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GENERALIZED MEASURES OF FUZZY DIRECTED-DIVERGENCE, TOTAL AMBIGUITY AND INFORMATION IMPROVEMENT

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Abstract

Uncertainty and fuzziness are the basic nature of human thinking and of many real world objectives. In the present communication, we have introduced two new generalized measures of fuzzy directed divergence with the proof of their validity. Further, we studied some measures of total ambiguity and new generalized measures of fuzzy information improvement. We have also discussed particular cases of corresponding directed divergence and symmetric divergence.

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1. INTRODUCTION

Shannon (1948) used "entropy" to measure uncertain degree of the randomness in a probability distribution. Let X is a discrete random variable with probability distribution $P = (p_1, p_2, ..., p_n)$ in an experiment. The information contained in this experiment is given by

$$H(P) = -\sum_{i=1}^{n} p_i \log p_i, \tag{1}$$

which is well known Shannon entropy.

The concept of entropy has been widely used in different areas, e.g. communication theory, statistical mechanics, finance, pattern recognition, and neural network etc. Fuzzy set theory developed by Lofti A. Zadeh (1965) has found wide applications in many areas of science and technology, e.g. clustering, image processing, decision making etc. because of its capability to model non-statistical imprecision or vague concepts.

It may be recalled that a fuzzy subset A in U (universe of discourse) is characterized by a *membership function* $\mu_A : U \to [0, 1]$ which represents the *grade of*

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membership of $x \in U$ in A as follows:

$$\mu_A(x) = \begin{cases} 0, & \text{if } x \notin A \text{ and there is no ambiguity,} \\ 1, & \text{if } x \in A \text{ and there is no ambiguity,} \\ 0.5, & \text{if maximum ambiguity i.e. } x \in A \text{ or } x \notin A \end{cases}$$

In fact $\mu_A(x)$ associates with each $x \in U$, a grade of membership in the set A. When $\mu_A(x)$ is valued in $\{0,\ 1\}$, it is the characteristic function of a crisp (i.e. nonfuzzy) set.

A fuzzy set A^* is called a *sharpened* version of A if the following conditions are satisfied:

$$\mu_{A^*}(x_1) \leq \mu_A(x_1), \text{ if } \mu_A(x_1) \leq 0.5; \quad \forall i$$

and

$$\mu_{A^*}(x_i) \geq \mu_A(x_i), \text{ if } \mu_A(x_i) \leq 0.5; \quad \forall i.$$

Since $\mu_A(x)$ and $1 - \mu_A(x)$ gives the same degree of fuzziness, therefore, corresponding to the entropy due to Shannon (1948), De Luca and Termini (1972) suggested the following measure of fuzzy entropy:

$$\begin{aligned} H(A) \;=\; -\sum_{i=1}^{n} [&\mu_A(x_i) \log \mu_A(x_i) \\ &+ (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i))] \,. \end{aligned} \tag{2}$$

De Luca and Termini introduced a set of four properties and these properties are widely accepted as a criterion for defining any new fuzzy entropy. In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity/difficulty in making a decision whether an element belongs to a set or not. So, a measure of average fuzziness $H(A)$ in a fuzzy set should have at least the following properties to be valid fuzzy entropy:

—P1 (*Sharpness*): $H(A)$ is minimum if and only if A is a crisp set, i.e. $\mu_A(x) = 0$ or $1; \forall x$.
—P2 (*Maximality*): $H(A)$ is maximum if and only if A is most fuzzy set, i.e. $\mu_A(x) = 0.5;\ \ \forall x$.
—P3 (*Resolution*): $H(A) \geq H(A^*)$, where A^* is sharpened version of A.
—P4 (*Symmetry*): $H(A) = H(\bar{A})$, where $\bar{A}$ is the complement of A i.e. $\mu_{\bar{A}}(x_i) = 1 - \mu_A(x_i)$.

Later on Bhandari and Pal (1993) made a survey on information measures on fuzzy sets and gave some new measures of fuzzy entropy. Corresponding to Renyi's (1961) entropy they have suggested the following measure:

$$H_\alpha(A) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \log \left[\mu_A^\alpha(x_i) + (1 - \mu_A(x_i))^\alpha\right], \tag{3}$$

$\alpha \neq 1,\ \alpha > 0$.

Also, Hooda and Bajaj (2008) proposed a new generalized R-norm measure of fuzzy information and a generalized fuzzy directed divergence measure analogous to a R-norm directed divergence.

