ on } [0, T_0] \text{ for all } w \in L^2(\Omega). \end{aligned} \quad (4.37)$$

Regularity properties are easy consequences of (4.27), (4.29), (4.30) and (4.32).

Moreover, since

$$|u|^\alpha u \in C([0, T_0]; L^2(\Omega))$$

by (4.35), (B.4) follows from (4.37).

### UNIQUENESS

Let  $u$  and  $v$  be two solutions of (1.1)-(1.3) on  $[0, T_0]$ . Subtracting the two distributive solutions (1.1) with  $u$  and  $v$  we get that  $w = u - v$  satisfies

$$\begin{aligned} w_{tt} + M(a(u))(-\Delta w + w) &= \{M(a(u)) - M(a(v))\}(\Delta v - v) + \\ &+ \left(|v|^\alpha v - |u|^\alpha u\right) \end{aligned} \quad (4.38)$$

in  $\Omega \times [0, T_0]$ , with  $\gamma_1 \frac{\partial w}{\partial \nu} + \gamma_2 w = 0$  on  $\partial\Omega \times [0, T_0]$  and  $w(x, 0) = 0, w_t(x, 0) = 0$  in  $\Omega$ .

Taking the  $L^2(\Omega)$  – inner product of (4.38) with  $w_t$  leads to

$$\frac{d}{dt} \left\{ \|w_t\|^2 + M(a(u)) \left( a(w) + \beta \left\| \gamma^{\frac{1}{2}} w \right\|_{L^2(\partial\Omega)}^2 \right) \right\} =$$

$$\begin{aligned}
 &= 2M'(a(u))a(u, u_t) \left( a(w) + \beta \left\| \frac{1}{\gamma^2} w \right\|_{L^2(\partial\Omega)}^2 \right) + \\
 &+ 2\{M(a(u)) - M(a(v))\}(\Delta v - v, w_t) + 2(|v|^\alpha v - |u|^\alpha u, w_t). \quad (4.39)
 \end{aligned}$$

The first and second terms in the right-hand side of (4.39) are bounded by

$$C\|w(t)\|_{Y_k}^2 \quad \text{and} \quad C\|w(t)\|_{Y_k}\|w_t(t)\|.$$

Making use of (4.36) (with  $u_m$  replaced by  $v$ ) one can see that the last term in (4.39) is bounded by

$$C\|w(t)\|_{Y_k}\|w_t(t)\|.$$

Thus, integrating (4.39) over  $(0, t)$  yields

$$\|w_t(t)\|^2 + m_0\|w(t)\|_{Y_k}^2 \leq C \int_0^t \|w(s)\|_{H^1(\Omega)}^2 + \|w_t(s)\|^2 ds.$$

We take

$$\|w_t(t)\|^2 + m_0\|w(t)\|_{Y_k}^2 \leq C \int_0^t \left\{ \|w_t(s)\|^2 + m_0\|w(s)\|_{Y_k}^2 \right\} ds, \quad \text{if } m_0 \geq 1$$

$$\text{and } m_0 \left\{ \|w_t(t)\|^2 + \|w(t)\|_{Y_k}^2 \right\} \leq C \int_0^t \left\{ \|w_t(s)\|^2 + \|w(s)\|_{Y_k}^2 \right\} ds, \quad \text{if } m_0 < 1.$$

Gronwall's lemma with regard to the boundary and initial conditions implies  $w \equiv 0$ .

The proof of Theorem 2.1. is complete.

## REFERENCES

- [1] ĎURIKOVIČ, V., *Funkcionálna analýza Nelineárne metódy*, Univerzita Komenského, Bratislava, 1989.
- [2] GREGUŠ, G., ŠVEC, M. AND ŠEDA, V., *Obyčajné diferenciálne rovnice*, Alfa, Bratislava, SNTL, Praha, 1985.
- [3] HOSOYA, M. AND YAMADA, Y., *On some nonlinear wave equations I: local existence and regularity of solutions*, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. 38, 1991.
- [4] LADYŽENSKAJA, O. A., URALCEVA, N. N., *Linejnyje i kvazilinejnyje uravnenija eliptičeskovo tipa*, Nauka, Moskava, 1973.
- [5] MARTIN, R. H., *Nonlinear Operators and Differential Equations in Banach Spaces*, John Wiley and Sons, New York, London, Sydney, Toronto, 1976.



THE GALERKIN METHOD FOR SOLUTIONS OF NONLINEAR VIBRATION  
PROBLEMS

Vladimír Ďurikovič  
Department of Applied Mathematics  
St. Cyril and Method University  
Square J. Herda 2  
917 01 Trnava, Slovak Republic

Júlia Gajdošová  
Department of Applied Mathematics  
St. Cyril and Method University  
Square J. Herda 2  
917 01 Trnava, Slovak Republic

Received September 2009