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ON INTEGRAL COMPLETE $R-$PARTITE GRAPHS

PAVEL HÍC AND MILAN POKORNÝ

Abstract

A graph G is called integral if all the eigenvalues of its adjacency matrix are integers. In this paper we investigate integral complete $r-$partite graphs $K_{p_1,p_2,...,p_r} = K_{a_1p_1,a_2p_2,...,a_sp_s}$ with $s \leq 4$. New sufficient conditions for complete 3-partite graphs and complete 4-partite graphs to be integral are given. From these conditions we construct infinitely many new classes of integral complete $r-$partite graphs for $s = 3, 4$. Moreover, the summary of our results about integral complete 3-partite graphs and integral complete 4-partite graphs can be found in the paper.

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1. INTRODUCTION

The notion of integral graphs was introduced by F. Harary and A.J.Schwenk in 1974 (see [5]). A graph G is called integral if all the zeros of the characteristic polynomial $P(G, x)$ are integers. In general, the problem of characterizing integral graphs seems to be very difficult. Thus, it makes sense to restrict our investigations to some families of graphs, such as cubic graphs, trees, etc. The survey of results on integral trees and integral graphs were presented in [2, 11].

It has recently been discovered that integral graphs may be of interest for designing the network topology of perfect state transfer networks, see [1, 3].

Complete $n-$partite graphs presents an important family of graphs for which the problem has been considered in [8, 9, 10, 11, 12, 13]. A complete $r-$partite graph $K_{p_1,p_2,...,p_r}$ is a graph with a set $V = V_1 \cup V_2 \cup ... \cup V_r$ of $p_1 + p_2 + ... + p_r = n$ vertices, where V_i's are nonempty disjoint sets, $|V_i| = p_i$ for $1 \leq i \leq r$, such that two vertices in V are adjacent if and only if they belong to different V_i's. Assume that the number of distinct integers of $p_1, p_2, ..., p_r$ is s. Without loss of generality, assume that the first s ones are the distinct integers such that $p_1 < p_2 < ... < p_s$. Suppose that a_i is the multiplicity of p_i for $i = 1, 2, ..., s$. The complete $r-$partite graph $K_{p_1,p_2,...,p_r} = K_{p_1,..,p_1,...,p_s,...,p_s}$ is also denoted by $K_{a_1p_1,a_2p_2,...,a_sp_s}$, where $r = \sum_{i=1}^{s} a_i$ and $|V| = n = \sum_{i=1}^{s} a_ip_i$.

In this paper we give a survey of results on integral complete $r-$partite graphs. We can construct infinitely many new classes of integral complete multipartite graphs for $s = 3, 4$ from known integral complete multipartite graphs by solv-

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ing some certain Diophantine equations. These results are different from those in the existing literature. The problem of existence of integral complete multipartite graphs $K_{a_1p_1,a_2p_2,...,a_sp_s}$ with arbitrarily large number $s \geq 5$ remain open.

We shall consider only simple undirected graphs. For a graph G, let $V(G)$ denote the vertex set and $E(G)$ the edge set. The characteristic polynomial $|xI - A|$ of the adjacency matrix A of G is called the characteristic polynomial of G and is denoted by $P(G, x)$. The spectrum of A is also called the spectrum of G. For all other facts on terminology of graph spectra see [4].

2. PRELIMINARIES

Now, we shall state some known results on integral complete multipartite graphs.

Theorem 2.1 ([12]) The complete $r-$partite graph $K_{p_1,p_2,...,p_r} = K_{a_1p_1,a_2p_2,...,a_sp_s}$ on n vertices is integral if and only if there exist integers u_i and positive integers p_i $(i = 1, 2, ..., s)$ such that $-p_s < u_s < -p_{s-1} < u_{s-1} < ... < -p_2 < u_2 < -p_1 < u_1 < +\infty$ and

$$a_k = \frac{\prod_{i=1}^{s}(p_k+u_i)}{p_k \prod_{i=1, i\neq k}^{s}(p_k-p_i)}, k = 1, 2, ..., s \tag{2.1}$$

are positive integers.

Theorem 2.2 ([12]) For any positive integer q, the complete multipartite graph $K_{a_1p_1q,a_2p_2q,...,a_sp_sq}$ is integral if and only if the complete $r-$partite graph $K_{a_1p_1,a_2p_2,...,a_sp_s}$ is integral.

