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ASYMPTOTIC STABILITY OF LINEAR CONTINUOUS TIME-VARYING SYSTEMS WITH STATE DELAYS IN HILBERT SPACES

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Abstract

This paper studies the stabilization of the infinite-dimensional linear time-varying system with state delays

$$\dot{x} = A(t)x + A_1(t)x(t-h) + B(t)u\,.$$

The operator $A(t)$ is assumed to be the generator of a strong evolution operator. In contrast to the previous results, the stabilizability conditions are obtained via solving a Riccati differential equation and do not involve any stability property of the evolution operator. Our conditions are easy to construct and to verify. We provide a step-by-step procedure for finding feedback controllers and state stability conditions for some linear delay control systems with nonlinear perturbations.

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1. INTRODUCTION

Consider a linear control system with state delays

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + B(t)u(t), \quad t \geq t_0, x(t) = \phi(t), \quad t \in [-h, t_0], \tag{1.1}$$

where $x \in X$ is the state, $u \in U$ is the control, $h \geq 0$. The stabilizability question consists on finding a feedback control $u(t) = K(t)x(t)$ for keeping the closed-loop system

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t) + A_1(t)x(t-h)$$

asymptotically stable in the Lyapunov sense. In the qualitative theory of dynamical systems, the stabilizability is one of the most important properties of the systems and has attracted the attention of many researchers; see for example [1; 7; 10; 16; 17; 21] and references therein. It is well known that the main technique for solving stabilizability for control systems is the Lyapunov function method, but finding Lyapunov functions is still a difficult task (see, e.g. [3; 13; 15; 19; 20; 22]). However, for linear control system (1.1), the system can be made exponentially stabilizable if the underlying system $\dot{x}(t) = A(t)x(t)$ is asymptotically stable. In other words, if the evolution operator $E(t, s)$ generated by $A(t)$ is stable, then the delay control system (1.1) is asymptotically stabilizable under appropriate conditions on $A_1(t)$ (see [1; 17; 22]). For infinite-dimensional control systems, the investigation of stabilizability is more complicated and requires sophisticated techniques from semigroup theory. The difficulties increase to the same extent as passing from time-invariant to time-varying systems. Some results have been given in [2; 4; 9; 17] for time-invariant systems in Hilbert spaces.

This paper considers linear abstract control systems with both time-varying and time-delayed states and the object is to find stabilizability conditions based on the global controllability of undelayed control system $[A(t), B(t)]$. In contrast to [1; 17; 19], the stabilizability conditions obtained in this paper are derived by solving Riccati differential equations and do not involve any stability assumption on the evolution operator $E(t,s)$. New sufficient conditions for the stabilizability of a class of linear systems with nonlinear delay perturbations in Hilbert spaces are also established. The main results of the paper are further generalizations to infinite-dimensional case and can be regarded as extensions of the results of [7; 12; 14; 21].

The paper is organized as follows. In Section 2 we give the notation, and definitions to be used in this paper. Auxiliary propositions are given in Section 3. Sufficient conditions for the stabilizability are presented in Section 4.

2. NOTATION AND DEFINITIONS

We will use the following notation: $\mathbb{R}^+$ denotes the set of all non-negative real numbers. X denotes a Hilbert space with the norm $\|.\|_X$ and the inner product $\langle ., . \rangle_X$, etc. $L(X)$ (respectively, $L(X,Y)$) denotes the Banach space of all linear bounded operators S mapping X into X (respectively, X into Y) endowed with the norm

$$\|S\| = \sup\{\|Sx\| : x \in X, \|x\| \leq 1\}.$$

$L_2([t,s], X)$ denotes the set of all strongly measurable square integrable X-valued functions on $[t,s]$. $D(A), Im(A), A^*$ and A^{-1} denote the domain, the image, the adjoint and the inverse of the operator A, respectively. If A is a matrix, then A^T denotes the conjugate transpose of A. $B_1 = \{x \in X : \quad \|x\| = 1\}$. $\mathrm{cl}\, M$ denotes the closure of a set M; I denotes the identity operator. $C_{[t,s],X}$ denotes the set of all X-valued continuous functions on $[t,s]$. Let X, U be Hilbert spaces. Consider a linear time-varying control undelayed system $[A(t), B(t)]$ given by

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad t \geq t_0, x(t_0) = x_0, \tag{2.1}$$

where $x(t) \in X$, $u(t) \in U$; $A(t) : X \to X$; $B(t) \in L(U, X)$.