Kullback and Leibler (1951) obtained the measure of directed divergence of probability distribution $P = (p_1, p_2, \ldots, p_n)$ from the probability distribution $Q = (q_1, q_2, \ldots, q_n)$ as

$$D(P:Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}. \tag{4}$$

Kullback (1959) suggested the measure of symmetric divergence as

$$J(P:Q) = \sum_{i=1}^{n} (p_i - q_i) \log \frac{p_i}{q_i}. \tag{5}$$

Let A and B be two standard fuzzy sets with same supporting points x_1, x_2, $\ldots$,x_n and with fuzzy vectors

$$\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)$$

and $\mu_B(x_1), \mu_B(x_2), \ldots, \mu_B(x_n)$. The simplest measure of fuzzy directed divergence as suggested by Bhandari and Pal (1993), is

$$\begin{aligned} I(A,B) \ = \ & \sum_{i=1}^{n} \Bigg[\mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} \\ & + (1 - \mu_A(x_i)) \log \frac{(1 - \mu_A(x_i))}{(1 - \mu_B(x_i))} \Bigg] \end{aligned} \tag{6}$$

and the corresponding symmetric divergence measure by

$$J(A,B) = I(A,B) + I(B,A),$$

which on simplification gives

$$\begin{aligned} J(A,B) \ = \ & \sum_{i=1}^{n} [(\mu_A(x_i) - \mu_B(x_i)] \\ & \times \log \frac{\mu_A(x_i)(1 - \mu_A(x_i))}{\mu_B(x_i)(1 - \mu_B(x_i))}. \end{aligned} \tag{7}$$

It is important to notice that if we take $B = A_F$ (the most fuzzy set) i.e. $\mu_B(x_i) = 0.5; \forall\, i$, then from (6) and (2) we have

$$\begin{aligned} I(A, A_F) \ = \ & n \log 2 - \Big[- \sum_{i=1}^{n} \mu_A(x_i) \log \mu_A(x_i) \\ & + (1 - \mu_A(x_i)) \log (1 - \mu_A(x_i)) \Big] \end{aligned}$$

i.e.,

$$I(A, A_F) = n \log 2 - H(A). \tag{8}$$

The uncertainty is the state of being uncertain (i.e. not certain to occur) which gives rise to fuzziness and ambiguity. Ambiguity can be viewed in non-specificity (i.e. indistinguishable alternatives) and conflict (i.e. distinguishable alternatives) while fuzziness can be viewed as lack of distinction between a set and its complement and vagueness is non-specific knowledge about lack of distinction. Kapur (1997) suggested the measure of total fuzzy ambiguity which can be obtained by taking the sum of measure of fuzzy directed divergence and corresponding measure of fuzzy entropy. From (2) and (6), we have

$$\begin{aligned} TA &= -\sum_{i=1}^{n} \mu_A(x_i) \log \mu_B(x_i) \\ &\quad -\sum_{i=1}^{n}(1-\mu_A(x_i)) \log(1-\mu_B(x_i)). \end{aligned} \tag{9}$$

and measure of fuzzy information improvement can be obtained if we subtract directed divergence of fuzzy set A and fuzzy set C from the directed divergence of fuzzy set A and fuzzy set B, which gives the reduction in ambiguity in revising B to C.

In the present paper, two new generalized measures of fuzzy directed divergence are introduced and studied in Section 2. Their particular cases have also been discussed. In Section 3, we have obtained the measures of total ambiguity. In Section 4, new generalized measures of fuzzy information improvement are defined and discussed.

2. GENERALIZED MEASURES OF FUZZY DIRECTED DIVERGENCE

Harvda and Charvat (1967) defined the directed divergence measure of a probability distribution $P = (p_1, p_2, \ldots, p_n)$ from another probability distribution $Q = (q_1, q_2, \ldots, q_n)$ as

$$D^{\beta}(P:Q) = \frac{1}{\beta-1}\sum_{i=1}^{n}(p_i^{\beta} q_i^{1-\beta} - 1); \quad \beta > 0, \, \beta \neq 1, \tag{10}$$

which is called the generalized directed divergence of degree β.

The following measure of symmetric divergence was proposed by Kullback (1959):

$$\begin{aligned} J^{\beta}(P:Q) &= D^{\beta}(P:Q) + D^{\beta}(Q:P) \\ &= \frac{1}{\beta-1}\sum_{i=1}^{n}(p_i^{\beta} q_i^{1-\beta} + q_i^{\beta} p_i^{1-\beta} - 2), \end{aligned} \tag{11}$$

which is also called a distance measure of degree β.

Corresponding to the measure (10) and (11), Hooda (2004) suggested the following measures of fuzzy directed divergence:

$$\begin{aligned} I^{\beta}(A,B) &= \frac{1}{\beta-1}\sum_{i=1}^{n}\Big[\mu_A^{\beta}(x_i)\mu_B^{1-\beta}(x_i) \\ &\quad +(1-\mu_A x_i))^{\beta}(1-\mu_B(x_i))^{1-\beta} - 1\Big] \end{aligned} \tag{12}$$

and

$$J^{\beta}(A,B) = I^{\beta}(A,B) + I^{\beta}(B,A) \tag{13}$$

respectively.

