ON INTEGRAL COMPLETE R−PARTITE GRAPHS

PAVEL HÍC AND MILAN POKORNÝ

Abstract

A graph $G$ is called integral if all the eigenvalues of its adjacency matrix are integers. In this paper we investigate integral complete $r$−partite graphs $K_{p_1,p_2,...,p_r} = K_{a_1p_1,a_2p_2,...,a_sp_s}$, with $s \leq 4$. New sufficient conditions for complete 3-partite graphs and complete 4-partite graphs to be integral are given. From these conditions we construct infinitely many new classes of integral complete $r$−partite graphs for $s = 3, 4$. Moreover, the summary of our results about integral complete 3-partite graphs and integral complete 4-partite graphs can be found in the paper.

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Additional Key Words and Phrases: Integral Graph, Graph Spectrum, Complete $r$−partite Graph, Diophantine Equation

1. INTRODUCTION

The notion of integral graphs was introduced by F. Harary and A.J.Schwenk in 1974 (see [5]). A graph $G$ is called integral if all the zeros of the characteristic polynomial $P(G, x)$ are integers. In general, the problem of characterizing integral graphs seems to be very difficult. Thus, it makes sense to restrict our investigations to some families of graphs, such as cubic graphs, trees, etc. The survey of results on integral complete graphs and integral complete multigraphs were presented in [2, 11].

It has recently been discovered that integral graphs may be of interest for designing the network topology of perfect state transfer networks, see [1, 3].

Complete $n$−partite graphs presents an important family of graphs for which the problem has been considered in [8, 9, 10, 11, 12, 13]. A complete $r$−partite graph $K_{p_1,p_2,...,p_r}$ is a graph with a set $V = V_1 \cup V_2 \cup ... \cup V_r$ of $p_1 + p_2 + ... + p_r = n$ vertices, where $V_i$'s are nonempty disjoint sets, $|V_i| = p_i$ for $1 \leq i \leq r$, such that two vertices in $V$ are adjacent if and only if they belong to different $V_i$'s. Assume that the number of distinct integers of $p_1, p_2, ..., p_r$ is $s$. Without loss of generality, assume that the first $s$ ones are the distinct integers such that $p_1 < p_2 < ... < p_s$.

Suppose that $a_i$ is the multiplicity of $p_i$ for $i = 1, 2, ..., s$. The complete $r$−partite graph $K_{p_1,p_2,...,p_r} = K_{p_1p_1,...,p_sp_s}$ is also denoted by $K_{a_1p_1,a_2p_2,...,a_sp_s}$, where $r = \sum_{i=1}^{s} a_i$ and $|V| = n = \sum_{i=1}^{s} a_i p_i$.

In this paper we give a survey of results on integral complete $r$−partite graphs. We can construct infinitely many new classes of integral complete multipartite graphs for $s = 3, 4$ from known integral complete multipartite graphs by solv-
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...ing some certain Diophantine equations. These results are different from those in the existing literature. The problem of existence of integral complete multipartite graphs $K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s}$ with arbitrarily large number $s \geq 5$ remain open.

We shall consider only simple undirected graphs. For a graph $G$, let $V(G)$ denote the vertex set and $E(G)$ the edge set. The characteristic polynomial $|xI - A|$ of the adjacency matrix $A$ of $G$ is called the characteristic polynomial of $G$ and is denoted by $P(G, x)$. The spectrum of $A$ is also called the spectrum of $G$. For all other facts on terminology of graph spectra see [4].

2. PRELIMINARIES

Now, we shall state some known results on integral complete multipartite graphs.

Theorem 2.1 ([12]) The complete $r$-partite graph $K_{p_1, p_2, \ldots, p_r} = K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s}$ on $n$ vertices is integral if and only if there exist integers $u_i$ and positive integers $p_i (i = 1, 2, \ldots, s)$ such that $-p_s < u_s < -p_{s-1} < u_{s-1} < \ldots < -p_2 < u_2 < -p_1 < u_1 < +\infty$ and

$$a_k = \prod_{i=1}^{s}(p_k + u_i) \prod_{i \neq k}(p_k - p_i), k = 1, 2, \ldots, s$$

are positive integers.