In the sequel, we say that control $u(t)$ is admissible if $u(t) \in L_2([t_0, \infty), U)$.

We make the following assumptions on the system (2.1):

(i) $B(t) \in L(U, X)$ and $B(.)u \in C_{[t_0,\infty),X}$ for all $u \in U$.

(ii) The operator $A(t) : D(A(t)) \subset X \to X$, $\mathrm{cl}\, D(A(t)) = X$ is a bounded function in $t \in [t_0, \infty)$ and generates a strong evolution operator $E(t,\tau) : \{(t,\tau) : t \geq \tau \geq t_0\} \to L(X)$ (see, e.g. [5; 6]):

$$E(t,t) = I, \quad t \geq t_0, \quad E(t,\tau)E(\tau,r) = E(t,r), \quad \forall t \geq \tau \geq r \geq t_0\,,$$

$E(t,\tau)$ is continuous in t and τ, $E(t,t_0)x = x + \int_{t_0}^t E(t,\tau)A(\tau)x d\tau$, for all $x \in D(A(t))$, so that the system (2.1), for every admissible control $u(t)$ has a unique solution given by

$$x(t) = E(t, t_0)x_0 + \int_{t_0}^{t} E(t, \tau)B(\tau)u(\tau)d\tau.$$

Definition The system $[A(t), B(t)]$ is called globally null-controllable in time $T > 0$, if every state can be transferred to 0 in time T by some admissible control $u(t)$, i.e.,

$$\operatorname{Im} U(T, t_0) \subset L_T(L_2([t_0, T), U),$$

where $L_T = \int_{t_0}^{T} E(T, s)B(s)ds$.

Definition The system $[A(t), B(t)]$ is called stabilizable if there exists an operator function $K(t) \in L(X, U)$ such that the zero solution of the closed loop system $\dot{x} = [A(t) + B(t)K(t)]x$ is asymptotically stable in the Lyapunov sense.

Following the setting in [2], we give a concept of the Riccati differential equation in a Hilbert space. Consider a differential operator equation

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) - P(t)B(t)R^{-1}B^*(t)P(t) + Q(t) = 0, \qquad (2.2)$$

where $P(t), Q(t) \in L(X)$ and $R > 0$ is a constant operator.

Definition An operator $P(t) \in L(X)$ is said to be a solution of the Riccati differential equation (2.2) if for all $t \geq t_0$ and all $x \in D(A(t))$,

$$\langle \dot{P}x, x\rangle + \langle PAx, x\rangle + \langle Px, Ax\rangle - \langle PBR^{-1}B^*Px, x\rangle + \langle Qx, x\rangle = 0\,.$$

An operator $Q \in L(X)$ is said to be non-negative definite ,denote by $Q \geq 0$, if $\langle Qx, x\rangle \geq 0$, for all $x \in X$. If for some $c > 0, \langle Qx, x\rangle > c\|x\|^2$ for all $x \in X$, then Q is called positive definite and is denote by $Q > 0$. Operator $Q \in L(X)$ is called self-adjoint if $Q = Q^*$. The self-adjoint operator is characterized by the fact that its inner product $\langle Qx, x\rangle$ takes only real values and its spectrum is a bounded closed set on the real axis. The least segment that contains the spectrum is $[\lambda_{\min}(Q), \lambda_{\max}(Q)]$, where

$$\lambda_{\min}(Q) = \inf\{\langle Qx, x\rangle : x \in B_1\}, \lambda_{\max}(Q) = \sup\{\langle Qx, x\rangle : x \in B_1\} = \|Q\|.$$

We denote by $BC([t, \infty], X^+)$ the set of all linear bounded self-adjoint non-negative definite operators in $L(X)$ that are continuous and bounded on $[t, \infty)$.