Remark. Theorem 2.2 shows that it is reasonable to study (2.1) only when $(p_1, p_2, ..., p_s) = 1$. Let us call such vector primitive. So, in general, the primitive vectors are only ones which are of interest.

3. INTEGRAL COMPLETE GRAPHS AND INTEGRAL COMPLETE BIPARTITE GRAPHS

It is very easy to verify that the following theorems are true (see for example [4,5,11,12].

Theorem 3.1 A complete graph K_{p_1} is integral for every positive integer p_1.

Theorem 3.2 For the complete bipartite graph $K_{p_1,p_2} = K_{a_1p_1,a_2p_2}$ on $n = p_1+p_2$ vertices, $a_1 = a_2 = 1$, the graph K_{p_1,p_2} is integral if and only if p_1p_2 is a perfect square.

From above theorems it follows that it is interesting to study only complete multipartite graphs $K_{a_1p_1,a_2p_2,...,a_sp_s}$ with $s \geq 3$.

4. INTEGRAL COMPLETE 3-PARTITE GRAPHS

In this part we give some results concerning to integral 3-partite graphs. First infinite family of integral complete 3-partite graphs was constructed in [10], where the author mentioned the general problem on integral complete multipartite graphs.

Theorem 4.1 ([10]) Let $p_1 = 4u^2(u^2+v^2)^3, p_2 = 3u^2v^2(u^2+6uv+v^2)(-u^2+6uv-v^2), p_3 = 4v^2(u^2+v^2)^3$ such that $(3-\sqrt{8})v < u < v$, and let $u_1 = 24u^2v^2(u^2+v^2)^2, u_2 = -2uv(u^2+v^2)^2(-u^2+6uv-v^2), u_3 = -2uv(u^2+v^2)^2(u^2+6uv+v^2)$, such that u, v are positive integers. Then K_{p_1,p_2,p_3} is integral.

The above theorem gives infinitely many integral complete 3-partite graphs K_{p_1,p_2,p_3}.

From the theory of divisors and co-divisors of a graph follows that the divisor of a graph K_{p_1,p_2,p_3} has the characteristic polynomial $P(D,x) = x^3 - (p_1p_2 + p_1p_3 + p_2p_3)x - 2p_1p_2p_3$. Moreover, the characteristic polynomial of the co-divisor is $P(C,x) = x^{p_1+p_2+p_3-3}$. More details about the theory of divisors and co-divisors can be found in [4,6,7]. Thus, the following theorem holds:

Theorem 4.2 ([9]) A graph K_{p_1,p_2,p_3} is integral if and only if the characteristic polynomial of its divisor has only integer zeros, which means that the zeros u_1, u_2, u_3 have to satisfy the following equations:

$$\begin{aligned} u_1 + u_2 + u_3 = 0; u_2 < 0; u_3 < 0 \\ u_1u_2 + u_1u_3 + u_2u_3 = -p_1p_2 - p_1p_3 - p_2p_3 \\ u_1u_2u_3 = 2p_1p_2p_3. \end{aligned} \tag{4.1}$$

From 2.2 it follows that if $p_1 < p_2 < p_3$ and u_1, u_2, u_3 are the zeros of $P(D,x)$, then

$$-p_3 < u_3 < -p_2 < u_2 < -p_1 < 0 < u_1. \tag{4.2}$$

Using a computer we have found 137 primitive integral complete 3-partite graphs K_{p_1,p_2,p_3} , where $p_1 < p_2 < p_3$, $1 \le p_1 \le 1000$, $p_1 + 1 \le p_2 \le 1000$, $p_2 + 1 \le p_3 \le 1000$. Some of them are in Table 1.

$No.$	p_1	p_2	p_3	u_1	u_2	u_3	$No.$	p_1	p_2	p_3	u_1	u_2	u_3
1	3	17	65	39	-5	-34	13	112	288	585	576	-156	-420
2	4	13	48	32	-6	-26	14	144	441	980	882	-210	-672
3	5	53	207	115	-9	-106	15	200	225	252	450	-210	-240
4	7	13	45	35	-9	-26	16	125	357	500	600	-175	-425
5	7	109	429	231	-13	-218	17	5	12	77	40	-7	-33
6	8	65	252	144	-14	-130	18	25	297	675	495	-45	-450
7	13	24	28	42	-16	-26	19	33	98	833	357	-49	-308
8	17	33	35	55	-21	-34	20	45	136	396	306	-66	-240
9	29	36	80	90	-32	-58	21	49	220	441	392	-77	-315
10	29	39	77	91	-33	-58	22	216	385	684	798	-270	-528
11	5	8	12	16	-6	-10	23	297	437	585	858	-345	-513
12	44	200	525	400	-70	-330	24	693	760	828	1515	-720	-798

Table 1

Analyzing these 137 primitive integral complete 3-partite graphs, one can see that 62 of them have the property $u_3 = -2p_2$ (see for example graphs No. 1-6 in Table 1), 62 of them have the property $u_3 = -2p_1$ (see for example graphs No. 7-11 in Table 1).

