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INTEGRAL TRANSFORMS OF THE GENERALIZED MATHIEU SERIES

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Abstract

The main object of this paper is to investigate the Mellin–Barnes type integral representations for the general Mathieu–type series and to apply them in deriving their integral transforms. The Mathieu–type series defined in terms of Mellin–Barnes integrals have more freedom in the parameters and the variable in comparison to their series definition. Thus this approach in terms of Mellin–Barnes integral representation generalizes their definitions in terms of the generalized H–function, that is the $\overline{H}$–function introduced by Inayat–Hussain. The results obtained in this article are believed to be new.

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1. INTRODUCTION AND PRELIMINARIES

The familiar infinite series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} \qquad r \in \mathbb{R}_+ \tag{1}$$

is named after Émile Léonard Mathieu [1835–1890], who introduced it in his 1890 book [11] on elasticity of solid bodies. Alternating form of the above series is given by

$$\widetilde{S}(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{(n^2+r^2)^2} \qquad r \in \mathbb{R}_+, \tag{2}$$

which is recently defined and studied by Pogány *et al.* [14].

A useful integral representation for $S(r)$ is given by

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{x \sin(rx)}{e^x - 1}\, dx, \tag{3}$$

which was obtained by Emersleben [6] and

$$\widetilde{S}(r) = \frac{1}{r} \int_0^{\infty} \frac{x \sin(rx)}{e^x + 1}\, dx, \tag{4}$$

derived for the first time in [14, p. 72, Eq. (2.8)]. A number of remarkable and useful results for slightly generalized Mathieu–type series

$$S_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^\mu} \qquad \mu - 1, r \in \mathbb{R}_+ \tag{5}$$

are available in the works of Cerone and Lenard [2], Choi and Srivastava [3], Diananda [4], Elezović *et al.* [5], Srivastava and Tomovski [19], Tomovski [20] and Tomovski and Trenčevski [21] among others.

The Mathieu $(\boldsymbol{a}, \boldsymbol{\lambda})$–series was introduced by Pogány [12] in the form of a convergent series

$$\mathfrak{S}_\mu(\boldsymbol{a}, \boldsymbol{\lambda}; r) = \sum_{n=0}^{\infty} \frac{a_n}{(\lambda_n + r)^\mu} \qquad r, \mu > 0\,. \tag{6}$$

deriving closed form integral representation and bilateral bounding inequalities for $\mathfrak{S}_\mu(\boldsymbol{a}, \boldsymbol{\lambda}; r)$. Here the sequences $\boldsymbol{a} := (a_n)_{n\in\mathbb{N}_0}, \boldsymbol{\lambda} := (\lambda_n)_{n\in\mathbb{N}_0}$ are positive. Following the convention that (λ_n) is monotone increasing divergent, we have

$$\boldsymbol{\lambda}\colon \quad 0 \le \lambda_0 < \lambda_1 < \cdots < \lambda_n \xrightarrow[n\to\infty]{} \infty\,. \tag{7}$$

In this article we present the Mellin–Barnes type integrals for the Mathieu series defined by (2) and (5) and a Laplace–Mellin transform formula for (6). These integrals are further employed in obtaining the Mellin, Laplace, Euler and Hankel transforms of these series.

2. THE $\overline{H}$–FUNCTION

Inayat–Hussain [9, p. 4126] introduced a generalization of the Fox' H–function in the form

$$\overline{H}(z) = \overline{H}{}^{m,n}_{p,q}[z] = \overline{H}{}^{m,n}_{p,q}\left[z \,\middle|\, \begin{matrix} (a_j, A_j; \alpha_j)_{\overline{1,n}}, & (a_j, A_j)_{\overline{n+1,p}} \\ (b_j, B_j)_{\overline{1,m}}, & (b_j, B_j; \beta_j)_{\overline{m+1,q}} \end{matrix} \right] \tag{8}$$

$$:= \frac{1}{2\pi \mathrm{i}} \int_{\mathfrak{L}} \chi(s) z^s \,\mathrm{d}s \tag{9}$$

for all $z \neq 0$, where

$$\chi(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s) \cdot \prod_{j=1}^n \left\{\Gamma(1 - a_j + A_j s)\right\}^{\alpha_j}}{\prod_{j=n+1}^p \Gamma(a_j - A_j s) \cdot \prod_{j=m+1}^q \left\{\Gamma(1 - b_j + B_j s)\right\}^{\beta_j}}\,, \tag{10}$$

which contains fractional powers of some of the gamma functions. The parameters $A_i, B_j > 0\,(i = \overline{1,p}, j = \overline{1,q})$ and the exponents $\alpha_i, \beta_j (i = \overline{1,p}, j = \overline{1,q})$ can take noninteger values; $\mathfrak{L} = \mathfrak{L}_{\mathrm{i}\tau\infty}$ is a contour starting at the point $\tau - \mathrm{i}\infty$, terminating at the point $\tau + \mathrm{i}\infty$ with $\tau \in \mathbb{R}$.