Further, it is proved that $I^\beta(A,B) \geq 0$ for all $\beta(\neq 1) > 0$ and it vanishes only when $A = B$, which shows that (12) is a valid measure of fuzzy directed divergence of fuzzy set A and B. Hence $J^\beta(A,B)$ is a valid symmetric divergence measure.

2.1 Fuzzy Directed Divergence corresponding to Renyi's measure

Corresponding to Renyi's measure (1961) of directed divergence

$$D_\alpha(P,Q) = \frac{1}{\alpha-1}\log\left(\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}\right), \tag{14}$$

$\alpha \neq 1, \alpha > 0$, we define the measure of fuzzy directed divergence of fuzzy set A from fuzzy set B,

$$\begin{aligned} I_\alpha(A,B) \;=\; & \frac{1}{\alpha-1}\sum_{i=1}^{n}\log\Big[\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) \\ & +(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\Big], \end{aligned} \tag{15}$$

where $\alpha \neq 1,\ \alpha > 0$ and measure of fuzzy symmetric divergence

$$J_\alpha(A,B) = I_\alpha(A,B) + I_\alpha(B,A). \tag{16}$$

The measures (15) and (16) are called the generalized measures of order α.

Next, we show that $I_\alpha(A,B)$ is a valid fuzzy directed divergence measure. $I_\alpha(A,B)$ is a valid measure only if it is non-negative. So it is proved that $I_\alpha(A,B) \geq 0$ with equality if $\mu_A(x_i) = \mu_B(x_i)$ for each $i = 1,2,\ldots,n$.

Let $\sum\limits_{i=1}^{n}\mu_A(x_i) = s, \quad \sum\limits_{i=1}^{n}\mu_B(x_i) = t.$

Then

$$\frac{1}{\alpha-1}\left[\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)}{s}\right)^\alpha\left(\frac{\mu_B(x_i)}{t}\right)^{1-\alpha} - 1\right] \geq 0$$

or

$$\frac{1}{\alpha-1}\sum_{i=1}^{n}\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) \quad \geq \quad \frac{1}{\alpha-1}s^\alpha t^{1-\alpha}\,. \tag{17}$$

Similarly,

$$\begin{aligned} & \frac{1}{\alpha-1}\sum_{i=1}^{n}(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha} \\ & \qquad \geq \frac{1}{\alpha-1}(n-s)^\alpha(n-t)^{1-\alpha}. \end{aligned} \tag{18}$$

***Case 1*:** When $0 < \alpha < 1$,
i.e. $\frac{1}{\alpha-1} < 0$ then from (17) and (18) we have

$$\begin{aligned} & \sum_{i=1}^{n}\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha} \\ & \qquad \leq s^\alpha t^{1-\alpha} + (n-s)^\alpha(n-t)^{1-\alpha}, \end{aligned} \tag{19}$$

where $\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha} < 1$ but very close to 1, $\forall i = 1, 2, \ldots, n$, and $s^\alpha t^{1-\alpha} + (n-s)^\alpha(n-t)^{1-\alpha} < n$ but very close to n. This implies

$$\sum_{i=1}^n \log\left[\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\right]$$
$$<< \sum_{i=1}^n \mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}.$$

or

$$\sum_{i=1}^n \log\left[\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\right]$$
$$\leq s^\alpha t^{1-\alpha} + (n-s)^\alpha(n-t)^{1-\alpha} - n. \tag{20}$$

Though left hand side and right hand side of (20) are negative but the absolute value of right hand side is less than the absolute value of left hand side. Therefore, multiplying both sides of (20) by $\frac{1}{\alpha-1}$, we get

$$I_\alpha(A,B) \geq \frac{1}{\alpha-1}[s^\alpha t^{1-\alpha} + (n-s)^\alpha(n-t)^{1-\alpha} - n]. \tag{21}$$

For the sake of simplicity and constructing Table 1, we denote $a = s^\alpha t^{1-\alpha} + (n-s)^\alpha(n-t)^{1-\alpha} - n$; $b_i = \mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^\alpha(1 - \mu_B(x_i))^{1-\alpha}$, $i = 1, 2, \ldots, n$; $c = \sum_{i=1}^n \log(b_i)$ and $d = I_\alpha(A,B)$.