Theorem 2.2 ([12]) For any positive integer $q$, the complete multipartite graph $K_{a_1 p_1 q, a_2 p_2 q, \ldots, a_s p_s q}$ is integral if and only if the complete $r$-partite graph $K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s}$ is integral.

Remark. Theorem 2.2 shows that it is reasonable to study (2.1) only when $(p_1, p_2, \ldots, p_s) = 1$. Let us call such vector primitive. So, in general, the primitive vectors are only ones which are of interest.

3. INTEGRAL COMPLETE GRAPHS AND INTEGRAL COMPLETE BIPARTITE GRAPHS

It is very easy to verify that the following theorems are true (see for example [4, 5, 11, 12].

Theorem 3.1 A complete graph $K_{p_1}$ is integral for every positive integer $p_1$.

Theorem 3.2 For the complete bipartite graph $K_{p_1, p_2} = K_{a_1 p_1, a_2 p_2}$ on $n = p_1 + p_2$ vertices, $a_1 = a_2 = 1$, the graph $K_{p_1, p_2}$ is integral if and only if $p_1 p_2$ is a perfect square.

From above theorems it follows that it is interesting to study only complete multipartite graphs $K_{a_1 p_1, a_2 p_2, \ldots, a_s p_s}$ with $s \geq 3$.

4. INTEGRAL COMPLETE 3-PARTITE GRAPHS

In this part we give some results concerning to integral 3-partite graphs. First infinite family of integral complete 3-partite graphs was constructed in [10], where the author mentioned the general problem on integral complete multipartite graphs.
ON INTEGRAL COMPLETE $\tau$-PARTITE GRAPHS

Theorem 4.1 ([10]) Let $p_1 = 4u^2(u^2 + v^2)^3, p_2 = 3u^2v^2(u^2 + 6uv + v^2)(-u^2 + 6uv - v^2), p_3 = 4v^2(u^2 + v^2)^3$ such that $3 - \sqrt{8}v < u < v$, and let $u_1 = 24u^2v^2(u^2 + v^2)^2, u_2 = -2uv(u^2 + v^2)^2(-u^2 + 6uv - v^2), u_3 = -2uv(u^2 + v^2)^2(u^2 + 6uv + v^2)$, such that $u, v$ are positive integers. Then $K_{p_1, p_2, p_3}$ is integral.

The above theorem gives infinitely many integral complete 3-partite graphs $K_{p_1, p_2, p_3}$.

From the theory of divisors and co-divisors of a graph follows that the divisor of a graph $K_{p_1, p_2, p_3}$ has the characteristic polynomial $P(D, x) = x^3 - (p_1 p_2 + p_1 p_3 + p_2 p_3)x - 2p_1 p_2 p_3$. Moreover, the characteristic polynomial of the co-divisor is $P(C, x) = x^{p_1 + p_2 + p_3 - 3}$. More details about the theory of divisors and co-divisors can be found in [4,6,7]. Thus, the following theorem holds:

Theorem 4.2 ([9]) A graph $K_{p_1, p_2, p_3}$ is integral if and only if the characteristic polynomial of its divisor has only integer zeros, which means that the zeros $u_1, u_2, u_3$ have to satisfy the following equations:

\[
\begin{align*}
&u_1 + u_2 + u_3 = 0; u_2 < 0; u_3 < 0 \\
&u_1 u_2 + u_1 u_3 + u_2 u_3 = -p_1 p_2 - p_1 p_3 - p_2 p_3 \\
&u_1 u_2 u_3 = 2p_1 p_2 p_3.
\end{align*}
\]

