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TWO NOTES ON SIMILARITIES

JANA DOBRAKOVOVÁ

Abstract

In this paper we study two problems concerning real numbers similarities, associate fuzzy quantities and pseudometrics. The first is concerning the relationship between similarities and s-generating fuzzy quantities, the other relates to pseudometrics built by means of similarities and respective convergence.

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1. INTRODUCTION

In this article we deal with some types of binary fuzzy relations modeling equivalence, proximity, or similarity of real numbers. There are several kinds of fuzzy relations, fuzzyfying these notions. One significant type is represented by fuzzy relations, called **nearnesses** (see for example [2], [3], [4], [5]), the other by **fuzzy equivalence relations**.

These are studied since 1965, when Lotfi Zadeh, in his first paper on fuzzy sets, defined the notion of a fuzzy binary relation.

In what follows we will deal with one particular kind of fuzzy equivalence relations, introduced also by Lotfi Zadeh (1971), called **similarity relations**.

We will study similarity relations defined on the universe of real numbers $\mathbb{R}$ by means of fuzzy quantities. In the first part we discuss a problem concerning correspondence between a fuzzy quantity and the associate similarity relation. In the second part relationship between similarities and derived pseudometrics.

First, let us briefly recall some basic concepts. For more details see for instance [6].

2. PRELIMINARY RESULTS

Definition 1.

A binary fuzzy relation S on the universe $\mathbb{R}$ is called a similarity relation (or briefly similarity) on $\mathbb{R}$ if and only if it is reflexive, symmetric and transitive w.r.t.minimum, it means, if and only if for any $x, y, z \in \mathbb{R}$:

(S1) $S(x, x) = 1$

(S2) $S(x, y) = S(y, x)$

(S3) $\min(S(x, z), S(y, z)) \leq S(x, y)$.

It is clear that real binary fuzzy relation S defined by :

$$S_f(x,y) = \begin{cases} 1, & \text{if } x = y, \\ \min(f(x), f(y)), & \text{if } x \neq y \end{cases} \tag{1}$$

where f is a fuzzy quantity, that is to say a function defined on $\mathbb{R}$, with values from the interval $\langle 0, 1\rangle$, is a similarity.

In what follows we will call such a fuzzy quantity f the **S-generating quantity**.

It is obvious that properties of such a similarity S_f depend on properties of the S-generating quantity f (see [1], [2]).

Definition 2.
A real function of a real variable f is said to be lower semicontinuous at a point $x_0 \in \mathbb{R}$ if

$$f(x_0) \leq \liminf_{x->x_0} f(x)$$

and it is said to be lower semicontinuous, if it is lower semicontinuous at each point of its domain of definition.

3. FROM FUZZY QUANTITY TO SIMILARITY AND BACKWARDS

Proposition 1.
Let f be a fuzzy quantity and S_f be the associate similarity. If g is a fuzzy quantity, defined for each $x \in \mathbb{R}$ by

$$g(x) = \sup(S_f(x,y), y \neq x) \tag{2}$$

then $g(x) \leq f(x)$, for each $x \in \mathbb{R}$ and $S_g(x,y) = S_f(x,y)$, for each couple $x, y \in \mathbb{R}$. If moreover f is a lower semicontinuous function, then $g(x) = f(x)$, for each $x \in \mathbb{R}$.

Proof. $g(x) \leq f(x)$, and $S_g(x,y) = S_f(x,y)$ follows quite easily from the definitions. Let us suppose that f is lower semicontinuous, let $x_0 \in \mathbb{R}$, then

$$g(x_0) = \sup(S_f(x_0,x), x \neq x_0) = \sup(\min(f(x_0), f(x)), x \neq x_0) \geq$$

$$\geq \limsup_{x->x_0} f(x) \geq \liminf_{x->x_0} f(x) \geq f(x_0)$$

As the following example shows, the assumption of lower semicontinuity for the equality $g(x) = f(x)$ is not necessary.

Example 1.
Consider two real numbers a, b, such that $0 \leq a < b \leq 1$ and a fuzzy quantity f, defined by

$$f(x) = \begin{cases} a, & \text{if } x < 0, \\ b, & \text{if } x \geq 0 \end{cases}$$

Then

$$S_f(x,y) = \begin{cases} a, & \text{if } x \neq y \wedge \min(x,y) < 0, \\ b, & \text{if } x \neq y \wedge \min(x,y) \geq 0, \\ 1, & \text{if } x = y \end{cases}$$

Since $\liminf_{x\to 0} f(x) = a < b = f(0)$, f is not lower semicontinuous at 0.

If we define a fuzzy quantity g according the formula (2), it can be easily seen that $g(x) = f(x)$ for each $x \in \mathbb{R}$.

Figure 1 shows graph of S_f ($S_f \equiv S_g$) for values $a = \frac{1}{4}$, $b = \frac{3}{4}$ on the interval $\langle -3, 4\rangle \times \langle -3, 4\rangle$.