It has been established by Buschman and Srivastava [1, p. 4708] that the sufficient conditions for the absolute convergence of the contour integral (9) is given by

$$\Lambda = \sum_{j=1}^m B_j + \sum_{j=1}^n |\alpha_j| A_j - \sum_{j=m+1}^q |\beta_j| B_j - \sum_{j=n+1}^p A_j > 0\,. \tag{11}$$

This condition provides exponential decay of the integrand in (9), and region of absolute convergence of the contour integral (9) is given by

$$|\arg z| < \frac{\pi}{2}\,\Lambda. \tag{12}$$

Remark 2.1. It is interesting to note that for $\alpha_i = \beta_j = 1\,(i = \overline{1, n}, j = \overline{m+1, q})$, the $\overline{H}$–function reduces to the familiar Fox' H–function [10].

Remark 2.2. Functional relations for the $\overline{H}$–function are given by Saxena [15]. Fractional intergation operators associated with the $\overline{H}$–function are defined and studied by Saxena and Singh [18] and Saxena et al. [16]. Bivariate distribution associated with this function are discussed by Saxena et al. [17]. Gupta and Soni [7] have given Riemann–Liouville fractional integral formulæ for this function. Mathieu series of $\overline{H}$–function terms are considered very recently by Pogány and Saxena [13].

In what follows, $\Gamma(a \pm b)$ will stand for the product $\Gamma(a-b)\Gamma(a+b)$ and $\mathrm{B}(p, q)$, $\min\big(\Re\{p\}, \Re\{q\}\big) > 0$ will denote the Beta–function.

3. MELLIN–BARNES INTEGRAL REPRESENTATION FOR MATHIEU SERIES $\widetilde{S}$

The following results will be established in this section, which gives the Mellin–Barnes respesetation for the alternating Mathieu–series (2).

THEOREM 3.1. *Let*

$$\widetilde{S}(r, z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2nz^{n-1}}{(n^2 + r^2)^2} \qquad r > 0\,. \tag{13}$$

Then there holds the following integral representation

$$\widetilde{S}(r, z) = \frac{1}{\pi \mathrm{i}} \int_{\tau - \mathrm{i}\infty}^{\tau + \mathrm{i}\infty} \frac{\Gamma(s)\Gamma(2-s)\Gamma^2(1 \pm \mathrm{i}r - s)}{\Gamma^2(2 \pm \mathrm{i}r - s)}\, z^{-s}\, ds \qquad |\arg(-z)| < \pi\,. \tag{14}$$

PROOF. The contour of intergation extends from $\tau - \mathrm{i}\infty$ to $\tau + \mathrm{i}\infty$, such that all the poles of the Gamma function $\Gamma(2-s)$ at the points $s = k+2$, $k \in \mathbb{N}$ are separated from the poles of the $\Gamma(s)$ at the points $s = -k$, $k \in \mathbb{N}$. Assuming that the poles of the integrand are simple and calculating the residues $\mathrm{res}\big[\Gamma; -k\big]$ at the poles of $\Gamma(s)$, we find that

$$\begin{aligned}
\widetilde{S}(r, z) &= 2 \sum_{k=0}^{\infty} \lim_{s \to -k} \frac{(s+k)\Gamma(s)\Gamma(2-s)\Gamma^2(1 \pm \mathrm{i}r - s)}{\Gamma^2(2 \pm \mathrm{i}r - s)}\, z^{-s} \\
&= 2 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2+k)\Gamma^2(1 \pm \mathrm{i}r + k)}{k!\,\Gamma^2(2 \pm \mathrm{i}r + k)}\, z^k \\
&= 2 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2+k)}{k!(1 - \mathrm{i}r + k)^2(1 + \mathrm{i}r + k)^2}\, z^k \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2kz^{k-1}}{(k^2 + r^2)^2}\,,
\end{aligned}$$

where we have applied the well–known formula

$$\text{res}\big[\Gamma; -k\big] = \lim_{s\to -k}(s+k)\Gamma(s) = \frac{(-1)^k}{k!}, \qquad \Gamma(x+1) = x\Gamma(x)\,.$$