***Case 2*:** When $\alpha > 1$,
i.e. $\frac{1}{\alpha-1} > 0$, then from (17) and (18) we have

$$\sum_{i=1}^n b_i \geq s^\alpha t^{1-\alpha} + (n-s)^\alpha(n-t)^{1-\alpha}.$$

Similarly, it can be verified that

$$\sum_{i=1}^n \log(b_i) \geq [s^\alpha t^{1-\alpha} + (n-s)^\alpha(n-t)^{1-\alpha} - n]. \tag{22}$$

Multiplying both sides of (20) by $\frac{1}{\alpha-1}$, we get (21) in *Case 2* also.

$$\text{Further let} \quad \psi(s) = \frac{1}{\alpha-1}[s^\alpha t^{1-\alpha} + (n-s)^\alpha(n-t)^{1-\alpha} - n],$$

then

$$\psi'(s) = \frac{1}{\alpha-1}\left[\alpha\left(\frac{s}{t}\right)^{\alpha-1} - \alpha\left(\frac{n-s}{n-t}\right)^{\alpha-1}\right],$$

and

$$\psi''(s) = \left[\frac{\alpha}{t}\left(\frac{s}{t}\right)^{\alpha-2} + \frac{\alpha}{n-t}\left(\frac{n-s}{n-t}\right)^{\alpha-2}\right].$$

Clearly, $\psi''(s) > 0$ which shows that $\psi(s)$ is a convex function of s whose minimum value arises when $\frac{s}{t} = \frac{n-s}{n-t} = \frac{n}{n} = 1$. Now if $A = B$ (i.e. at $s = t$), then $\psi(s) = 0$. Hence $\psi(s) > 0$ and vanishes only when $s = t$.

Thus for all $\alpha > 0, I_\alpha(A,B) \geq 0$ and vanishes only when $A = B$.

Therefore, $I_\alpha(A,B)$ is a valid measure of directed divergence of fuzzy sets A and B; and consequently, $J_\alpha(A,B)$ is a valid measure of symmetric divergence.

Particular Cases:

— $\lim_{\alpha\to 1} I_\alpha(A,B) = I(A,B)$ and $\lim_{\alpha\to 1} J_\alpha(A,B) = J(A,B)$, where $I(A,B)$ and $J(A,B)$ are the fuzzy directed divergence and symmetric divergence measures given by (6) and (7) respectively.

—Let $B = A_F$, the most fuzzy set, i.e. $\mu_B(x_i) = 0.5 \quad \forall x_i$, then

$$\begin{aligned}
I_\alpha(A, A_F) &= \frac{1}{\alpha-1}\sum_{i=1}^{n}\log\Big[\mu_A^\alpha(x_i)(0.5)^{1-\alpha} \\
&\quad +(1-\mu_A(x_i))^\alpha(0.5)^{1-\alpha}\Big] \\
&= \frac{1}{\alpha-1}\Bigg[\sum_{i=1}^{n}\log(0.5)^{1-\alpha} \\
&\quad +\sum_{i=1}^{n}\log\Big[\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\Big]\Bigg] \\
&= \frac{1}{\alpha-1}\Bigg[\sum_{i=1}^{n}(\alpha-1)\log 2 \\
&\quad +\sum_{i=1}^{n}\log\Big[\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha\Big]\Bigg] \\
&= n\log 2 - \frac{1}{1-\alpha}\log\sum_{i=1}^{n}\Big[\mu_A^\alpha(x_i) \\
&\quad +(1-\mu_A(x_i))^\alpha\Big].
\end{aligned}$$

Thus $I_\alpha(A, A_F) = n\log 2 -$ (Entropy of the fuzzy set).

Example 1: Let A = (0.1, 0.3, 0.4, 0.2, 0.1) and B = (0.3, 0.5, 0.3, 0.1, 0.2) be two arbitrary standard fuzzy sets. *Case 1* and *Case 2* both can be verified from the computed Table 1 given below:

Table 1

α	a	b_1	b_2	b_3	b_4	b_5	c	d
0.1	-0.0044	0.9866	0.9922	0.9980	0.9966	0.9961	-0.0440	0.0489
0.2	-0.0078	0.9769	0.9862	0.9965	0.9939	0.9932	-0.0774	0.0967
0.5	-0.0120	0.9669	0.9789	0.9945	0.9899	0.9899	-0.1164	0.2327
0.7	-0.0100	0.9737	0.9825	0.9953	0.9912	0.9919	-0.0952	0.3175
0.9	-0.0042	0.9893	0.9926	0.9980	0.9961	0.9966	-0.0397	0.3976
1.2	0.0111	1.0267	1.0196	1.0055	1.0111	1.0085	0.1020	0.5100
1.5	0.0343	1.0782	1.0606	1.0174	1.0371	1.0253	0.3070	0.6140
2.0	0.0893	1.1905	1.1600	1.0476	1.1111	1.0625	0.7722	0.7722
5.0	0.7909	2.4606	2.7280	1.5881	3.6994	1.4479	5.8353	1.4588
10.0	3.1409	8.6406	14.4658	5.4772	102.6772	2.5981	17.4785	1.9420

2.2 Fuzzy Directed Divergence Corresponding to Sharma and Mittal's Measure

Sharma and Mittal (1975) characterized non-additive entropy of discrete probability distribution given by

$$H_\alpha^\beta(P) = \frac{1}{2^{1-\beta}-1}\left[\left(\sum_{i=1}^{n} p_i^\alpha\right)^{\frac{\beta-1}{\alpha-1}} - 1\right]; \tag{23}$$

where $\alpha \neq 1,\ \ \alpha > 0,\ \beta > 0,\ \beta \neq 1$.