From 2.2 it follows that if $p_1 < p_2 < p_3$ and $u_1, u_2, u_3$ are the zeros of $P(D, x)$, then

\[
\begin{align*}
&u_1 u_2 u_3 = 2p_1 p_2 p_3. \\
&-p_3 < u_3 < -p_2 < u_2 < -p_1 < 0 < u_1.
\end{align*}
\]

Using a computer we have found 137 primitive integral complete 3-partite graphs $K_{p_1, p_2, p_3}$, where $p_1 < p_2 < p_3$, $1 \leq p_1 \leq 1000, p_1 + 1 \leq p_2 \leq 1000, p_2 + 1 \leq p_3 \leq 1000$. Some of them are in Table 1.

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<th>No.</th>
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<th>$p_3$</th>
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Table 1

Analyzing these 137 primitive integral complete 3-partite graphs, one can see that 62 of them have the property $u_3 = -2p_2$ (see for example graphs No. 1-6 in Table 1), 62 of them have the property $u_3 = -2p_1$ (see for example graphs No. 7-11 in Table 1).

Based on similar exploration, but on a sample of only 34 complete 3-partite...
integral graphs, the authors of [12] proved the following theorem using (4.1) and
the assumption \( u_3 = -2p_1 \) or \( u_3 = -2p_2 \).

Theorem 4.3 ([12]) Let \( p_1 = \frac{a^2 + b^2}{d}, \ p_2 = \frac{2a(2a + b)}{d}, \ p_3 = \frac{2b(2b - a)}{d}, \ u_1 = \frac{2b(b + 2a)}{d}, \ u_2 = -\frac{2a(2b - a)}{d}, \ u_3 = -\frac{2(a^2 + b^2)}{d} \), where \( a > 0, b > 0, d = (2b(2b - a), 2a(2a + b), a^2 + b^2) \). Then \( K_{p_1, p_2, p_3} \) is integral, where \( d \in \{ 1, 2, 5, 10 \} \).

Remark. From 137 primitive integral complete 3-partite graphs found by computer, all 124 graphs \( K_{p_1, p_2, p_3} \), where \( u_3 = -2p_1 \) or \( u_3 = -2p_2 \) can be obtained using Theorem 4.3 for suitable \( a \) and \( b \). Moreover, the graph No. 16 from Table 1 can be obtained by Theorem 4.1.

Using (4.1) and (4.2) it is clear that none of the following cases can happen: \( u_3 = -2p_3, u_2 = -2p_1, u_2 = -2p_2, u_2 = -2p_3, u_1 = 2p_1, u_1 = 2p_3 \). From Table 1 we can see that there exist integral complete 3-partite graphs where \( u_1 = 2p_2 \).

In [9] these graphs are investigated and sufficient conditions for these graphs to be integral are given.

Theorem 4.4 ([9]) Let \( p_1 = \frac{2s^2(s^2 - t^2)}{dk^2}, p_2 = \frac{4t^2}{dk^2}, p_3 = \frac{2t^2(9t^2 - s^2)}{dk^2}, \) where \( t < s < \sqrt{3}, dk^2 = (2s^2(s^2 - t^2), 4t^2s^2, 2t^2(9t^2 - s^2)) \). Then the graph \( K_{p_1, p_2, p_3} \) is integral and zeros of its divisor are \( u_1 = \frac{8t^2}{dk^2}, u_2 = \frac{2s^2(s^2 - 3t^2)(s + t)}{dk^2}, u_3 = \frac{2t^2(s^2 + 3t^2)(t - s)}{dk^2} \).

