Journal of Applied Mathematics, Statistics and Informatics (JAMSI), **6** (2010), No. 1

VARIATIONAL ITERATION METHOD FOR SOLVING KLEIN-GORDON EQUATIONS

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Abstract

In this paper, we apply variational iteration method (VIM) for solving Klein-Gordon equations which arise in quantum field theory, relativistic physics, wave theory and other physical phenomena. The suggested algorithm is more efficient and easier to handle as compare to the decomposition method. Numerical results show the efficiency of the proposed algorithm.

Mathematics Subject Classification 2000: 35C05,35C10
Additional Key Words and Phrases: Variational iteration method, Lagrange multiplier, Klein-Gordon equations.

1. INTRODUCTION

The Klein-Gordon equations appear in quantum field theory, relativistic physics, dispersive wave-phenomena, plasma physics , nonlinear optics and applied and physical sciences [1, 13, 28] and are of the form

$$u_{tt}(x,\,t) - u_{xx}(x,\,t) + au(x,\,t) = h(x,\,t)\,,$$

with initial conditions

$$u(x,\,0) = f(x)\,, u_t(x,\,0) = g(x)\,.$$

Several techniques including finite difference, collocation, finite element, inverse scattering, decomposition and variational iteration using Adomian's polynomials have been used to handle such equations [1, 13, 28]. He [4-11] developed the variational iteration method for solving various physical problems. The method has been extremely useful for diversified physical problems and has the potential to cope with the versatility of the complex problems, see [1-30]. It is worth mentioning that convergence of variational iteration method (VIM) has been proved in [7-11, 13, 14, 27]. Moreover, [7-11, 13, 14, 27] it has been shown that VIM finds the convergent series solution in terms of easily computable components. Inspired and motivated by the ongoing research in this area, we apply He's variational iteration method (VIM) to solve Klein-Gordon equations. Numerical results show the complete reliability of the proposed technique.

2. VARIATIONAL ITERATION METHOD (VIM)

To illustrate the basic concept of the He's VIM, we consider the following general differential equation

$$L\,u + N\,u = g(x), \tag{1}$$

where L is a linear operator, N a nonlinear operator and g(x) is the inhomogeneous term. According to variational iteration method [1-30], we can construct a correction functional as follows

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \left(L\,u_n(s) + N\,\tilde{u}_n(s) - g(s) \right) ds, \tag{2}$$

where λ is a Lagrange multiplier [4-11], which can be identified optimally via variational iteration method. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. i.e.$\delta\,\tilde{u}_n = 0$; (2) is called a correction functional. The solution of the linear problems can be solved in a single iteration step due to the exact identification of the Lagrange multiplier. The principles of variational iteration method and its applicability for various kinds of differential equations are given in [4-11]. In this method, it is required first to determine the Lagrange multiplier λ optimally. The successive approximation u_{n+1}, of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0,consequently, the solution is given by $u = \lim\limits_{n\to\infty} u_n$.

3. NUMERICAL APPLICATIONS

In this section, we apply the variational iteration method (VIM) for solving Klein-Gordon equations. Numerical results are very encouraging.

Example 3.1 Consider the following Linear Klein-Gordon equation

$$u_{tt}\,(x,\,t)\; - \;u_{xx}\,(x,\,t)\; + u\,(x,\,t)\; = \;o,$$

with initial conditions

$$u\,(x,\,0)\; = 0, u_t\,(x,\,0)\; = x.$$

The correction functional is given by

$$u_{n+1}\,(x,\,t)\; = \;xt\; + \int_0^t \lambda\,(s) \left(\frac{\partial^2 u_n}{\partial s^2} \; - \; \frac{\partial^2 \tilde{u}_n}{\partial x^2} \; + \; \tilde{u}_n\,(s,\,t) \right) ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda\,(s)\; = \;(s - \,t)$, we obtain the following iterative formula

$$u_{n+1}\,(x,\,t)\; = \;xt\; + \int_0^t (s\; - \;t) \left(\frac{\partial^2 u_n}{\partial s^2} \; - \; \frac{\partial^2 u_n}{\partial x^2} \; + \; u_n\,(s,\,t) \right) ds.$$