3. AUXILIARY PROPOSITIONS

To prove the main results we need the following propositions.

Proposition 3.1 [5]. *If $Q \in L(X)$ is a self-adjoint positive definite operator, then $\lambda_{\min}(Q) > 0$ and*

$$\lambda_{\min}(Q)\|x\|^2 \leq \langle Qx, x\rangle \leq \lambda_{\max}(Q)\|x\|^2, \quad \forall x \in X.$$

Proposition 3.2 [2; 5]. *The system $[A(t), B(t)]$ is globally null-controllable in time $T > 0$ if and only if one of the following conditions hold:*

(i) There is a number $c > 0$ such that

$$\int_{t_0}^{T} \|B^*(s)E^*(T, s)x\|^2 ds \geq c\|E^*(T, t_0)x\|^2, \quad \forall x \in X.$$

(ii) The operator $\int_{t_0}^{T} E(T,s)B(s)B^*(s)E^*(T,s)ds$ *is positive definite.*

Associated with control system $[A(t), B(t)]$ we consider the cost functional

$$J(u) = \int_0^\infty [\langle Ru(t), u(t)\rangle + \langle Q(t)x(t), x(t)\rangle]dt, \tag{3.1}$$

where $R > 0$, $Q(t) \in BC([t_0, \infty), X^+)$. The following proposition solves the optimal quadratic problem (2.1)–(3.1).

PROPOSITION 3.3 [18]. *Assume that the optimal quadratic problem (2.1)-(3.1) is solved in the sense that for every initial state* x_0 *there is an admissible control* $u(t)$ *such that the cost functional (3.1) exists and is finite. Then the Riccati differential equation (2.2) has a solution* $P(t) \in BC([t_0, \infty), X^+)$*. Moreover, the control* $u(t)$ *is given in the feedback form*

$$u(t) = -R^{-1}B^*(t)P(t)x(t), \quad t \geq t_0 \tag{3.2}$$

minimizes functional (3.1).

For the finite-dimensional case, it is well known [12; 14] that if system $[A, B]$ is globally null-controllable then the control

$$u(t) = -B^T P^{-1}(T)x(t), \quad T > t_0,$$

where $P(T) > 0$ is the solution of the Riccati equation

$$\dot{P}(t) + AP(t) + P(t)A^T + P(t)QP(t) + BR^{-1}B^T, \quad P(t_0) = 0,$$

for a matrix $Q > 0$, minimizes the cost functional (3.1). In the proposition below, we extend this assertion to the infinite-dimensional case based on solving an optimal quadratic control problem.

PROPOSITION 3.4. *If control system* $[A(t), B(t)]$ *is globally null-controllable in finite time, then for every operator* $Q(t) \in BC([t_0, \infty), X^+)$*, Riccati differential equation (2.2) has a solution* $P(t) \in BC([t_0, \infty), X^+)$ *and the feedback control (3.2) minimizes the cost functional (3.1).*

Proof. Assume that the system is globally null-controllable in some $T > t_0$. Let us take operators $R > 0, Q(t) \in BC([t_0, \infty), X^+)$ and consider a linear optimal quadratic control problem for the system $[A(t), B(t)]$ with the cost functional (3.1). Due to the global null-controllability, for every initial state $x_0 \in X$ there is an admissible control $u(t) \in L_2([t_0, T], U)$ such that the solution $x(t)$ of the system, according to the control $u(t)$, satisfies

$$x(t_0) = x_0, \quad x(T) = 0.$$

Let $u_x(t)$ denote an admissible control according to the solution $x(t)$ of the system. Define

$$\tilde{u}(t) = u_x(t), t \in [t_0, T], \tilde{u}(t) = 0, \quad t > T.$$

If $\tilde{x}(.)$ is the solution corresponding to $\tilde{u}_{\tilde{x}}(.)$, then $\tilde{x}(t) = 0$ for all $t > T$. Therefore, for every initial state x_0, we have

$$J(u) = \int_{t_0}^\infty [\langle Q(s)\tilde{x}(s), \tilde{x}(s)\rangle + \langle R\tilde{u}_{\tilde{x}}(s), \tilde{u}_{\tilde{x}}(s)]ds < +\infty.$$