This completes the proof of Theorem 3.1. $\square$

NOTE 3.2. *As $z \to 1$ in* (18)*, then we obtain*

$$\lim_{z\to 1} \widetilde{S}(r,z) = \widetilde{S}(r)\,.$$

The following relationship between $\widetilde{S}(r)$ and the $\overline{H}$–function can be established by comparing the definition (8) and (14):

$$\widetilde{S}(r,z) = \overline{H}{}^{1,2}_{2,2}\left[z \,\middle|\, \begin{matrix} (-1,1;1),\ (\pm\mathrm{i}r,1;2) \\ (0,1),\ (\pm\mathrm{i}r-1,1;2) \end{matrix} \right]. \tag{15}$$

3.1 Analytic continuation formula

Calculating the residues at the poles of $\Gamma(2+s)$ of the integrand of (18) at the points $s=-2-n$, $n\in\mathbb{N}$, we arrive at the result:

$$\widetilde{S}(r,z) = \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{2(n+1)}{((n+1)^2+r^2)^2}\left(\frac{1}{z}\right)^n \qquad r>0\,. \tag{16}$$

This shows that

$$\widetilde{S}(r,z) = \mathcal{O}\big(z^{-2}\big) \qquad z\to\infty\,. \tag{17}$$

4. LAPLACE–MELLIN TRANSFORM OF $\mathfrak{S}_\mu(\boldsymbol{A},\boldsymbol{\lambda};R,Z)$

Let us introduce the function

$$\mathfrak{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};r,z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{(\lambda_n+r)^\mu} \qquad r,\mu>0\,, \tag{18}$$

and write

$$\mathcal{L}_s\big[f\big] = \int_0^\infty e^{-sx} f(x)\,dx, \qquad \mathcal{M}_\nu\big[f\big] = \int_0^\infty x^{\nu-1} f(x)\,dx$$

the Laplace and the Mellin transform of some function f, respectively. The integral transform of function f with respect to the kernel $e^{-sx}x^{\nu-1}$, that is the integral

$$\mathcal{L}_s\big[x^{\nu-1}f(x)\big] \equiv \mathcal{M}_\nu\big[e^{-sx}\,f(x)\big] = \int_0^\infty e^{-sx}\,x^{\nu-1}\,f(x)dx$$

we call *Laplace–Mellin transform* of the function f.

THEOREM 4.1. *Assume that $\Re\{s\},\Re\{\nu\},r,\mu>0$ and the sequences $\boldsymbol{a},\boldsymbol{\lambda}$ are positive and $\boldsymbol{\lambda}$ satisfies* (7)*. Then the Laplace–Mellin transform of the function $\mathfrak{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};r,z)$ becomes*

$$\int_0^\infty e^{-sx}x^{\nu-1}\mathfrak{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};r,x)\,dx = \frac{\Gamma(\nu)}{s^\nu}\sum_{n=0}^{\infty}\frac{a_n(\nu)_n}{(\lambda_n+r)^\mu}\left(\frac{1}{s}\right)^n, \tag{19}$$

where $(\nu)_n$ denotes the Pochhammer symbol.

PROOF. By virtue of the well–known Gamma function formula, *viz.*

$$\Gamma(\nu)s^{-\nu} = \int_0^\infty \mathrm{e}^{-sx}\, x^{\nu-1}\, \mathrm{d}x \qquad \min\Big(\Re\{s\}, \Re\{\nu\}\Big) > 0$$

we transform the integrand on the left in (19), expand the exponential function into Maclaurin series and reversing the order of integration and summation, having in mind the definition of Ψ–function, we arrive at the desired result. Indeed, we have

$$\begin{aligned}\int_0^\infty e^{-sx}\, x^{\nu-1}\mathfrak{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};r,x)\,dx &= \sum_{n=0}^\infty \frac{a_n}{(\lambda_n+r)^\mu}\int_0^\infty e^{-sx}\,x^{n+\nu-1}\,dx\\ &= \frac{1}{s^\nu}\sum_{n=0}^\infty \frac{a_n\Gamma(n+\nu)s^{-n}}{(\lambda_n+r)^\mu} = \frac{\Gamma(\nu)}{s^\nu}\sum_{n=0}^\infty \frac{a_n(\nu)_n s^{-n}}{(\lambda_n+r)^\mu}\,.\end{aligned}$$

This completes the the proof. □

The following interesting consequence holds letting $s \to \infty$ in (19).