Consequently, following approximants are obtained

$$u_0\,(x,\,t)\; = \;xt,$$

$$u_1\,(x,\,t)\; = \;xt\; - \;\frac{1}{3\,!} xt^3,$$

$$u_2(x, t) = xt - \frac{1}{3!}xt^3 + \frac{1}{5!}xt^5,$$

$$u_3(x, t) = xt - \frac{1}{3!}xt^3 + \frac{1}{5!}xt^5 - \frac{1}{7!}xt^7,$$

$$\vdots$$

The series solution is given by

$$u(x, t) = xt - \frac{1}{3!}xt^3 + \frac{1}{5!}xt^5 - \frac{1}{7!}xt^7 + \cdots,$$

and the closed form solution is given as

$$u(x, t) = x \sin t.$$

Example 3.2 Consider the following linear inhomogeneous Klein-Gordon equation

$$u_{tt}(x, t) - u_{xx}(x, t) + u(x, t) = 2 \sin x,$$

with initial conditions

$$u(x, 0) = \sin x, u_t(x, 0) = 1.$$

The correction functional is given by

$$u_{n+1}(x, t) = \sin x + t + \int_0^t \lambda(s) \left(\frac{\partial^2 u_n}{\partial s^2} - \frac{\partial^2 \tilde{u}_n}{\partial x^2} + \tilde{u}_n(s, t) - 2 \sin s \right) ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) = (s - t)$, we obtain the following iterative formula

$$u_{n+1}(x, t) = \sin x + t + \int_0^t (s - t) \left(\frac{\partial^2 u_n}{\partial s^2} - \frac{\partial^2 u_n}{\partial x^2} + u_n(s, t) - 2 \sin s \right) ds.$$

Consequently, following approximants are obtained

$$u_0(x, t) = \sin x + t + t^2 \sin x,$$

$$u_1(x, t) = \sin x + t + t^2 \sin x - t^2 \sin x - \frac{1}{3!}t^4 \sin x - \frac{1}{3!}t^3,$$

$$u_2(x, t) = \sin x + t + t^2 \sin x - t^2 \sin x - \tfrac{1}{3!}t^4 \sin x - \tfrac{1}{3!}t^3 +$$
$$+ \tfrac{1}{3!}t^4 \sin x + \tfrac{1}{90}t^6 \sin x + \tfrac{1}{5!}t^5,$$

$$\vdots$$

The series solution is given by

$$u(x, t) = \sin x + t + t^2 \sin x - t^2 \sin x - \frac{1}{3!}t^4 \sin x - \frac{1}{3!}t^3 + \frac{1}{3!}t^4 \sin x + \frac{1}{90}t^6 \sin x + \frac{1}{5!}t^5 + \cdots,$$

and the closed form solution is given as

$$u(x,\, t) \;=\; \sin x \;+ \sin t.$$

Example 3.3 Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x,\, t) \;-\; u_{xx}(x,\, t) \;+ u^2(x,\, t) \;=\; x^2t^2,$$

with initial conditions

$$u(x,\, 0) \;= 0, u_t(x,\, 0) \;= x.$$

The correction functional is given by

$$u_{n+1}(x,\, t) \;=\; x\,t \;+\; \int_0^t \lambda(s) \left(\frac{\partial^2 u_n}{\partial s^2} \;-\; \frac{\partial^2 \tilde{u}_n}{\partial x^2} \;+\; \tilde{u}_n^2(s,\, t) -\; s^2t^2 \right) ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) \;=\; (s -\; t)$, we obtain the following iterative formula

$$u_{n+1}(x,\, t) \;=\; x\,t \;+\; \int_0^t (s - t) \left(\frac{\partial^2 u_n}{\partial s^2} \;-\; \frac{\partial^2 u_n}{\partial x^2} \;+\; u_n^2(s,\, t) -\; s^2t^2 \right) ds.$$

Consequently, following approximants are obtained

$$\begin{aligned} u_0(x,\, t) \;&=\; xt\,, \\ u_1(x,\, t) \;&=\; 0, \\ u_2(x,\, t) \;&=\; 0, \\ &\vdots \end{aligned}$$

The closed form solution is given by

$$u(x,\, t) \;=\; x\,t.$$

Example 3.4 Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x,\, t) \;-\; u_{xx}(x,\, t) \;+ u^2(x,\, t) \;=\; 2x^2 - 2t^2 + x^4t^4,$$

with initial conditions

$$u(x,\, 0) \;= 0, u_t(x,\, 0) \;= 0.$$

The correction functional is given by

$$u_{n+1}(x,\, t) \;=\; x^2t^2 - \frac{1}{6}t^4 + \frac{1}{30}x^4t^6 \;+\; \int_0^t \lambda(s) \left(\frac{\partial^2 u_n}{\partial s^2} \;-\; \frac{\partial^2 \tilde{u}_n}{\partial x^2} \;+\; \tilde{u}_n^2(s,\, t) \right) ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) \;=\; (s -\; t)$, we obtain the following iterative formula

$$u_{n+1}(x,\, t) \;=\; \int_0^t (s - t) \left(\frac{\partial^2 u_n}{\partial s^2} \;-\; \frac{\partial^2 u_n}{\partial x^2} \;+\; u_n^2(s,\, t) \;-\; \left(2x^2 - 2s^2 + x^4s^4\right) \right) ds.$$