Hooda (2004) suggested the following fuzzy entropy corresponding to (23):

$$H_\alpha^\beta(A) = \frac{1}{2^{1-\beta}-1}\sum_{i=1}^{n}\left[(\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha)^{\frac{\beta-1}{\alpha-1}} - 1\right], \tag{24}$$

where $\alpha \neq 1,\ \ \alpha > 0,\ \beta > 0,\ \beta \neq 1$.

Sharma and Mittal (1977) also studied the following generalized measure of directed divergence:

$$\frac{1}{1-2^{1-\beta}}\left[\left(\sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 1\right], \tag{25}$$

where $\alpha \neq 1,\ \ \alpha > 0,\ \beta > 0,\ \beta \neq 1$. Corresponding to (25), we define the following measures of fuzzy directed divergence and symmetric fuzzy directed divergence

$$I_\alpha^\beta(A,B) = \frac{1}{2^{\beta-1}-1}\sum_{i=1}^{n}\left[\left(\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i)\right.\right.$$
$$\left.\left. +(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\right)^{\frac{\beta-1}{\alpha-1}} - 1\right], \tag{26}$$

where $\alpha \neq 1$, $\alpha > 0$, $\beta > 0$, $\beta \neq 1$ and

$$J_\alpha^\beta(A,B) = I_\alpha^\beta(A,B) + I_\alpha^\beta(B,A) \tag{27}$$

respectively.

$I_\alpha^\beta(A,B)$ is a valid measure only if it is non-negative. We claim that $I_\alpha^\beta(A,B) \geq 0$ with equality if $\mu_A(x_i) = \mu_B(x_i)$ for each $i = 1, 2, \ldots, n$. From (26) it is obvious that $I_\alpha^\beta(A,B) = 0$ if $\mu_A(x_i) = \mu_B(x_i)$. Next, we prove that $I_\alpha^\beta(A,B) > 0$ for two different fuzzy sets empirically.

For the sake of simplicity and constructing the Tables 2 to 5, we denote

$$\begin{aligned} s &= \sum_{i=1}^{n}\mu_A(x_i); \\ t &= \sum_{i=1}^{n}\mu_B(x_i); \\ a &= s^\alpha t^{1-\alpha} + (n-s)^\alpha(n-t)^{1-\alpha} - n; \\ f &= I_\alpha^\beta(A,B); \end{aligned}$$

and

$$e = \sum_{i=1}^{n} e_i,$$

where

$$\begin{aligned} e_i &= (b_i)^{\frac{\beta-1}{\alpha-1}} - 1 \\ &= \Big[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) \\ &\quad +(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}\Big]^{\frac{\beta-1}{\alpha-1}} - 1, \end{aligned}$$

for $i = 1, 2, \ldots, n$. Further taking all the possible cases of α and β with two above considered standard fuzzy sets in Example 1 of section 2. Next, we tabulate the values given in Table 2 to 5.

Case 1: $0 < \alpha < 1$ and $0 < \beta < 1$. In this case $\frac{1}{2^{\beta-1}-1} < 0$, $\frac{\beta-1}{\alpha-1} > 0$ and the computed values are presented in Table 2.

Table 2

α	β	e_1	e_2	e_3	e_4	e_5	e	$I_\alpha^\beta(A,B)$
0.2	0.6	-0.0116	-0.0069	-0.0017	-0.0030	-0.0034	-0.0267	0.110357
0.4	0.8	-0.0110	-0.0068	-0.0017	-0.0031	-0.0032	-0.0260	0.201301
0.1	0.5	-0.0074	-0.0043	-0.0010	-0.0018	-0.0021	-0.0169	0.057806
0.3	0.9	-0.0042	-0.0025	-0.0006	-0.0011	-0.0012	-0.0099	0.148134
0.6	0.4	-0.0459	-0.0300	-0.0079	-0.0147	-0.0141	-0.1128	0.331761
0.8	0.1	-0.0849	-0.0584	-0.016	-0.0303	-0.0270	-0.2168	0.467185
0.7	0.2	-0.0686	-0.0460	-0.0124	-0.0232	-0.0215	-0.1719	0.403862
0.5	0.7	-0.0199	-0.0127	-0.0033	-0.0060	-0.0060	-0.0480	0.256106
0.9	0.3	-0.0727	-0.0509	-0.0140	-0.0270	-0.0232	-0.1880	0.48921

Observation: In this case, $I_\alpha^\beta(A,B)$ is a monotonic increasing function with respect to α but not with respect to β.