For the values (\( s, t \)) \( \in \{ (2, 3), (3, 4), (3, 5), (5, 6), (7, 9) \} \) we get graphs No. 11-15 in Table 1. Another graphs for \( t < 9 \) can be found in Table 2. Let us remark that for graph No. 11 in Table 1 it holds both \( u_1 = 2p_2 \) and \( u_3 = -2p_1 \).

| No. | t | s | \( p_1dk^2 \) | \( p_2dk^2 \) | \( p_3dk^2 \) | \( dk^2 \) | \( p_1 \) | \( p_2 \) | \( p_3 \) | \( u_1 \) | \( u_2 \) | \( u_3 \) |
|-----|---|---|----------------|----------------|----------------|-------|-------|-------|-------|-------|-------|
| 1   | 2 | 3 | 90             | 144            | 216            | 18    | 5     | 8     | 12    | 16    | -10   | -6    |
| 2   | 3 | 4 | 224            | 576            | 1170           | 2     | 112   | 288   | 585   | 576   | -420  | -156  |
| 3   | 3 | 5 | 800            | 900            | 1008           | 4     | 200   | 225   | 252   | 450   | -240  | -210  |
| 4   | 4 | 5 | 1600           | 1600           | 3808           | 2     | 225   | 800   | 1904  | 1600  | -1260 | -340  |
| 5   | 5 | 6 | 792            | 3600           | 9450           | 18    | 44    | 200   | 525   | 400   | -330  | -70   |
| 6   | 5 | 7 | 2352           | 4900           | 8800           | 4     | 588   | 1225  | 2200  | 2450  | -1680 | -770  |
| 7   | 5 | 8 | 4992           | 6400           | 8050           | 2     | 2496  | 3200  | 4025  | 6400  | -3640 | -2760 |
| 8   | 6 | 7 | 1274           | 7056           | 19800          | 2     | 637   | 3528  | 9900  | 7056  | -6060 | -1050 |
| 9   | 7 | 8 | 1920           | 12544          | 36946          | 2     | 960   | 6272  | 18473 | 12544 | -10920| -1624 |
| 10  | 7 | 9 | 5184           | 15876          | 35280          | 36    | 144   | 441   | 980   | 882   | -672  | -210  |
| 11  | 7 | 10 | 10200         | 19600          | 33418          | 2     | 5100  | 9800  | 16709 | 19600  | -13090| -6510 |
| 12  | 7 | 11 | 17424         | 23716          | 31360          | 4     | 4356  | 5929  | 7840  | 11858  | -6930  | -4928 |
| 13  | 7 | 12 | 27360         | 28224          | 29106          | 18    | 1520  | 1568  | 1617  | 3136  | -1596  | -1540 |
| 14  | 8 | 9 | 2754           | 20736          | 63360          | 18    | 153   | 1152  | 3520  | 2304   | -2040  | -264  |
| 15  | 8 | 11 | 13794         | 30976          | 58240          | 2     | 6897  | 15488 | 29120 | 30976  | -21736 | -9240 |
| 16  | 8 | 13 | 35490         | 32364          | 52096          | 2     | 17475 | 21632 | 26048 | 43264  | -24024 | -19240 |

Table 2

Using Theorem 4.2 in Theorem 4.4 we get the following result.
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Corollary 4.5 For every $p_1, p_2, p_3$, where $p_1 = \frac{2s^2(t^2-t^3)}{ak^2}, p_2 = \frac{q^2s^2}{ak^2}, p_3 = \frac{2t^2(9t^2-s^2)}{ak^2}, t < s < \sqrt{3}tk^2 = (2s^2(t^2-t^2), 4t^2s^2, 2t^2(9t^2-s^2))$ and for arbitrary $q \in N$ the graph $K_{p_1q,p_2q,p_3q}$ is integral complete 3-partite graph.

The following theorem shows how we can construct an infinite class of integral graphs $K_{a_1p_1,a_2p_2,a_3p_3}$ from the known integral graph $K_{p_1,p_2,p_3}$ using Diophantine equations.