COROLLARY 4.2. *Let the situation be the same as in* Theorem 4.1. *Then for all* s, $\Re\{s\} > 0$ *we have*

$$\int_0^\infty e^{-sx}\, x^{\nu-1}\mathfrak{S}_\mu(\boldsymbol{a},\boldsymbol{\lambda};r,x)\,dx = \mathcal{O}\big(s^{-\nu}\big) \qquad s\to\infty\,. \tag{20}$$

Also, specifying sequences $\boldsymbol{a}, \boldsymbol{\lambda}$ we can deduce a new summation formula. However, we leave to interested reader to show the following example.

COROLLARY 4.3. *When* $\Re\{\nu\}, \Re\{s\}, \mu > 0$*, we have*

$$\begin{aligned}\Gamma(\mu)\int_0^\infty & e^{-x}\, x^{\nu-1}\mathfrak{S}_\mu((n!)^{-1},\mathbb{N};r,x/s)\,dx\\ &= \int_0^\infty e^{-rx}\,x^{\mu-1}\,\overline{H}_{1,1}^{1,1}\Big[\begin{array}{c}(1-\nu,1;1)\\(0,1)\end{array}\Big|\frac{1}{se^x}\Big]\,dx\,.\end{aligned} \tag{21}$$

Remark 4.4. We note that the results of this section are general in nature and depend on the nature of the sequences $\boldsymbol{a}$ and $\boldsymbol{\lambda}$.

5. INTEGRAL TRANSFORMS OF $\widetilde{S}(R,Z)$

Here we are presenting more specific results involving the Laplace, Mellin, Euler and Hankel transform of the function $\widetilde{S}(r,z)$. First we derive the Mellin transform formula of $\widetilde{S}(r,z)$.

PROPOSITION 5.1. *For all* $0 < \Re\{s\} < 2$, $\Re\{a\} > 0$*, we have*

$$\int_0^\infty x^{s-1}\,\widetilde{S}(r,ax)\,dx = \frac{2\mathrm{B}(s,2-s)}{a^s\big((s-1)^2+r^2\big)^2}\,. \tag{22}$$

PROOF. By the application of the well–known Mellin inversion formula, relation (14) yields

$$\int_0^\infty x^{s-1}\,\widetilde{S}(r,ax)\,dx = \frac{2\Gamma(s)\Gamma(2-s)\Gamma^2(1\pm \mathrm{i}r-s)}{a^s\Gamma^2(2\pm\mathrm{i}r-s)}\,,$$

which is the same as (22). □

Next, we obtain the Laplace–Mellin transform of $\widetilde{S}(r,z)$ which generalizes the Mellin transform formula (22).

PROPOSITION 5.2. *There holds the formula*

$$\int_0^\infty e^{-px}\,x^{\rho-1}\widetilde{S}(r,ax^\lambda)dx = \frac{2}{p^\rho}\,\overline{H}^{1,3}_{3,3}\left[\frac{a}{p^\lambda}\middle|\begin{matrix}(1-\rho,\lambda),(-1,1),(\pm \mathrm{i}r,1;2)\\ (0,1),\ \ (1\pm \mathrm{i}r,1;2)\end{matrix}\right], \tag{23}$$

where $\min\big(\Re\{a\},\Re\{p\},\Re\{\rho\}\big)>0,\ \lambda>0.$

PROOF. By virtue of the result (14), it follows that the integral is equal to

$$\int_0^\infty e^{-px}\,x^{\rho-1}\frac{1}{\pi\mathrm{i}}\int_{\tau-\mathrm{i}\infty}^{\tau+\mathrm{i}\infty}\frac{\Gamma(s)\Gamma(2-s)\Gamma^2(1\pm \mathrm{i}r-s)}{\Gamma^2(2\pm \mathrm{i}r-s)}\big(ax^\lambda\big)^{-s}\,dsdx.$$

Interchanging the order of integration, it is found that the above expression becomes

$$\int_{\tau-\mathrm{i}\infty}^{\tau+\mathrm{i}\infty}\frac{\Gamma(s)\Gamma(2-s)\Gamma^2(1\pm \mathrm{i}r-s)}{\Gamma^2(2\pm \mathrm{i}r-s)}\,a^{-s}\int_0^\infty e^{-px}\,x^{\rho-\lambda s-1}\,dxds\,.$$