The assumption of Proposition 3.3 for the optimal quadratic problem (2.1), (3.1) is satisfied and hence there is an operator function $P(t) \in BC([t_0,\infty), X^+)$, which is a solution of the Riccati equation (2.2) and the control (3.2) minimizes the cost functional (3.1). Proposition is proved. $\diamondsuit$

We conclude this section with a Lyapunov stability result on functional differential equations. Consider a general functional differential equation of the form

$$\dot{x}(t) = f(t, x_t), \quad t \geq t_0, x(t) = \phi(t), \quad t \in [-h, t_0], \tag{3.3}$$

where $\phi(t) \in C_{[-h,t_0],X}$, $x_t(s) = x(t+s)$, $-h \leq s \leq t_0$. Define

$$\|x_t\| = \sup_{s\in[-h,t_0]} \|x(t+s)\|.$$

PROPOSITION 3.5 [11]. *Assume that there exist a function*

$$V(t, x_t) : R^+ \times C([t_0, -h]) \to R^+$$

and numbers $c_1 > 0, c_2 > 0, c_3 > 0$ such that

(i) $c_1\|x(t)\|^2 \leq V(t, x_t) \leq c_2\|x_t\|^2$, *for all* $t \geq t_0$.

(ii) $\frac{d}{dt}V(t, x_t) \leq -c_3\|x(t)\|^2$, *for all* $t \geq t_0$.

Then the system (3.3) is asymptotically stable.

4. STABILIZABILITY CONDITIONS

Consider the linear control delay system (1.1), where $x(t) \in X$, $u(t) \in U$; X, U are infinite-dimensional Hilbert spaces; $A_1(t) : X \to X$ and $A(t), B(t)$ satisfy the assumptions stated in Section 2 so that the control system (1.1) has a unique solution for every initial condition $\phi(t) \in C_{[0,\infty),X}$ and admissible control $u(t)$. Let

$$p = \sup_{t\in[t_0,\infty)} \|P(t)\|.$$

We denote by $BC([t,\infty], X^+)$ the set of all linear bounded self-adjoint non-negative definite operators in $L(X)$ that are continuous and bounded on $[t,\infty)$.

THEOREM 4.1. *Assume that for some self-adjoint constant positive definite operator $Q \in L(X)$, the Riccati differential equation (2.2), where $R = I$ has a solution $P(t) \in BC([t_0,\infty), X^+)$ such that*

$$a_1 := \sup_{t\in[t_0,\infty)} \|A_1(t)\| < \frac{\sqrt{\lambda_{\min}(Q)}}{2p}. \tag{4.1}$$

Then the control delay system (1.1) is stabilizable.

Proof. For simplicity of expression, let $t_0 = 0$. Let $0 < Q \in L(X)$, $P(t) \in BC([0,\infty), X^+)$ satisfy the Riccati equation (2.2), where $R = I$. Let

$$u(t) = K(t)x(t), \tag{4.2}$$

where $K(t) = -\frac{1}{2}B^*(t)P(t)$, $t \geq 0$.