Case 2: $0 < \alpha < 1$ and $\beta > 1$. In this case $\frac{1}{2^{\beta-1}-1} > 0$, $\frac{\beta-1}{\alpha-1} < 0$ and computed values are presented in Table 3.

Table 3

α	β	e_1	e_2	e_3	e_4	e_5	e	$I_\alpha^\beta(A,B)$
0.2	1.5	0.0147	0.0086	0.0022	0.0038	0.0043	0.0337	0.08135
0.4	2	0.0568	0.0349	0.0088	0.0159	0.0166	0.1332	0.13323
0.1	2.5	0.0227	0.0131	0.0032	0.0056	0.0065	0.0513	0.028058
0.3	3	0.0890	0.0530	0.0132	0.0236	0.0255	0.2044	0.068157
0.6	3.5	0.2166	0.1356	0.0339	0.0637	0.0612	0.5111	0.109754
0.8	4	0.3445	0.2221	0.0552	0.1081	0.0958	0.8259	0.117996
0.7	4.5	0.3650	0.2290	0.0562	0.1083	0.0999	0.8586	0.083249
0.5	5	0.3086	0.1859	0.0452	0.0841	0.0841	0.7082	0.047215
0.9	5.5	0.6244	0.3993	0.0954	0.1930	0.1635	1.4758	0.068238

Observation: In this case, $I_\alpha^\beta(A,B)$ is not a monotonic function with respect to either α or with respect to β.

Case 3: $\alpha > 1$ and $0 < \beta < 1$. In this case $\frac{1}{2^{\beta-1}-1} < 0$, $\frac{\beta-1}{\alpha-1} < 0$ and the computed values are presented in Table 4.

Table 4

α	β	e_1	e_2	e_3	e_4	e_5	e	$I_\alpha^\beta(A,B)$
1.5	0.1	-0.1268	-0.1005	-0.0305	-0.0635	-0.0440	-0.3653	0.787088
2	0.2	-0.1302	-0.1120	-0.0365	-0.0808	-0.0473	-0.4069	0.955849
2.5	0.3	-0.1250	-0.1149	-0.0402	-0.0942	-0.0472	-0.4216	1.096671
3	0.4	-0.1143	-0.1110	-0.0415	-0.1021	-0.0446	-0.4135	1.215378
3.5	0.5	-0.1000	-0.1014	-0.0404	-0.1036	-0.0399	-0.3853	1.315536
4	0.6	-0.0831	-0.0874	-0.0369	-0.0977	-0.0338	-0.3389	1.399456
4.5	0.7	-0.0643	-0.0697	-0.0310	-0.0843	-0.0265	-0.2758	1.468905
5	0.8	-0.0440	-0.0489	-0.0229	-0.0633	-0.0183	-0.1975	1.525482
5.5	0.9	-0.0225	-0.0256	-0.0125	-0.0351	-0.0095	-0.1052	1.570742

Observation: In this case, $I_\alpha^\beta(A,B)$ is a monotonic increasing function with respect to α and β together. In other words, if we do not take any one parameter in increasing order, then monotonicity may lost.

Case 4: $\alpha > 1$ and $\beta > 1$. In this case $\frac{1}{2^{\beta-1}-1} > 0$, $\frac{\beta-1}{\alpha-1} > 0$ and the computed values are presented in Table 5.

Table 5

α	β	e_1	e_2	e_3	e_4	e_5	e	$I_\alpha^\beta(A,B)$
1.5	2.5	0.2536	0.1931	0.0530	0.1154	0.0779	0.6930	0.37903
2	3.5	0.5463	0.4493	0.1233	0.3013	0.1636	1.5839	0.34013
2.5	2	0.2102	0.1905	0.0604	0.1518	0.0716	0.6845	0.68450
3	4	0.8350	0.8005	0.2363	0.7138	0.2559	2.8416	0.40594
3.5	1.5	0.1111	0.1129	0.0421	0.1155	0.0416	0.4232	1.02168
4	5	1.3806	1.4957	0.4568	1.7954	0.4100	5.5384	0.36923
4.5	4.5	1.1711	1.3229	0.4446	1.7925	0.3680	5.0991	0.49440
5	5.5	1.7537	2.0926	0.6826	3.3566	0.5164	8.4020	0.38849
5.5	3	0.5776	0.6797	0.2855	1.0441	0.2093	2.7962	0.93208

Observation: In this case, $I_\alpha^\beta(A,B)$ is not a monotonic function with respect to either α or with respect to β.