Now, if we evaluate the x–integral by the Gamma function formula and interpret the result thus obtained with the help (8), the assertion of the proposition readily follows. The interchange of the order of integration is permissible under the conditions given with (23). □

Setting $\rho=1$ in (23) we obtain the Laplace transform formula

$$\mathcal{L}_p\big[\widetilde{S}(r,ax^\lambda)\big]=\frac{2}{p}\,\overline{H}^{1,3}_{3,3}\left[\frac{a}{p^\lambda}\middle|\begin{matrix}(0,\lambda),(-1,1),(\pm \mathrm{i}r,1;2)\\ (0,1),\ \ (1\pm \mathrm{i}r,1;2)\end{matrix}\right],$$

where $\min\big(\Re\{a\},\Re\{p\},\lambda\big)>0.$

Following a similar procedure and using the familiar Beta function formula, we obtain the Euler transform result for the alternating Mathieu power series $\widetilde{S}(r,x)$.

PROPOSITION 5.3. *Suppose that* $\min\big(\Re\{\rho\},\Re\{\sigma\},\lambda,x\big)>0$. *Then we have*

$$\begin{aligned}\int_0^x & t^{\rho-1}(x-t)^{\sigma-1}\widetilde{S}(r,ta^\lambda)\,dt\\ &=\frac{2\Gamma(\sigma)}{x^{1-\lambda-\sigma}}\,\overline{H}^{1,3}_{3,2}\left[ax^\lambda\middle|\begin{matrix}(1-\rho,\lambda;1),(-1,1;1),(\pm \mathrm{i}r,1;2)\\ (0,1),(\pm \mathrm{i}r-1,1;2),(1-\rho-\sigma,\lambda;1)\end{matrix}\right].\end{aligned} \tag{24}$$

Finally, we obtain the Hankel transform of $\widetilde{S}(r,z)$.

PROPOSITION 5.4. *Let* $a,\sigma>0,\ \Re\{\lambda+\nu\}>0,\ \Re\{b\}>0,\ \Re\{\lambda\}-2\sigma<3/2.$ *Then it follows*

$$\int_0^\infty x^{\lambda-1}J_\nu(ax)\widetilde{S}(r,bx^\sigma)dx = \tag{25}$$

$$\Big(\frac{2}{a}\Big)^\lambda\overline{H}^{1,3}_{4,2}\left[b\Big(\frac{x}{a}\Big)^\sigma\middle|\begin{matrix}(-1,1;1),(\pm \mathrm{i}r,1;2),\Big(\frac{2-\lambda-\nu}{2},\frac{\sigma}{2}\Big),\Big(\frac{2+\nu-\lambda}{2},\frac{\sigma}{2}\Big)\\ (0,1),\quad (\pm \mathrm{i}r-1,1;2)\end{matrix}\right],$$

where $J_\nu(\cdot)$ *denotes the Bessel function of the first kind of order* ν.

PROOF. By means of the integral for the Bessel function of the first kind [10, p. 57, Eq. (2.46)]

$$\int_o^\infty x^{\lambda-1}\, J_\nu(bx)\, dx = \frac{2^{\lambda-1}\Gamma\big((\lambda+\nu)/2\big)}{a^\lambda\,\Gamma\big(1+(\nu-\lambda)/2\big)} \qquad b>0,\ -\Re\{\nu\}<\Re\{\lambda\}<3/2\,,$$

and by means of the formula (15), it is found the asserted result (25). □

When $\nu = 1/2$, then by virtue of the identity

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\cdot \sin x,$$

the above integral yields

$$\int_0^\infty x^{\lambda-1}\sin(ax)\widetilde{S}(r, bx^\sigma)dx = \tag{26}$$
$$\sqrt{\pi}\left(\frac{2}{a}\right)^\lambda \overline{H}_{4,2}^{1,3}\left[b\left(\frac{x}{a}\right)^\sigma \middle| \begin{matrix} (-1,1;1), (\pm \mathrm{i}r, 1; 2), \left(\frac{1-\lambda}{2}, \frac{\sigma}{2}\right), \left(\frac{2-\lambda}{2}, \frac{\sigma}{2}\right) \\ (0,1), \quad (\pm \mathrm{i}r - 1, 1; 2) \end{matrix} \right],$$

where $a, \sigma > 0, \Re\{\lambda\} > -1$, $\Re\{\lambda\} - 2\sigma < 1$ and $\Re\{b\} > 0$.