For some number $\alpha \in (0,1)$ to be chosen later, we consider a Lyapunov function, for the delay system (1.1),

$$V(t,x_t) = \langle P(t)x(t), x(t)\rangle + \alpha \int_{t-h}^{t} \langle Qx(s), x(s)\rangle ds.$$

Since $Q > 0$ and $P(t) \in BC([0,\infty), X^+)$, it is easy to verify that

$$c_1\|x(t)\|^2 \le V(t,x_t) \le c_2\|x_t\|^2,$$

for some positive constants c_1, c_2. On the other hand, taking the derivative of $V(t,x_t)$ along the solution $x(t)$ of the system, we have

$$\dot{V}(t,x_t) = \langle \dot{P}(t)x(t), x(t)\rangle + 2\langle P(t)\dot{x}(t), x(t)\rangle + \alpha[\langle Qx(t), x(t)\rangle - \langle Qx(t-h), x(t-h)\rangle]. \tag{4.3}$$

Substituting the control (4.2) into (4.3) gives

$$\begin{aligned}\dot{V}(t,x_t) &= -(1-\alpha)\langle Qx(t), x(t)\rangle + 2\langle P(t)A_1(t)x(t-h), x(t)\rangle \\ &\quad -\alpha\langle Qx(t-h), x(t-h)\rangle.\end{aligned}$$

From Proposition 3.1 it follows that

$$\lambda_{\min}(Q)\|x\|^2 \le \langle Qx, x\rangle \le \lambda_{\max}(Q)\|x\|^2, \quad x \in X,$$

where $\lambda_{\min}(Q) > 0$. Therefore,

$$\dot{V}(t,x_t) \le -\lambda_{\min}(Q)(1-\alpha)\|x\|^2 + 2pa_1\|x(t-h)\|\|x(t)\| - \lambda_{\min}(Q)\alpha\|x(t-h)\|^2.$$

By completing the square, we obtain

$$\begin{aligned}&2pa_1\|x(t-h)\|\|x(t)\| - \lambda_{\min}(Q)\alpha\|x(t-h)\|^2 \\ &= -\Big[\sqrt{\alpha\lambda_{\min}(Q)}\|x(t-h)\| - \frac{pa_1}{\sqrt{\alpha\lambda_{\min}(Q)}}\|x(t)\|\Big]^2 + \frac{p^2a_1^2}{\alpha\lambda_{\min}(Q)}\|x(t)\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\dot{V}(t,x_t) &\le -\lambda_{\min}(Q)(1-\alpha)\|x(t)\|^2 + \frac{p^2a_1^2}{\alpha\lambda_{\min}(Q)}\|x(t)\|^2 \\ &= -\Big[\lambda_{\min}(Q)(1-\alpha) - \frac{1}{\alpha\lambda_{\min}(Q)}p^2a_1^2\Big]\|x(t)\|^2.\end{aligned}$$

Since the maximum value of $\alpha(1-\alpha)$ in $(0,1)$ is attained at $\alpha = 1/2$, from (4.1) it follows that for some $c_3 > 0$,

$$\dot{V}(t,x_t) \le -c_3\|x(t)\|^2, \quad \forall t \ge t_0.$$

The the present proof is complete by using Proposition 3.5. $\diamondsuit$

The following theorem shows that if the system $[A(t), B(t)]$ is globally null-controllable then the delay system (1.1) is stabilizable under an appropriate condition on $A_1(t)$.

THEOREM 4.2. *Assume that $[A(t), B(t)]$ is globally null-controllable in finite time. Then the delay system (1.1) is stablizable if (4.1) holds, where $Q(t) = I$, and $P(t)$ satisfies the Riccati equation (2.2). Moreover, the feedback control is given by*

$$u(t) = -\frac{1}{2}B^*(t)P(t)x(t), \quad t \ge 0.$$

Proof. By assumption, the system $[A(t), B(t)]$ is globally null-controllable in some $T > 0$ time. This means that for every initial state $x_0 \in X$ there is an admissible control $u(t) \in L_2([0,T], U)$ such that the solution $x(t)$ of the system according to the control $u(t)$ satisfies

$$x(0) = x_0, \quad x(T) = 0.$$

Define an admissible control $\tilde{u}(t)$, $t \geq 0$ by

$$\tilde{u}(t) = u(t), \quad t \in [0,T], \tilde{u}(t) = 0, \quad t > T.$$

Denoting by $\tilde{x}(t)$ the solution under to the control $\tilde{u}(t)$, we have

$$\begin{aligned} J(\tilde{u}) &= \int_0^\infty [\|\tilde{u}(t)\|^2 + \|\tilde{x}(t)\|^2] dt \\ &= \int_0^T [\|u(t)\|^2 + \|x(t)\|^2] dt < +\infty. \end{aligned}$$