Based on similar exploration, but on a sample of only 34 complete 3-partite integral graphs, the authors of [12] proved the following theorem using (4.1) and the assumption $u_3 = -2p_1$ or $u_3 = -2p_2$.

Theorem 4.3 ([12]) Let $p_1 = \frac{a^2+b^2}{d}$, $p_2 = \frac{2a(2a+b)}{d}$, $p_3 = \frac{2b(2b-a)}{d}$, $u_1 = \frac{2b(b+2a)}{d}$, $u_2 = -\frac{2a(2b-a)}{d}$, $u_3 = -\frac{2(a^2+b^2)}{d}$, where $a > 0, b > 0, d = (2b(2b-a), 2a(2a+b), a^2+b^2)$. Then K_{p_1,p_2,p_3} is integral, where $d \in \{1, 2, 5, 10\}$.

Remark. From 137 primitive integral complete 3-partite graphs found by computer, all 124 graphs K_{p_1,p_2,p_3}, where $u_3 = -2p_1$ or $u_3 = -2p_2$ can be obtained using Theorem 4.3 for suitable a and b. Moreover, the graph No. 16 from Table 1 can be obtained by Theorem 4.1.

Using (4.1) and (4.2) it is clear that none of the following cases can happen: $u_3 = -2p_3, u_2 = -2p_1, u_2 = -2p_2, u_2 = -2p_3, u_1 = 2p_1, u_1 = 2p_3$. From Table 1 we can see that there exist integral complete 3-partite graphs where $u_1 = 2p_2$.

In [9] these graphs are investigated and sufficient conditions for these graphs to be integral are given.

Theorem 4.4 ([9]) Let $p_1 = \frac{2s^2(s^2-t^2)}{dk^2}, p_2 = \frac{4t^2s^2}{dk^2}, p_3 = \frac{2t^2(9t^2-s^2)}{dk^2}$, where $t < s < \sqrt{3}t, dk^2 = (2s^2(s^2-t^2), 4t^2s^2, 2t^2(9t^2-s^2))$. Then the graph K_{p_1,p_2,p_3} is integral and zeros of its divisor are $u_1 = \frac{8t^2s^2}{dk^2}, u_2 = \frac{2ts(s-3t)(t+s)}{dk^2}, u_3 = \frac{2ts(s+3t)(t-s)}{dk^2}$.

For the values $(s,t) \in \{(2,3),(3,4),(3,5),(5,6),(7,9)\}$ we get graphs No. 11-15 in Table 1. Another graphs for $t < 9$ can be found in Table 2. Let us remark that for graph No. 11 in Table 1 it holds both $u_1 = 2p_2$ and $u_3 = -2p_1$.

$No.$	t	s	p_1dk^2	p_2dk^2	p_3dk^2	dk^2	p_1	p_2	p_3	u_1	u_2	u_3
1	2	3	90	144	216	18	5	8	12	16	−10	−6
2	3	4	224	576	1170	2	112	288	585	576	−420	−156
3	3	5	800	900	1008	4	200	225	252	450	−240	−210
4	4	5	450	1600	3808	2	225	800	1904	1600	−1260	−340
5	5	6	792	3600	9450	18	44	200	525	400	−330	−70
6	5	7	2352	4900	8800	4	588	1225	2200	2450	−1680	−770
7	5	8	4992	6400	8050	2	2496	3200	4025	6400	−3640	−2760
8	6	7	1274	7056	19800	2	637	3528	9900	7056	−6006	−1050
9	7	8	1920	12544	36946	2	960	6272	18473	12544	−10920	−1624
10	7	9	5184	15876	35280	36	144	441	980	882	−672	−210
11	7	10	10200	19600	33418	2	5100	9800	16709	19600	−13090	−6510
12	7	11	17424	23716	31360	4	4356	5929	7840	11858	−6930	−4928
13	7	12	27360	28224	29106	18	1520	1568	1617	3136	−1596	−1540
14	8	9	2754	20736	63360	18	153	1152	3520	2304	−2040	−264
15	8	11	13794	30976	58240	2	6897	15488	29120	30976	−21736	−9240
16	8	13	35490	43264	52096	2	17745	21632	26048	43264	−24024	−19240

Table 2

Using Theorem 4.2 in Theorem 4.4 we get the following result.