Theorem 4.6 Let $p_i > 0, u_i (i = 1, 2, 3)$ be those of No.1 in table 1. Then for any nonnegative $t$ the graph $K_{a_1p_1,a_2p_2,a_3p_3}$ is integral if $p_i, a_i, u_i (i = 1, 2, 3)$ are given by the following:

$$p_1 = 3, p_2 = 17, p_3 = 65,$$
$$u_1 = 39 + 2184t, u_2 = -5, u_3 = -34,$$
$$a_1 = 1 + 52t, a_2 = 1 + 39t, a_3 = 1 + 21t.$$

Proof.
From table 1 No. 1 we have $p_1 = 3, p_2 = 17, p_3 = 65, u_1 = 39, u_2 = -5, u_3 = -34$. By (2.1) in Theorem 2.1, we get

$$a_1 = \frac{(p_1+u_1)(p_1+u_2)(p_1+u_3)}{p_1(p_1-p_2)(p_1-p_3)} = \frac{3+u_1}{42},$$
$$a_2 = \frac{(p_2+u_1)(p_2+u_2)(p_2+u_3)}{p_2(p_2-p_1)(p_2-p_3)} = \frac{17+u_2}{56},$$
$$a_3 = \frac{(p_3+u_1)(p_3+u_2)(p_3+u_3)}{p_3(p_3-p_1)(p_3-p_2)} = \frac{65+u_3}{104}.$$

So, $K_{a_1p_1,a_2p_2,a_3p_3}$ is integral if and only if $a_i (i = 1, 2, 3)$ and $u_3$ must be positive integers. From (4.3) and (4.4) we get the Diophantine equation

$$4a_2 = 3a_1 = 1.$$

From elementary number theory follows that all positive integral solutions of (4.6) are given by $a_1 = 1 + 4t_1, a_2 = 1 + 3t_1, u_1 = 39 + 168t_1$, where $t_1$ is a nonnegative integer.

Using $u_1 = 39 + 168t_1$ in (4.5) we get $a_3 = 1 + \frac{21}{4}t_1$. As $a_3$ must be a nonnegative integer, $t_1 = 13t$. Then we get $a_1 = 1 + 52t, a_2 = 1 + 39t, a_3 = 1 + 21t, u_1 = 39 + 2184t$.

From theorem 2.1 it follows that $K_{a_1p_1,a_2p_2,a_3p_3}$ is an integral graph.

Remarks:
1. It is possible to use the analogous procedure on each graph from tables 1, 2, as well as on each of 137 integral complete 3-partite graphs mentioned below Theorem 4.2, and we get from each of these graphs the infinite class of integral graphs.
2. Similarly, using the analogous procedure on each graph from Theorems 4.1, 4.3, and 4.4 we get the infinite class of integral graphs.
3. In [12,13] other classes of integral graphs $K_{a_1p_1,a_2p_2,a_3p_3}$ different from those in this paper are constructed.

5. INTEGRAL COMPLETE $r$-PARTITE GRAPHS

Roitman in [10] conjectured that for $r > 3$ there exist an infinite number of integral $r$-partite graphs. However, first integral complete 4-partite graphs was given in
From the theory of divisors and co-divisors of a graph follows that the divisor of the graph $K_{p_1,p_2,p_3,p_4}$ has characteristic polynomial

$$P(D, x) = x^4 - (p_1p_2 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4)x^2 - 2(p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4)x - 3p_1p_2p_3p_4.$$ 

Moreover, the characteristic polynomial of the co-divisor is $P(C, x) = x^{p_1+p_2+p_3+p_4-4}$. 

The following theorem holds.

**Theorem 5.1** ([8]) A graph $K_{p_1,p_2,p_3,p_4}$ is integral if and only if the characteristic polynomial $P(D, x)$ of its divisor has only integer zeros.

From Theorem 2.1 it follows that if $p_1 < p_2 < p_3 < p_4$ and $u_1, u_2, u_3$, and $u_4$ are the zeros of $P(D, x)$, then $-p_4 < u_1 < -p_3 < u_2 < -p_2 < u_3 < -p_1 < 0 < u_1$ and $u_1 + u_2 + u_3 + u_4 = 0$.