Consequently, following approximants are obtained

$$u_0(x,\, t) \;=\; x^2t^2,$$

$$\begin{aligned} u_1(x,\, t) &= 0,\\ u_2(x,\, t) &= 0,\\ &\vdots \end{aligned}$$

The closed form solution is given as

$$u(x,\, t) \;=\; x^2t^2.$$

Example 3.5 Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x,\, t) \;-\; u_{xx}(x,\, t) \;+ u^2(x,\, t) \;=\; 6xt\,\left(x^2 - t^2\right) + x^6t^8,$$

with initial conditions

$$u(x,\, 0) \;= 0, u_t(x,\, 0) \;= 0.$$

The correction functional is given by

$$u_{n+1}(x,\, t) \;=\; \int_0^t \lambda(s)\left(\frac{\partial^2 u_n}{\partial s^2} \;-\; \frac{\partial^2 \tilde{u}_n}{\partial x^2} \;+\; \tilde{u}_n^2(s,\, t) - 6xt\,\left(x^2 - s^2\right) - x^6s^8\right)\, ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s) \;=\; (s -\, t)$, we obtain the following iterative formula

$$u_{n+1}(x,\, t) \;=\; \int_0^t (s-t)\left(\frac{\partial^2 u_n}{\partial s^2} \;-\; \frac{\partial^2 u_n}{\partial x^2} \;+\; u_n^2(s,\, t) - 6xt\,\left(x^2 - s^2\right) - x^6s^8\right)\, ds.$$

Consequently, following approximants are obtained

$$u_0(x,\, t) \;= x^2t^3,$$

$$\begin{aligned} u_1(x,\, t) &= 0,\\ u_2(x,\, t) &= 0,\\ &\vdots \end{aligned}$$

The solution is given by

$$u(x,\, t) \;= x^2t^3.$$

Example 3.6 Consider the following nonlinear Klein-Gordon equation

$$u_{tt}(x,\, t) \;-\; u_{xx}(x,\, t) \;+ u^2(x,\, t) \;=\; -x\,\cos t \;+\; x^2\cos^2 t,$$

with initial conditions

$$u(x,\, 0) \;= x, u_t(x,\, 0) \;= 0.$$

The correction functional is given by

$$u_{n+1}(x,\,t)\;=\;x+\int\limits_{0}^{t}\lambda(s)\left(\frac{\partial^2 u_n}{\partial s^2}\;-\frac{\partial^2 \tilde{u}_n}{\partial x^2}\;+\;\tilde{u}_n^2(s,\,t)+x\,\cos s\;-x^2\cos^2 s\right)ds.$$

Making the correction functional stationary, the Lagrange multiplier can be identified as $\lambda(s)\;=\;(s-\,t)$, we obtain the following iterative formula

$$u_{n+1}(x,\,t)\;=\;x+\int\limits_{0}^{t}(s-t)\left(\frac{\partial^2 u_n}{\partial s^2}\;-\frac{\partial^2 u_n}{\partial x^2}\;+\;u_n^2(s,\,t)+x\,\cos s\;-x^2\cos^2 s\right)ds.$$

Proceeding as before, the solution is given as

$$u(x,\,t)\;=\;x\,\cos t.$$

4. CONLUSION

In this paper, we applied variational iteration method (VIM) for solving Klein Gordon equations. The method is applied in a direct way without using linearization, transformation, perturbation, discretization or restrictive assumptions. The fact that the proposed VIM solves nonlinear problems without using Adomian's polynomials is a clear advantage of this algorithm over the decomposition method.

5. REFERENCES

[1] S. Abbasbandy, Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method, Internat. J. Numer. Meth. Engrg. 70 (2007), 876-881.

[2] M. A. Abdou and A. A. Soliman, Variational iteration method for solving Burger's and coupled Burger's equations, J. Comput. Appl. Math. 181 (2005), 245-251.

[3] M. A. Abdou and A. A. Soliman, New applications of variational iteration method, Phys. D 211 (1-2) (2005), 1-8.

[4] J. H. He, Variational iteration method, A kind of non-linear analytical technique, some examples, Internat. J. Nonlin. Mech. 34 (4) (1999), 699-708.

[5] J. H. He, Variational iteration method for autonomous ordinary differential systems, Comput. Math. Appl. 114 (2-3) (2000), 115-123.