Corollary 4.5 For every p_1, p_2, p_3, where $p_1 = \frac{2s^2(s^2-t^2)}{dk^2}, p_2 = \frac{4t^2s^2}{dk^2}, p_3 = \frac{2t^2(9t^2-s^2)}{dk^2}$, $t < s < \sqrt{3}t$,$dk^2 = (2s^2(s^2-t^2), 4t^2s^2, 2t^2(9t^2-s^2))$ and for arbitrary $q \in N$ the graph K_{p_1q,p_2q,p_3q} is integral complete 3-partite graph.

The following theorem shows how we can construct an infinite class of integral graphs $K_{a_1p_1,a_2p_2,a_3p_3}$ from the known integral graph K_{p_1,p_2,p_3} using Diophantine equations.

Theorem 4.6 Let $p_i > 0, u_i (i = 1, 2, 3)$ be those of No.1 in table 1. Then for any nonnegative t the graph $K_{a_1p_1,a_2p_2,a_3p_3}$ is integral if $p_i, a_i, u_i (i = 1, 2, 3)$ are given by the following:
$p_1 = 3, p_2 = 17, p_3 = 65,$
$u_1 = 39 + 2184t, u_2 = -5, u_3 = -34,$
$a_1 = 1 + 52t, a_2 = 1 + 39t, a_3 = 1 + 21t.$

Proof.
From table 1 No. 1 we have $p_1 = 3, p_2 = 17, p_3 = 65, u_1 = 39, u_2 = -5, u_3 = -34.$ By (2.1) in Theorem 2.1, we get

$$a_1 = \frac{(p_1+u_1)(p_1+u_2)(p_1+u_3)}{p_1(p_1-p_2)(p_1-p_3)} = \frac{3+u_1}{42}, \quad (4.3)$$

$$a_2 = \frac{(p_2+u_1)(p_2+u_2)(p_2+u_3)}{p_2(p_2-p_1)(p_2-p_3)} = \frac{17+u_1}{56}, \quad (4.4)$$

$$a_3 = \frac{(p_3+u_1)(p_3+u_2)(p_3+u_3)}{p_3(p_3-p_1)(p_3-p_2)} = \frac{65+u_1}{104}. \quad (4.5)$$

So, $K_{a_1p_1,a_2p_2,a_3p_3}$ is integral if and only if $a_i (i = 1, 2, 3)$ and u_1 must be positive integers. From (4.3) and (4.4) we get the Diophantine equation

$$4a_2 - 3a_1 = 1. \quad (4.6)$$

From elementary number theory follows that all positive integral solutions of (4.6) are given by $a_1 = 1 + 4t_1, a_2 = 1 + 3t_1, u_1 = 39 + 168t_1$, where t_1 is a nonnegative integer.

Using $u_1 = 39+168t_1$ in (4.5) we get $a_3 = 1+\frac{21}{13}t_1$. As a_3 must be a nonnegative integer, $t_1 = 13t$.Then we get $a_1 = 1+52t, a_2 = 1+39t, a_3 = 1+21t, u_1 = 39+2184t.$ From theorem 2.1 it follows that $K_{a_1p_1,a_2p_2,a_3p_3}$ is an integral graph. □

Remarks:
1. It is possible to use the analogous procedure on each graph from tables 1,2, as well as on each of 137 integral complete 3-partite graphs mentioned below Theorem 4.2, and we get from each of these graphs the infinite class of integral graphs.
2. Similarly, using the analogous procedure on each graph from Theorems 4.1, 4.3, and 4.4 we get the infinite class of integral graphs.
3. In [12,13] other classes of integral graphs $K_{a_1p_1q,a_2p_2q,a_3p_3q}$ different from those in this paper are constructed.

5. INTEGRAL COMPLETE 4-PARTITE GRAPHS

Roitman in [10] conjectured that for $r > 3$ there exist an infinite number of integral $r-$partite graphs. However, first integral complete 4-partite graphs was given in

[8].