Therefore, by Proposition 3.4, there is $P(t) \in BC([0,\infty), X^+)$ satisfying the Riccati differential equation (2.2), where $Q = R = I$. Based on the condition (4.1) the proof is completed by the same arguments used in the proof of Theorem 4.1, where we use the same feedback control operator $K(t)$ and the Lyapunov function $V(t, x_t)$.

Remark Note that when $Q = I$, then the condition (4.1) is replaced by the condition

$$\sup_{t\in[0,\infty)} \|A_1(t)\| < \frac{1}{2p}. \tag{4.4}$$

Therefore, when the controllability problem of the linear control system is solvable, the following step-by-step procedure can be used to find the feedback controller for system (1.1).

Step 1: Verify the controllability conditions by Proposition 3.1.

Step 2: Find a solution $P(t) \in BC([t_0,\infty), X^+)$ to the Riccati differential equation

$$\dot{P}(t) + A^*(t)P(t) + P(t)A(t) - P(t)B(t)B^*(t)P(t) + I = 0 \tag{4.5}$$

Step 3: Compute $a_1 = \sup_{t\in[0,\infty)}\{\|A_1(t)\|\}$ and $p = \sup_{t\in[0,\infty)}\{\|P(t)\|\}$.

Step 4: Verify the condition (4.4)

Step 5: The stabilizing controller is then defined by (4.2).

In the same way, Theorem 4.2 can be extended to the system with multiple delays

$$\dot{x}(t) = A(t)x(t) + \sum_{i=1}^r A_i(t)x(t-h_i) + B(t)u(t), \quad t \geq t_0, x(t) = \phi(t), \quad t \in [-h_r, t_0], \tag{4.6}$$

where $A_i(t) \in L(X)$, $0 \leq h_1 \leq \ldots \leq h_r$, $r \geq 1$.

THEOREM 4.3. *Let the control system $[A(t), B(t)]$ be globally null-controllable in some finite time. Assume that*

$$\sum_{i=1}^r \sup_{t\in[t_0,\infty)} \|A_i(t)\|^2 < \frac{2-r}{4p^2}.$$

Then the control delay system (4.6) is stabilizable.

The proof is similar to the proof of Theorem 4.2, with $Q = I$ and

$$V(t, x_t) = \langle P(t)x(t), x(t)\rangle + \frac{1}{2}\sum_{i=1}^{r}\int_{t-h_i}^{t}\|x(s)\|^2 ds.$$

Remark It is worth noting that although the Lyapunov function method is not used, the results obtained in [8, 9] give us explicit stabilizability conditions under a dissipative assumption on the operator $W(t) = A(t) + A_1(t-h) + B(t)K(t)$. In contrast to these conditions, our conditions are obtained via the controllability assumption and the solution of Riccati differential equation (4.5) and do not involve the stability of evolution operator $E(t, s)$ or the dissipative property of the operator $W(t)$, therefore they can be easily verified and constructed.

As an application, we consider the stabilization of the nonlinear control system in Hilbert spaces

$$\dot{x}(t) = A(t)x(t) + A_1(t)x(t-h) + B(t)u(t) + f(t, x(t), x(t-h), u(t)),$$

$$t \geq 0, x(t) = \phi(t), \quad t \in [-h, 0], \tag{4.7}$$

where $x \in X, u \in U$ and $f(t, x, y, u) : [0, \infty) \times X \times X \times U \to X$ is a given nonlinear function. We recall that nonlinear control system (4.7) is stabilizable by a feedback control $u(t) = K(t)x(t)$, where $K(t) \in L(X, U)$, if the closed-loop system