Moreover, if we set $\nu = -1/2$, make use of the relation

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cdot \cos x,$$

we deduce that

$$\int_0^\infty x^{\lambda-1}\cos(ax)\widetilde{S}(r, bx^\sigma)dx = \tag{27}$$
$$\sqrt{\pi}\left(\frac{2}{a}\right)^\lambda \overline{H}_{4,2}^{1,3}\left[b\left(\frac{x}{a}\right)^\sigma \middle| \begin{matrix} (-1,1;1), (\pm \mathrm{i}r, 1; 2), \left(\frac{2-\lambda}{2}, \frac{\sigma}{2}\right), \left(\frac{1-\lambda}{2}, \frac{\sigma}{2}\right) \\ (0,1), \quad (\pm \mathrm{i}r - 1, 1; 2) \end{matrix} \right],$$

under the same set of constraints upon the parameter as in previous case.

6. MELLIN–BARNES INTEGRAL REPRESENTATION FOR $S_\mu(R, Z)$

In this section we consider the generalized Mathieu power series

$$S_\mu(r,z) = \sum_{n=1}^\infty \frac{2nz^{n-1}}{(n^2+r^2)^\mu} \qquad r, \mu > 0\,. \tag{28}$$

THEOREM 6.1. *The following integral representation holds true*

$$S_\mu(r,z) = \frac{1}{\pi\,\mathrm{i}}\int_{\tau-\mathrm{i}\infty}^{\tau+\mathrm{i}\infty} \frac{\Gamma(s)\Gamma(2-s)\Gamma^\mu(1\pm \mathrm{i}r - s)}{\Gamma^\mu(2\pm\mathrm{i}r-s)}\,(-z)^{-s}\, ds \qquad |\arg(-z)|<\pi\,. \tag{29}$$

PROOF. Following the lines of proving procedure of the Theorem 1, using the same integration contour, assuming that the poles of the integrand are simple and calculating the residues $\operatorname{res}\big[\Gamma; -k\big]$ at the poles of $\Gamma(s)$, we obtain

$$\begin{aligned}\widetilde{S}_\mu(r,z) &= 2\sum_{k=0}^{\infty} \lim_{s\to -k} \frac{(s+k)\Gamma(s)\Gamma(2-s)\Gamma^\mu(1\pm \mathrm{i}r - s)}{\Gamma^\mu(2\pm \mathrm{i}r-s)}\,(-z)^{-s}\\ &= 2\sum_{k=0}^{\infty} \frac{\Gamma(2+k)}{k!(1-\mathrm{i}r+k)^\mu(1+\mathrm{i}r+k)^\mu}\,z^k\\ &= \sum_{k=1}^{\infty}\frac{2kz^{k-1}}{(k^2+r^2)^\mu}\,.\end{aligned}$$

This completes the proof of the formula (29). □

NOTE 6.2. *As $z \to 1$ in (29), we obtain*

$$\lim_{z\to 1} S_\mu(r,z) = S_\mu(r),$$

which was in focus of investigation e.g. by Cerone and Lenard [2].

Next, the relationship between the generalized Mathieu power series $S_\mu(r,z)$ and the $\overline{H}$–function is exhibited. On comparing (29) and (8), it is found that

$$S_\mu(r,z) = 2\overline{H}^{1,2}_{2,2}\Big[-z\Big|\begin{matrix}(-1,1;1),\ (\pm \mathrm{i}r,1;\mu)\\ (0,1),\ (\pm\mathrm{i}r-1,1;\mu)\end{matrix}\Big]\,. \tag{30}$$

6.1 Analytic continuation of $S_\mu(r,z)$

The residues at the poles of $\Gamma(2+s)$ of the integrand of (28) at the points $s = -2-n$, $n\in\mathbb{N}$ give us

$$S_\mu(r,z) = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{2(n+1)}{((n+1)^2+r^2)^\mu}\left(\frac{1}{z}\right)^n \qquad r>0\,. \tag{31}$$

The above result shows that

$$S_\mu(r,z) = \mathcal{O}\big(z^{-2}\big) \qquad z\to\infty\,. \tag{32}$$

7. INTEGRAL TRANSFORMS OF THE FUNCTION $S_\mu(R,Z)$

Our next goal is to exhibit integral transform formulæ for generalized Mathieu power series $S_\mu(r,z)$. The derivation procedure is similar to one used in section **5**, therefore detailed proofs are omitted.