From the theory of divisors and co-divisors of a graph follows that the divisor of the graph K_{p_1,p_2,p_3,p_4} has characteristic polynomial

$P(D,x) = x^4 - (p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_2p_4 + p_3p_4)x^2 - 2(p_1p_2p_3 + p_1p_2p_4 + p_1p_3p_4 + p_2p_3p_4)x - 3p_1p_2p_3p_4$.

Moreover, the characteristic polynomial of the co-divisor is $P(C,x) = x^{p_1+p_2+p_3+p_4-4}$.

The following theorem holds.

Theorem 5.1 ([8]) A graph K_{p_1,p_2,p_3,p_4} is integral if and only if the characteristic polynomial $P(D,x)$ of its divisor has only integer zeros.

From Theorem 2.1 it follows that if $p_1 < p_2 < p_3 < p_4$ and u_1, u_2, u_3, and u_4 are the zeros of $P(D,x)$, then $-p_4 < u_4 < -p_3 < u_3 < -p_2 < u_2 < -p_1 < 0 < u_1$ and $u_1 + u_2 + u_3 + u_4 = 0$.

Using a computer we found the first known integral complete 4-partite graphs. We explored all graphs K_{p_1,p_2,p_3,p_4}, where $p_1 < p_2 < p_3 < p_4 < 1000$, which means that we explored billions of graphs. For each of these graphs we calculated the characteristic polynomial of its divisor $P(D,x)$. Then we verified whether the polynomial has integer root u_4 satisfying a condition $-p_4 < u_4 < -p_3$ and in positive case we similarly verified integer roots u_3 and u_2 satisfying $-p_3 < u_3 < -p_2 < u_2 < -p_1$. Only two of the graphs we explored are integral. Their list is in the table 3.

p_1	117	234
p_2	261	522
p_3	352	704
p_4	495	990
u_1	870	1740
u_2	−144	−288
u_3	−297	−594
u_4	−429	−858

Table 3

Only the graph in the first column after the heading is primitive. The graph in the second column can be obtained using Theorem 5.2, which follows from Theorem 2.2, for $q = 2$. The spectrum $Spec(K_{117q,261q,352q,495q}) = \{-429q, -297q, -144q, 0, 870q\}$.

Theorem 5.2 ([8]) Let $p_1 = 117, p_2 = 261, p_3 = 352, p_4 = 495$. Then for any positive integer q the graph K_{p_1q,p_2q,p_3q,p_4q} is integral.

Using analogous procedure than in Theorem 4.6 we get the following theorem.

Theorem 5.3 Let $p_i > 0, u_i (i = 1,2,3,4)$ be those of Theorem 5.2.Then for any nonnegative t the graph $K_{a_1p_1,a_2p_2,a_3p_3,a_4p_4}$ is integral if $p_i, a_i, u_i (i = 1,2,3,4)$ are given by following:

$p_1 = 117, p_2 = 261, p_3 = 352, p_4 = 495$,

$u_1 = 870 + 48372870t, u_2 = -429, u_3 = -297, u_4 = -144$,

$a_1 = 1 + 49010t, a_2 = 1 + 42770t, a_3 = 1 + 39585t, a_4 = 1 + 35438t.$

Proof.

For $s = 4$ by Theorem 5.2, we have $p_1 = 117, p_2 = 261, p_3 = 352, p_4 = 495, u_2 = -429, u_3 = -297, u_4 = -144$. By (2.1) Theorem 2.1 we get

(5.1) $$a_1 = \frac{(p_1+u_1)(p_1+u_2)(p_1+u_3)(p_1+u_4)}{p_1(p_1-p_2)(p_1-p_3)(p_1-p_4)} = \frac{1}{987}(117 + u_1)$$

(5.2) $$a_2 = \frac{(p_2+u_1)(p_2+u_2)(p_2+u_3)(p_2+u_4)}{p_2(p_2-p_1)(p_2-p_3)(p_2-p_4)} = \frac{1}{1131}(261 + u_1)$$

(5.3) $$a_3 = \frac{(p_3+u_1)(p_3+u_2)(p_3+u_3)(p_3+u_4)}{p_3(p_3-p_1)(p_3-p_1)(p_3-p_4)} = \frac{1}{1222}(352 + u_1)$$

(5.4) $$a_4 = \frac{(p_4+u_1)(p_4+u_2)(p_4+u_3)(p_4+u_4)}{p_4(p_4-p_1)(p_4-p_2)(p_4-p_3)} = \frac{1}{1365}(495 + u_1)$$