$$\dot{x} = [A(t)x + K(t)B(t)]x + A_1(t)x(t-h) + f(t, x, x(t-h), K(t)x),$$

is asymptotically stable. Stabilizability of nonlinear control systems has been considered in [1; 14; 21] under the stability assumption on the evolution operator $E(t, s)$ and on the perturbation function $f(t, .)$ that for all $(t, x, y, u) \in [0, \infty) \times X \times X \times U$,

$$\|f(t, x, y, u)\| \leq a\|x\| + b\|y\| + c\|u\| \tag{4.8}$$

for some positive numbers a, b, c. In the following, in contrast to the mentioned above results, we give stabilizability conditions for nonlinear control system (4.7) via the global null-controllability of linear control system (2.1). Let

$$\beta = \sup_{t\in[0,+\infty)}\|B(t)\|, \quad a_1 = \sup_{t\in[0,+\infty)}\|A_1(t)\|, \quad p = \sup_{t\in[0,+\infty)}\|P(t)\|.$$

THEOREM 4.4. *Let the linear control system $[A(t), B(t)]$ be globally null-controllable in finite time. Assume that $a_1 \leq 1/(2p)$ and the condition (4.8) holds for positive numbers a, b, c satisfying*

$$a < \frac{1 - 4a_1^2p^2}{4p}, \quad 2b^2p^2 + c\beta p^2 + 4ba_1p^2 < \frac{1}{2} - 2ap - 2a_1^2p^2. \tag{4.9}$$

Then the nonlinear control system (4.7) is stabilizable.

Proof. Since the system $[A(t), B(t)]$ is globally null-controllable in finite time, by Proposition 3.4, for $Q = I$ there is an operator $P(t) \in BC([0, \infty), X^+)$ satisfying the Riccati equation (4.5). Let us consider the Lyapunov function

$$V(t, x_t) = \langle P(t)x(t), x(t)\rangle + \frac{1}{2}\int_{t-h}^{t}\|x(s)\|^2 ds$$

for the nonlinear control system (4.7). Taking the derivative of $V(t, x_t)$ along the solution $x(t)$ we have

$$\begin{aligned}
\frac{d}{dt}V(t,x_t)) &= \langle \dot{P}(t)x(t), x(t)\rangle + 2\langle P(t)\dot{x}(t), x(t)\rangle + \frac{1}{2}(\|x(t)\|^2 - \|x(t-h)\|^2) \\
&\le -\frac{1}{2}\|x(t)\|^2 + 2\langle P(t)f(x(t), x(t-h), u(t)), x(t)\rangle \\
&\quad +2\langle P(t)A_1(t)x(t-h), x(t)\rangle - \frac{1}{2}\|x(t-h)\|^2. \qquad (4.10)
\end{aligned}$$

Substituting the control $u(t) = -\frac{1}{2}B^*(t)P(t)x(t)$ in (4.10) gives

$$\begin{aligned}
\frac{d}{dt}V(t,x_t) &\le -\frac{1}{2}\|x(t)\|^2 + 2p\Big[a\|x(t)\| + b\|x(t-h)\| + \frac{c}{2}p\beta\|x(t)\|\Big]\|x(t)\| \\
&\quad +2pa_1\|x(t-h)\|\|x(t)\| - \frac{1}{2}\|x(t-h)\|^2 \\
&\le (-\frac{1}{2} + 2ap + c\beta p^2)\|x\|^2 + 2p\langle(b+a_1)\|x(t-h)\|\|x(t)\| \\
&\quad -\frac{1}{2}\|x(t-h)\|^2 \\
&\le -\Big[\frac{1}{2} - 2ap - c\beta p^2 - 2(b+a_1)^2p^2\Big]\|x(t)\|^2.
\end{aligned}$$

Therefore, from condition (4.9) it follows that there is a number $c_3 > 0$ such that

$$\frac{d}{dt}V(t,x(t)) \le -c_3\|x(t)\|^2, \quad \forall t \ge 0.$$

The proof is then completed by using Proposition 3.5. $\diamondsuit$

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