First, making use of the Mellin inversion formula (29) we deduce the following result.

PROPOSITION 7.1. *For all $0<\Re\{s\}<2$, $\Re\{a\}>0$, we have the Mellin transform formula*

$$\int_0^\infty x^{s-1}\,S_\mu(r,-ax)\,dx = \frac{2\mathrm{B}(s,2-s)}{a^s\big((s-1)^2+r^2\big)^\mu}\,. \tag{33}$$

The next subject is the Laplace–Mellin transform of $S_\mu(r,z)$ which generalizes (33) employing the Laplace–kernel e^{-px} in the integrand.

Proposition 7.2. *There holds the formula*

$$\int_0^\infty e^{-px}\, x^{\rho-1} S_\mu(r, ax^\lambda) dx = \frac{2}{p^\rho}\, \overline{H}_{3,3}^{1,3}\Big[-\frac{a}{p^\lambda}\Big| \begin{matrix} (1-\rho,\lambda), (-1,1), (\pm ir, 1;\mu) \\ (0,1),\ \ (1\pm ir, 1;\mu) \end{matrix}\Big], \tag{34}$$

where $\min\big(\Re\{a\}, \Re\{p\}, \Re\{\rho\}, \lambda\big) > 0$.

When we specify $\rho = 1$ in (34), it turns out to be the Laplace transform

$$\mathcal{L}_p\big[S_\mu(r, ax^\lambda)\big] = \frac{2}{p}\, \overline{H}_{3,3}^{1,3}\Big[-\frac{a}{p^\lambda}\Big| \begin{matrix} (0,\lambda), (-1,1), (\pm ir, 1;\mu) \\ (0,1),\ \ (1\pm ir, 1;\mu) \end{matrix}\Big],$$

where $\min\big(\Re\{a\}, \Re\{p\}, \lambda\big) > 0$.

In the next step, the Beta function formula helps in obtaining the Euler transform of the Mathieu power series $\widetilde{S}(r, x)$.

Proposition 7.3. *Let* $\min\big(\Re\{\rho\}, \Re\{\sigma\}, \lambda, x\big) > 0$. *Then we have*

$$\int_0^x t^{\rho-1}(x-t)^{\sigma-1} S_\mu(r, ta^\lambda)\, dt = \frac{2\Gamma(\sigma)}{x^{1-\lambda-\sigma}}\, \overline{H}_{3,2}^{1,3}\Big[-ax^\lambda\Big| \begin{matrix} (1-\rho,\lambda;1), (-1,1;1), (\pm ir, 1;\mu) \\ (0,1), (\pm ir-1, 1;\mu), (1-\rho-\sigma, \lambda; 1) \end{matrix}\Big]. \tag{35}$$

We finish the derivation procedures with the Hankel transform.

Proposition 7.4. *Let* a, $\sigma > 0$, $\Re\{\lambda+\nu\} > 0$, $\Re\{b\} > 0$, $\Re\{\lambda\} - 2\sigma < 3/2$. *Then it follows*

$$\int_0^\infty x^{\lambda-1} J_\nu(ax) S_\mu(r, bx^\sigma) dx = \tag{36}$$

$$\Big(\frac{2}{a}\Big)^\lambda \overline{H}_{4,2}^{1,3}\Bigg[-b\Big(\frac{x}{a}\Big)^\sigma \Bigg| \begin{matrix} (-1,1;1), (\pm ir, 1;\mu), \Big(\frac{2-\lambda-\nu}{2}, \frac{\sigma}{2}\Big), \Big(\frac{2+\nu-\lambda}{2}, \frac{\sigma}{2}\Big) \\ (0,1),\ \ (\pm ir - 1, 1;\mu) \end{matrix}\Bigg].$$

Letting $\nu = 1/2, -1/2$ the Bessel function of the first kind $J_\nu(\cdot)$ is expressible *via* the sine and cosine functions, therefore the above integral (36) yields respectively

$$\int_0^\infty x^{\lambda-1} \left\{ \begin{matrix} \sin(ax) \\ \cos(ax) \end{matrix} \right\} S_\mu(r, bx^\sigma) dx = \tag{37}$$