So, $K_{a_1p_1,a_2p_2,a_3p_3,a_4p_4}$ is integral if and only if $a_i (i = 1, 2, 3, 4)$ are positive integers, and u_1 must be a positive integer. From (5.1), (5.2), (5.3), and (5.4), we get the Diophantine equations

(5.5) $$377a_2 - 329a_1 = 48$$

(5.6) $$1365a_4 - 1222a_3 = 143.$$

From elementary number theory follows that all positive integral solutions of (5.5) and (5.6) are given respectively by $a_1 = 1 + 377t_1, a_2 = 1 + 329t_1, u_1 = 870 + 372099t_1$ and $a_3 = 1 + 1365t_2, a_4 = 1 + 1222t_2, u_1 = 870 + 1668030t_2$, where t_1 and t_2 are nonnegative integers. Hence $u_1 = 870 + 372099t_1 = 870 + 1668030t_2$ must be a positive integer. It deduces that $t_1 = 130t$ and $t_2 = 29t$, where t is a nonnegative integer. By above discussion, we have $a_1 = 1 + 49010t, a_2 = 1 + 42770t, a_3 = 1 + 39585t, a_4 = 1 + 35438t, u_1 = 870 + 48372870t$. Hence, when $p_1 = 117, p_2 = 261, p_3 = 352, p_4 = 495, a_1 = 1 + 49010t, a_2 = 1 + 42770t, a_3 = 1 + 39585t, a_4 = 1 + 35438t$, where t is nonnegative integer, then by Theorem 2.1 the graph $K_{a_1p_1,a_2p_2,a_3p_3,a_4p_4}$ is integral. □

By Theorem 2.3 we have the following theorem:

Theorem 5.4 Let $p_i > 0, u_i, a_i (i = 1, 2, 3, 4)$ be those of Theorem 5.3. Then for any positive integer q and any nonnegative t the graph $K_{a_1p_1q,a_2p_2q,a_3p_3q,a_4p_4q}$ is integral.

Remark. In [13] other classes of integral graphs $K_{a_1p_1q,a_2p_2q,a_3p_3q,a_4p_4q}$ different from these were constructed.

6. CONCLUSION

In this paper, the integral multipartite graphs $K_{a_1p_1,a_2p_2,...,a_sp_s}$ for $s = 3, 4$ are presented. For $s = 1, 2, 3, 4$, some results of such integral graphs can be found in [8, 9, 10, 11, 12, 13] and in the present paper. When $s \geq 5$, we have not found such integral graphs. Thus, we raise the following questions.

Question 6.1 (see also [12]) Are there any integral complete $r-$partite graphs $K_{p_1,p_2,...,p_r} = K_{a_1p_1,a_2p_2,...,a_sp_s}$ with arbitrarily large s?

For complete $r-$partite graphs $K_{p_1,p_2,...,p_r} = K_{a_1p_1,a_2p_2,...,a_sp_s}$, when $s = 1, 2, 3, 4$, let $a_1 = a_2 = ... = a_s = 1$, some results of such integral graphs can be found in [8, 9, 10, 12]. However, when $s \geq 5$, $a_1 = a_2 = ... = a_s = 1$, we have not found such integral graphs.

Hence, we have

Question 6.2 Are there any integral complete $r-$partite graphs $K_{p_1,p_2,...,p_r} = K_{a_1p_1,a_2p_2,...,a_sp_s}$ with $a_1 = a_2 = ... = a_s = 1$ when $s \geq 5$?

Theorems 4.1, 4.3, and 4.4 give sufficient conditions by which we can construct infinite classes of integral complete 3-partite graphs. However, by these conditions we are not able to generate integral graphs K_{p_1,p_2,p_3}, where
$(p_1, p_2, p_3) \in \{(5, 12, 77), (25, 297, 675), (33, 98, 833), (45, 136, 396), (49, 220, 441),$ $(216, 385, 684), (297, 437, 585), (693, 760, 828)\}$. These integral graphs were obtained using computers and are in Table 1, No. 17-24.

Question 6.3 Is it possible to find sufficient conditions by which we can generate these graphs?

Characterization of integral complete 3-partite graphs remains open.

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Pavel Híc,
Faculty of Education, Trnava University,
Priemyselná 4, P.O.BOX 9,
918 43 Trnava, Slovakia,
e-mail: phic@truni.sk

Milan Pokorný,
Faculty of Education, Trnava University,
Priemyselná 4, P.O.BOX 9,
918 43 Trnava, Slovakia,
e-mail: mpokorny@truni.sk

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