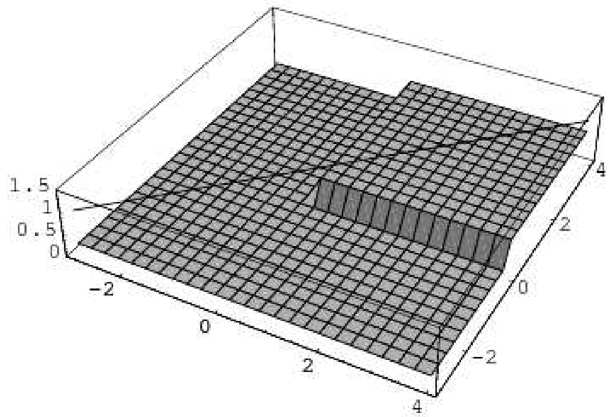

Fig. 1. Figure 1.

On the other hand, if we omit assumption of lower semicontinuity, the equality $f = g$ need not be valid.

Example 2.

Define a fuzzy quantity f by

$$f(x) = \begin{cases} \frac{1}{2+x^2}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0 \end{cases}$$

It is obvious that f is not lower semicontinuous at 0 and it is clear that the function g defined by means of (2) is not identical with f, since $g : y = \dfrac{1}{2 + x^2}$, hence $g(0) = \dfrac{1}{2}$. But in spite of that it holds:

$$S_f(x, y) = \begin{cases} \min(\frac{1}{2+x^2}, \frac{1}{2+y^2}), & \text{if } x \neq y, \\ 1, & \text{if } x = y \end{cases}$$

Graph of S_f ($S_f \equiv S_g$) on the interval $\langle -4, 4\rangle \times \langle -4, 4\rangle$ is shown in Figure 2.

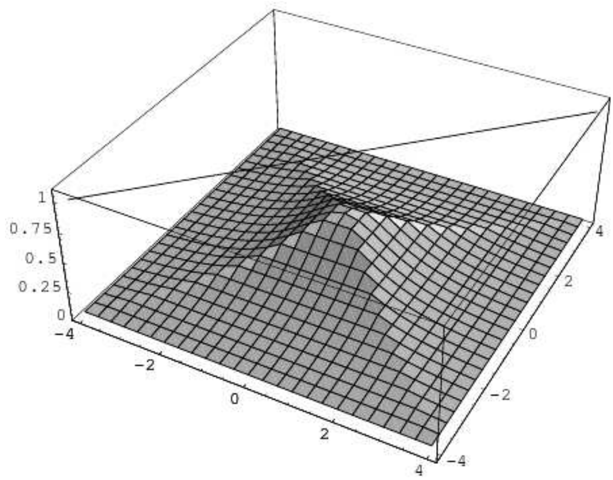

Fig. 2. Figure 2.

4. SIMILARITY TO PSEUDOMETRIC AND BACKWARDS

If we restrict ourselves to the pseudometrics with values from the interval $\langle 0, 1\rangle$, then a pseudometric d is a binary fuzzy relation, which is reflexive, symmetrical and satisfies the triangular inequality. In what follows we assume d to be such a pseudometric in the space of real numbers $\mathbb{R}$.

In [2] there are investigated some questions, concerning to the relation between similarities and pseudometrics in $\mathbb{R}$. There it is proved, that if S is a similarity, defined by means of (1), then the fuzzy relation $R = 1 - S$ is a pseudometric (Proposition 5 in [2]).

Continuing this investigation of relationship between fuzzy quantities, similarities and pseudometrics, we can simply prove

Proposition 2.
Let f be a fuzzy quantity. Then the real binary fuzzy relation d_f, defined for each $x, y \in \mathbb{R}$ by

$$d_f(x,y) = \begin{cases} 0, & \text{if } x = y, \\ \max(f(x), f(y)), & \text{if } x \neq y \end{cases} \tag{3}$$

is a pseudometric.

Proof. It follows easily from definition of the pseudometric.

In general, if d is a pseudometric, then the fuzzy relation $R = 1 - d$ need not be a similarity [Example 3 in [2]]. It always satisfies a little weaker type of fuzzy transitivity:

Proposition 3.
Let d be a pseudometric. Then the fuzzy relation R, defined for each $x, y \in$

$\mathbb{R}$ by $R = 1 - d$ is T_L-equivalence relation, i.e. fuzzy equivalence relation w.r.t. Lukasiewicz t-norm.

Proof. We have only to show, that R is T_L-transitive. From triangular inequality of d we have:

$$1 - R(x,z) + 1 - R(y,z) \geq 1 - R(x,y) \;\Rightarrow\; R(x,z) + R(y,z) - 1 \leq R(x,y)$$

It follows

$$T_L(R(x,z), R(y,z)) \leq R(x,y),\ \forall\, x,y,z \in \mathbb{R}$$

In case if pseudometric is defined like in Proposition 2, then there is a one-to-one correspondence between families of pseudometrics and similarities.