[6] J. H. He and X. H. Wu, Construction of solitary solution and compaction-like solution by variational iteration method, Chaos, Solitons & Fractals, 29 (1) (2006), 108-113.

[7] J. H. He, Variational iteration method — some recent results and new interpretations, J. Comput. Appl. Math. 207 (2007), 3-17.

[8] J. H. He and X. H. Wu, Variational iteration method: New development and applications, Comput. Math. Appl. 54(2007), 881-894.

[9] J. H. He, The variational iteration method for eighth-order initial boundary value problems, Phys. Scr. 76(6) (2007), 680-682.

[10] J. H. He, An elementary introduction of recently developed asymptotic methods and nanomechanics in textile engineering, Int. J. Mod. Phys. B 22 (21) (2008), 3487-4578.

[11] J. H. He, Some asymptotic methods for strongly nonlinear equation, Int. J. Mod. Phys. 20 (20)10 (2006), 1144-1199.

[12] S. T. Mohyud-Din, M. A. Noor and K. I. Noor, Travelling wave solutions of seventh-order generalized KdV equations by variational iteration method using Adomian's polynomials, Int. J. Mod. Phys. B, 23 (15) (2009), 3265-3277.

[13] S. T. Mohyud-Din, M. A. Noor and K. I. Noor, Some relatively new techniques for nonlinear problems, Math. Prob. Eng. 2009 (2009), Article ID 234849, 25 pages, doi:10.1155/2009/234849.

[14] S. T. Mohyud-Din, M. A. Noor and K. I. Noor, On the coupling of polynomials with correction functional, Int. J. Mod. Phys. B, (2009).

[15] M. A Noor and S. T. Mohyud-Din, Variational iteration technique for solving higher order boundary value problems, Appl. Math. Comput. 189 (2007), 1929-1942.

[16] M. A Noor and S. T. Mohyud-Din, An efficient method for fourth order boundary value problems, Comput. Math. Appl. 54 (2007), 1101-1111.

[17] M. A. Noor and S. T. Mohyud-Din, Variational iteration method for solving higher-order nonlinear boundary value problems using He's polynomials, Int. J. Nonlin. Sci. Num. Simul. 9 (2) (2008), 141-157.

[18] M. A. Noor and S. T. Mohyud-Din, Variational iteration method for solving fifth-order boundary value problems using He's polynomials, Math. Prob. Engg. (2008), Article ID 954794, doi: 10:1155/2008/954794.

[19] M. A. Noor and S. T. Mohyud-Din, Modified variational iteration method for heat and wave-like equations, Acta Applnda. Mathmtce. (2008), DOI: 10.1007/s10440-008-9255-x.

[20] M. A. Noor and S. T. Mohyud-Din, Modified variational iteration method for solving fourth-order boundary value problems, J. Appl. Math. Computg. (2008), Article ID: 90, DOI: 10.1007/s12190-008-0090-z.

[21] M. A. Noor and S. T. Mohyud-Din, Variational homotopy perturbation method for solving higher dimensional initial boundary value problems, Math. Prob. Engg. (2008), Article ID 696734, doi:10.1155/2008/696734.

[22] M. A. Noor and S. T. Mohyud-Din, Variational iteration method for unsteady flow of gas through a porous medium using He's polynomials and Pade approximants, Comput. Math. Appl. (2008).

[23] M. A. Noor and S. T. Mohyud-Din, Modified variational iteration method for solving Helmholtz equations, Comput. Math. Modlg. (2008).

[24] M. A. Noor and S. T. Mohyud-Din, Variational iteration method for solving twelfth-order boundary value problems using He's polynomials, Comput. Math. Modlg. (2008).

[25] M. A. Noor and S. T. Mohyud-Din, Solution of singular and nonsingular initial and boundary value problems by modified variational iteration method, Math. Prob. Engg. 2008 (2008), Article ID 917407, 23 pages, doi:10.1155/2008/917407.

[26] T. Özi and A.Yıldırım, A study of nonlinear oscillators with $u^{1/3}$ force by He's variational iteration method, Journal of Sound and Vibration 306 (2007) 372 -376.

[27] M. Tatari and M. Dehghan, On the convergence of He's variational iteration method, J. Comput. Appl. Math. 207 (2007), 121-128.

[28] A. M. Wazwaz, The modified decomposition method for analytic treatment of differential equations, Appl. Math. Comput. (2006), 165-176.

[29] L. Xu, The variational iteration method for solving integral equations, Comput. Math. Appl. 54 (2007), 1071-1078.

[30] A.Yıldırım, T. Özi, Solutions of singular IVPs of Lane-Emden type by the variational iteration method, Nonlinear Analysis: Theory Method Applications, (2008), in press.

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Received January 2010