$$\begin{cases} \sqrt{\pi}\,\Big(\frac{2}{a}\Big)^\lambda \overline{H}_{4,2}^{1,3}\Bigg[b\Big(\frac{x}{a}\Big)^\sigma \Bigg| \begin{matrix} (-1,1;1), (\pm ir, 1;\mu), \Big(\frac{1-\lambda}{2}, \frac{\sigma}{2}\Big), \Big(\frac{2-\lambda}{2}, \frac{\sigma}{2}\Big) \\ (0,1),\ \ (\pm ir - 1, 1;\mu) \end{matrix}\Bigg] \\ \\ \sqrt{\pi}\,\Big(\frac{2}{a}\Big)^\lambda \overline{H}_{4,2}^{1,3}\Bigg[b\Big(\frac{x}{a}\Big)^\sigma \Bigg| \begin{matrix} (-1,1;1), (\pm ir, 1;\mu), \Big(\frac{2-\lambda}{2}, \frac{\sigma}{2}\Big), \Big(\frac{1-\lambda}{2}, \frac{\sigma}{2}\Big) \\ (0,1),\ \ (\pm ir - 1, 1;\mu) \end{matrix}\Bigg] \end{cases},$$

where $a, \sigma > 0, \Re\{\lambda\} > -1$, $\Re\{\lambda\} - 2\sigma < 1$ and $\Re\{b\} > 0$.

8. LAPLACE–MELLIN TRANSFORM OF $\mathfrak{S}_\mu(\boldsymbol{A}, \boldsymbol{\lambda}; R)$

THEOREM 8.1. *Assume that* $\Re\{s\}, r > 0, \mu > \Re\{\nu\} > 0$, *the sequences* $\boldsymbol{a}, \boldsymbol{\lambda}$ *are positive and* $\boldsymbol{\lambda}$ *satisfies* (7). *Then we have*

$$\int_0^\infty e^{-sx}\, x^{\nu-1}\, \mathfrak{S}_\mu(\boldsymbol{a}, \boldsymbol{\lambda}; x)\, dx = \mathrm{B}(\nu, \mu-\nu) \sum_{n=1}^\infty \frac{a_n}{\lambda_n^{\mu-\nu}}\, {}_1F_1\big(\nu; 1+\nu-\mu; s\lambda_n\big)\,. \tag{38}$$

PROOF. By virtue of the Gamma function formula we transform the integrand on the left in (38), expand the exponential function into Maclaurin series and exchanging the integration and summation order, having on mind the definition of Ψ–function, we arrive at the desired result. Indeed, we have

$$\begin{aligned}\int_0^\infty e^{-sx} x^{\nu-1} \mathfrak{S}_\mu(\boldsymbol{a}, \boldsymbol{\lambda}; x)\, dx &= \sum_{n=1}^\infty a_n \int_0^\infty e^{-sx} \frac{x^{\nu-1}\, dx}{(\lambda_n + x)^\mu} \\ &= \sum_{n=1}^\infty a_n \sum_{m=0}^\infty \frac{(-s)^m}{m!} \int_0^\infty \frac{x^{m+\nu-1}\, dx}{(\lambda_n + x)^\mu} \\ &= \sum_{n=1}^\infty \frac{a_n}{\lambda_n^{\mu-\nu}} \sum_{m=0}^\infty \frac{(-s\lambda_n)^m}{m!} \int_0^\infty \frac{x^{m+\nu-1}\, dx}{(1+x)^\mu} \\ &= \sum_{n=1}^\infty \frac{a_n}{\lambda_n^{\mu-\nu}} \sum_{m=0}^\infty \frac{(-s\lambda_n)^m}{m!}\, \mathrm{B}\big(\nu+m, \mu-(\nu+m)\big) \\ &= \frac{1}{\Gamma(\mu)} \sum_{n=1}^\infty \frac{a_n}{\lambda_n^{\mu-\nu}} \sum_{m=0}^\infty \Gamma(\nu+m)\Gamma(\mu-\nu-m)\, \frac{(-s\lambda_n)^m}{m!} \\ &= \mathrm{B}(\nu, \mu-\nu) \sum_{n=1}^\infty \frac{a_n}{\lambda_n^{\mu-\nu}} \sum_{m=0}^\infty \frac{(\nu)_m}{(1+\nu-\mu)_m}\, \frac{(s\lambda_n)^m}{m!}\,,\end{aligned}$$

where the inner m–sum is the confluent hypergeometric ${}_1F_1$–function. Here we have used the well–known formula

$$(\alpha)_{-m} = \frac{(-1)^m}{(1-\alpha)_m} \qquad \alpha \in \mathbb{C},\, m \in \mathbb{N}_0\,.$$

This completes the proof of Theorem 8.1. □

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