Using a computer we found the first known integral complete 4-partite graphs. We explored all graphs $K_{p_1,p_2,p_3,p_4}$ where $p_1 < p_2 < p_3 < p_4 < 1000$, which means that we explored billions of graphs. For each of these graphs we calculated the characteristic polynomial of its divisor $P(D, x)$. Then we verified whether the polynomial has integer root $u_4$ satisfying a condition $-p_4 < u_4 < -p_3$ and in positive case we similarly verified integer roots $u_3$ and $u_2$ satisfying $-p_3 < u_3 < -p_2 < u_2 < -p_1$. Only two of the graphs we explored are integral. Their list is in the table 3.

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Table 3

Only the graph in the first column after the heading is primitive. The graph in the second column can be obtained using Theorem 5.2, which follows from Theorem 2.2, for $q = 2$. The spectrum $Spec(K_{117q,261q,352q,495q}) = \{-429q, -297q, -144q, 0, 870q\}$.

**Theorem 5.2** ([8]) Let $p_1 = 117, p_2 = 261, p_3 = 352, p_4 = 495$. Then for any positive integer $q$ the graph $K_{p_1q,p_2q,p_3q,p_4q}$ is integral.

Using analogous procedure than in Theorem 4.6 we get the following theorem.

**Theorem 5.3** Let $p_i > 0, u_i(i = 1, 2, 3, 4)$ be those of Theorem 5.2. Then for any nonnegative $t$ the graph $K_{a_1p_1,a_2p_2,a_3p_3,a_4p_4}$ is integral if $p_i, a_i(i = 1, 2, 3, 4)$ are given by following:

- $p_1 = 117, p_2 = 261, p_3 = 352, p_4 = 495,$
- $u_1 = 870 + 48372870t, u_2 = -429, u_3 = -297, u_4 = -144$.
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\[ a_1 = 1 + 49010t, \quad a_2 = 1 + 42770t, \quad a_3 = 1 + 39585t, \quad a_4 = 1 + 35438t. \]

Proof.

For \( s = 4 \) by Theorem 5.2, we have \( p_1 = 117, p_2 = 261, p_3 = 352, p_4 = 495, u_2 = -429, u_3 = -297, u_4 = -144 \). By (2.1) Theorem 2.1 we get

\[
\text{(5.1)} \quad a_1 = \frac{(p_1 + u_1)(p_1 + u_2)(p_1 + u_3)(p_1 + u_4)}{p_1(p_1 - p_2)(p_1 - p_3)(p_1 - p_4)} = \frac{1}{99}(117 + u_1)
\]

\[
\text{(5.2)} \quad a_2 = \frac{(p_2 + u_1)(p_2 + u_2)(p_2 + u_3)(p_2 + u_4)}{p_2(p_2 - p_3)(p_2 - p_4)} = \frac{1}{111}(261 + u_1)
\]

\[
\text{(5.3)} \quad a_3 = \frac{(p_3 + u_1)(p_3 + u_2)(p_3 + u_3)(p_3 + u_4)}{p_3(p_3 - p_4)} = \frac{1}{1222}(352 + u_1)
\]

\[
\text{(5.4)} \quad a_4 = \frac{(p_4 + u_1)(p_4 + u_2)(p_4 + u_3)(p_4 + u_4)}{p_4(p_4 - p_1)(p_4 - p_2)(p_4 - p_3)} = \frac{1}{1365}(495 + u_1)
\]

So, \( K_{a_1p_1, a_2p_2, a_3p_3, a_4p_4} \) is integral if and only if \( a_i (i = 1, 2, 3, 4) \) are positive integers, and \( u_1 \) must be a positive integer. From (5.1), (5.2), (5.3), and (5.4), we get the Diophantine equations

\[
\text{(5.5)} \quad 377a_2 - 329a_1 = 48
\]

\[
\text{(5.6)} \quad 1365a_4 - 1222a_3 = 143.
\]