In all the cases including the case of $\alpha = \beta$, we observe that $I_\alpha^\beta(A,B) > 0$. Hence for all $\alpha, \beta > 0$, $I_\alpha^\beta(A,B) \geq 0$ and vanishes only when $A = B$. Thus $I_\alpha^\beta(A,B)$ is a valid measure of directed divergence of fuzzy sets A and B and consequently $J_\alpha^\beta(A,B)$ is a valid measure of symmetric divergence. Note that, these tables can be constructed for any two standard fuzzy sets and we shall have the same facts as observed above.

Particular Cases:

—$\lim\limits_{\beta\to 1} I_\alpha^\beta(A,B) = I_\alpha(A,B)$ and $\lim\limits_{\beta\to 1} J_\alpha^\beta(A,B) = J_\alpha(A,B)$.

—Let $B = A_F$, the most fuzzy set, i.e. $\mu_B(x_i) = 0.5\ \forall x_i$, then

$$\begin{aligned}
I_\alpha^\beta(A, A_F) &= \frac{1}{2^{\beta-1}-1}\sum_{i=1}^{n}\Big[\Big(\mu_A^\alpha(x_i)(0.5)^{1-\alpha} \\
&\qquad +(1-\mu_A(x_i))^\alpha(0.5)^{1-\alpha}\Big)^{\frac{\beta-1}{\alpha-1}} - 1\Big] \\
&= \frac{1}{2^{\beta-1}-1}\cdot\frac{1}{2^{1-\beta}}\sum_{i=1}^{n}\Big[\Big(\mu_A^\alpha(x_i) \\
&\qquad +(1-\mu_A(x_i))^\alpha\Big)^{\frac{\beta-1}{\alpha-1}} - 2^{1-\beta}\Big] \\
&= \frac{1}{1-2^{1-\beta}}\sum_{i=1}^{n}\Big[\Big(\mu_A^\alpha(x_i) \\
&\qquad +(1-\mu_A(x_i))^\alpha\Big)^{\frac{\beta-1}{\alpha-1}} - 2^{1-\beta}\Big] \\
&= \frac{-n.2^{1-\beta}}{1-2^{1-\beta}} + \frac{1}{1-2^{1-\beta}}\sum_{i=1}^{n}\Big[\Big(\mu_A^\alpha(x_i) \\
&\qquad +(1-\mu_A(x_i))^\alpha\Big)^{\frac{\beta-1}{\alpha-1}} - 1\Big] \\
&\qquad + \frac{n}{1-2^{1-\beta}} \\
&= n - H_\alpha^\beta(A).
\end{aligned}$$

Thus $I_\alpha^\beta(A, A_F) = n-$ (Entropy of the fuzzy set).

3. MEASURES OF TOTAL AMBIGUITY

Let A and B be two fuzzy sets. The total ambiguity of the fuzzy set A about set B is the sum of two components:

—Fuzzy entropy present in the fuzzy set A.

—Fuzzy directed divergence of A from B measured by $I(A, B)$.

Using Havrda and Charvat measure, Kapur (1997) estimated the total ambiguity as

$$\begin{aligned} TA &= \frac{1}{1-\alpha}\sum_{i=1}^{n}\Big[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha - 1\Big] \\ &\quad + \frac{1}{\alpha-1}\Big[\sum_{i=1}^{n}\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) \\ &\quad + \sum_{i=1}^{n}(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha} - 1\Big] \\ &= \frac{1}{1-\alpha}\Big[\sum_{i=1}^{n}\mu_A^\alpha(x_i)(1-\mu_B^{1-\alpha}(x_i)) \\ &\quad + \sum_{i=1}^{n}(1-\mu_A(x_i))^\alpha\Big(1-(1-\mu_B(x_i))^{1-\alpha}\Big)\Big]. \end{aligned}$$

Analogously, corresponding to fuzzy entropy (3) and fuzzy directed divergence (15), we have total ambiguity given by

$$\begin{aligned} TA &= \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\Big[\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\Big] \\ &\quad + \frac{1}{\alpha-1}\sum_{i=1}^{n}\log\Big[\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) \\ &\quad + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\Big] \\ &= \frac{1}{\alpha-1}\log\Big[\Big(\frac{1}{\mu_A^\alpha(x_i)+(1-\mu_A(x_i))^\alpha}\Big) \\ &\qquad \times\Big(\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) \\ &\qquad\quad + (1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\Big)\Big]. \end{aligned}$$

Similarly, corresponding to the fuzzy entropy (24) and fuzzy directed divergence (26), we have following measure of total ambiguity:

$$TA = \frac{1}{2^{1-\beta}-1}\sum_{i=1}^{n}\Big[\Big(\mu_A^\alpha(x_i) + (1-\mu_A(x_i))^\alpha\Big)^{\frac{\beta-1}{\alpha-1}} - 1\Big]$$

$$+\frac{1}{2^{\beta-1}-1}\sum_{i=1}^{n}\Big[\Big(\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)$$
$$+(1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha}\Big)^{\frac{\beta-1}{\alpha-1}}-1\Big].$$

Total ambiguity is a fuzzy measure of inaccuracy analogous to Kerridge (1961) inaccuracy and is related to two fuzzy sets. It is not symmetric as we get something different if we interchange the role of the fuzzy sets A and B.