Proposition 4.
Let f be a fuzzy quantity and d_f be the pseudometric defined by (3). Then the fuzzy relation R, defined for each $x, y \in \mathbb{R}$ by $R = 1 - d_f$ is the similarity S_{1-f}, defined by (2) and conversely, if S_f is the similarity defined by (2), then the fuzzy relation R, defined for each $x, y \in \mathbb{R}$ by $R = 1 - S_f$ is the pseudometric d_{1-f} defined by (3).

Proof. It follows from the equality

$$\max(f(x), f(y)) = 1 - \min(1 - f(x), 1 - f(y)),$$

from Proposition 2 and finally from the fact, that f is a fuzzy quantity just if $1 - f$ is a fuzzy quantity.

Pseudometric d_f, for any fuzzy quantity f, seems to be a little unusual, compared with standard metrics or pseudometrics in $\mathbb{R}$. Directly from its definition it follows

Proposition 5.
Let d_f be a pseudometric defined by (3). Then

$$\max(d_f(x,z), d_f(y,z)) \geq d_f(x,y),\ \forall\, x,y,z \in \mathbb{R}$$

Corollary 1.
Let d_f be a pseudometric defined by (3). Then for arbitrary three real numbers x, y, z it holds:

$$d_f(x,y) = d_f(y,z) \geq d_f(x,z) \bigvee d_f(x,y) = d_f(x,z) \geq d_f(y,z) \bigvee$$

$$\bigvee d_f(x,z) = d_f(z,y) \geq d_f(x,y)$$

Corollary 2.
Let d_f be a pseudometric defined by (3). Then for arbitrary sequence of real numbers $\{x_n\}$ and for arbitrary real number x_0 it holds:

$$\lim_{n\to\infty} d_f(x_n, x_0) = 0 \;\Leftrightarrow\; \lim_{n\to\infty} f(x_n) = 0 \bigwedge f(x_0) = 0$$

It means, that the only convergent sequences in a pseudometric d_f are sequences with the correspondent sequence of functional values f converging to 0 and d_f-limits of these sequences are all numbers at which f vanishes. If $f(x_0) \neq 0$, then x_0 is a d_f-isolated point.

Moreover, if fuzzy quantity (function) $1 - f$ is convex, then as follows from Proposition 4 in [1], the following proposition is valid.

Proposition 6.
Let d_f be a pseudometric defined by (3), let $1-f$ be convex fuzzy quantity and let $x, y \in \mathbb{R},\ x < y$. Then in the interval $\langle x, y\rangle$ there exists a sequence of real numbers couples $\{x_n, y_n\}$, such that $\lim\limits_{n\to\infty} |x_n - y_n| = 0$ and $d_f(x_n, y_n) = d_f(x, y)$ for each natural n.

Geometric consequence of this proposition is illustrated with Figure 3 showing graph of d_f on $\langle -4, 4\rangle \times \langle -4, 4\rangle$ and contour lines of this graph, for

$$f(x) = \begin{cases} \frac{|x|-1}{|x|}, & \text{if } |x| > 1, \\ 0, & \text{if } |x| \leq 1 \end{cases}$$

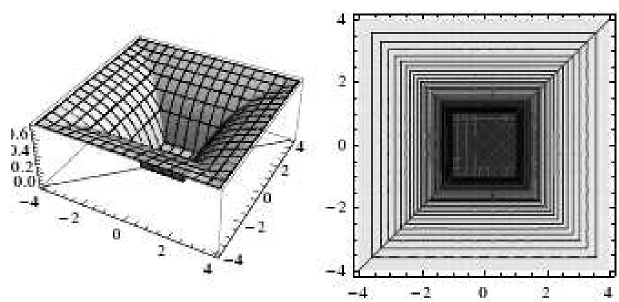

Fig. 3. Figure 3.

It is simple to see, that the function

$$1 - f(x) = \begin{cases} \frac{1}{|x|}, & \text{if } |x| > 1, \\ 1, & \text{if } |x| \leq 1 \end{cases}$$

is convex, hence the assumption and therefore also the assertion of Proposition 6 is fulfilled.

Notice that all contour lines are ïntersecting"the straight line $y = x$. It is straightforward geometrical consequence of Proposition 6.

5. ACKNOWLEDGEMENT

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REFERENCES

[1] Dobrakovová J.: *Order Preserving Similarities*, JAMSI Vol.4, No.2, 2008, pp. 175-182.
[2] Dobrakovová J.: *Pseudometrics and Fuzzy Relations*, APLIMAT 2009, Bratislava, pp. 375-380.
[3] Janiš V.: *Fixed points of fuzzy functions*, Tatra Mountains Math. Publ., 12, 1997, pp. 13-20.
[4] Kalina M.: *Derivatives of fuzzy functions and fuzzy derivatives*, Tatra Mountains math. Publ., 12, 1997, pp. 27-34.
[5] Kalina M.: *Nearness relations in linear spaces*, Kybernetika 40, 2004, pp. 441-458.
[6] Klement E. P., Mesiar R., Pap E.: *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.

Jana Dobrakovová,
Department of Mathematics, Slovak University of Technology,
nám. Slobody 17, 812 31 Bratislava,
Slovak Republic,
e-mail: jana.dobrakovova@stuba.sk

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