From elementary number theory follows that all positive integral solutions of (5.5) and (5.6) are given respectively by \( a_1 = 1 + 377t_1, a_2 = 1 + 329t_1, u_1 = 870 + 372099t_1 \) and \( a_1 = 1 + 1365t_2, a_3 = 1 + 1222t_2, u_1 = 870 + 1668030t_2 \), where \( t_1 \) and \( t_2 \) are nonnegative integers. Hence \( u_1 = 870 + 372099t_1 = 870 + 1668030t_2 \) must be a positive integer. It deduces that \( t_1 = 130t \) and \( t_2 = 29t \), where \( t \) is a nonnegative integer. By above discussion, we have \( a_1 = 1 + 49010t, a_2 = 1 + 42770t, a_3 = 1 + 39585t, a_4 = 1 + 35438t, u_1 = 870 + 48372870t \). Hence, when \( p_1 = 117, p_2 = 261, p_3 = 352, p_4 = 495, a_1 = 1 + 49010t, a_2 = 1 + 42770t, a_3 = 1 + 39585t, a_4 = 1 + 35438t, u_1 = 870 + 48372870t \). Hence, when \( p_1 = 117, p_2 = 261, p_3 = 352, p_4 = 495, a_1 = 1 + 49010t, a_2 = 1 + 42770t, a_3 = 1 + 39585t, a_4 = 1 + 35438t, u_1 = 870 + 48372870t \).

By Theorem 2.3 we have the following theorem:

Theorem 5.4 Let \( p_i > 0, u_i, a_i (i = 1, 2, 3, 4) \) be those of Theorem 5.3. Then for any positive integer \( q \) and any nonnegative \( t \) the graph \( K_{a_1p_1, a_2p_2, a_3p_3, a_4p_4} \) is integral.

Remark. In [13] other classes of integral graphs \( K_{a_1p_1q, a_2p_2q, a_3p_3q, a_4p_4q} \) different from these were constructed.

6. CONCLUSION

In this paper, the integral multipartite graphs \( K_{a_1p_1, a_2p_2, \ldots, a_sp_s} \) for \( s = 3, 4 \) are presented. For \( s = 1, 2, 3, 4 \), some results of such integral graphs can be found in [8, 9, 10, 11, 12, 13] and in the present paper. When \( s \geq 5 \), we have not found such integral graphs. Thus, we raise the following questions.

Question 6.1 (see also [12]) Are there any integral complete \( r \)-partite graphs \( K_{p_1, p_2, \ldots, p_r} = K_{a_1p_1, a_2p_2, \ldots, a_sp_s} \) with arbitrarily large \( s \)?

For complete \( r \)-partite graphs \( K_{p_1, p_2, \ldots, p_r} = K_{a_1p_1, a_2p_2, \ldots, a_sp_s} \), when \( s = 1, 2, 3, 4, \ldots \).
let \( a_1 = a_2 = \ldots = a_s = 1 \), some results of such integral graphs can be found in [8, 9, 10, 12]. However, when \( s \geq 5 \), \( a_1 = a_2 = \ldots = a_s = 1 \), we have not found such integral graphs.

Hence, we have

**Question 6.2** Are there any integral complete \( r \)-partite graphs \( K_{p_1,p_2,\ldots,p_r} = K_{a_1 p_1, a_2 p_2,\ldots,a_s p_s} \) with \( a_1 = a_2 = \ldots = a_s = 1 \) when \( s \geq 5 \)?

Theorems 4.1, 4.3, and 4.4 give sufficient conditions by which we can construct infinite classes of integral complete 3-partite graphs. However, by these conditions we are not able to generate integral graphs \( K_{p_1,p_2,p_3} \), where \((p_1,p_2,p_3) \in \{(5,12,77), (25,297,675), (33,98,833), (45,136,396), (49,220,441), (216,385,684), (297,437,585), (693,760,828)\} \). These integral graphs were obtained using computers and are in Table 1, No. 17-24.

**Question 6.3** Is it possible to find sufficient conditions by which we can generate these graphs?

Characterization of integral complete 3-partite graphs remains open.

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