Remarks

Fuzzy Information Measures have recently found applications to Engineering, Fuzzy traffic control, Fuzzy aircraft control, Medicines, Computer Science, Management and decision making, etc., which has been established by various authors. The applications of our generalized measures of fuzzy directed divergence and ambiguity in clustering, image thresholding, diagnosis process and object extraction problems can be studied on the lines followed by Bhandari et al. (1992). These are being studied separately and will be reported for publication in due course.

4. GENERALIZED FUZZY INFORMATION IMPROVEMENT MEASURES

Let P and Q be observed and predicted distributions respectively of a random variable. Let $R=(r_1,r_2,\ldots,r_n)$ be the revised probability distribution of Q, then

$$D(P:Q)-D(P:R)=\sum_{i=1}^{n}p_i\log\frac{r_i}{q_i} \tag{28}$$

which is known as Theil's (1967) measure of information improvement and has found wide applications in economics, accounts and financial management.

Similarly, suppose the correct fuzzy set is A and originally our estimate for it was the fuzzy set B that was revised to set C, the original ambiguity was $I(A,B)$ and finally ambiguity is $I(A,C)$, so the reduction in ambiguity is

$$\begin{aligned}I(A,B)-I(A,C) \;=\; &\sum_{i=1}^{n}\Big[\mu_A(x_i)\log\frac{\mu_C(x_i)}{\mu_B(x_i)}\\ &+(1-\mu_A(x_i))\log\frac{(1-\mu_C(x_i))}{(1-\mu_B(x_i))}\Big].\end{aligned} \tag{29}$$

Equation (29) can be called fuzzy information improvement measure.

In case of fuzzy directed divergence given by (15), the reduction in ambiguity is given by

$$\begin{aligned}&I_\alpha(A,B)-I_\alpha(A,C) \\ &\quad= I_\alpha(A,B,C)\\ &\quad= \frac{1}{\alpha-1}\Big[\sum_{i=1}^{n}\log\Big\{\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i)\end{aligned} \tag{30}$$

$$
\begin{aligned}
&+(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\Big\} \\
&-\sum_{i=1}^{n}\log\Big\{\mu_A^\alpha(x_i)\mu_C^{1-\alpha}(x_i) \\
&+(1-\mu_A(x_i))^\alpha(1-\mu_C(x_i))^{1-\alpha}\Big\}\Big] \\
= \;& \frac{1}{\alpha-1}\sum_{i=1}^{n}\log\Big[\Big\{\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) \\
&+(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\Big\}\Big\{\mu_A^\alpha(x_i)\mu_C^{1-\alpha}(x_i) \\
&+(1-\mu_A(x_i))^\alpha(1-\mu_C(x_i))^{1-\alpha}\Big\}^{-1}\Big],
\end{aligned} \tag{31}
$$

which can be called the generalized fuzzy information improvement measure of order α.

Corresponding to the fuzzy directed divergence (26), the reduction in ambiguity is given by

$$
\begin{aligned}
& I_\alpha^\beta(A,B) - I_\alpha^\beta(A,C) \\
= \;& I_\alpha^\beta(A,B,C) \\
= \;& \frac{1}{2^{\beta-1}-1}\sum_{i=1}^{n}\Big\{\Big(\mu_A^\alpha(x_i)\mu_B^{1-\alpha}(x_i) \\
&+(1-\mu_A(x_i))^\alpha(1-\mu_B(x_i))^{1-\alpha}\Big)^{\frac{\beta-1}{\alpha-1}} \\
&-\Big(\mu_A^\alpha(x_i)\mu_C^{1-\alpha}(x_i) \\
&+(1-\mu_A(x_i))^\alpha(1-\mu_C(x_i))^{1-\alpha}\Big)^{\frac{\beta-1}{\alpha-1}}\Big\},
\end{aligned}
$$

which can be called the generalized measure of order α and degree β.

5. CONCLUSION

The proposed two new generalized fuzzy directed divergence measures are valid measures. We have also discussed the particular cases and shown the computational structure of the directed divergence measures. Total ambiguity measures and fuzzy information improvement measures have also been introduced. Further comparative investigations for the amount of total ambiguity in different measures suggested for different pairs of fuzzy sets with different possible values of α and β can be computationally made and similar investigation can be done for the corresponding fuzzy information improvement measures suggested.

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