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MAJORIZATION FOR THE SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY LINEAR OPERATOR USING DIFFERENTIAL SUBORDINATION.

S. P. GOYAL, SANJAY KUMAR BANSAL AND PRANAY GOSWAMI

Abstract

In the present paper, we investigate majorization properties for the subclass of analytic functions defined by an extension, introduced by Saitoh, of the well-known Carlson-Shaffer linear operator, using differential subordination. Relevant connections of the main results obtained in this paper with those given by earlier workers are also pointed out.

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1. INTRODUCTION

Let $f(z)$ and $g(z)$ be analytic in the open unit disk

$$\Delta = \{z : z \in \mathcal{C}\ , |z| < 1\}. \tag{1.1}$$

We say that $f(z)$ is majorized by $g(z)$ in Δ (see [4]) and write

$$f(z) << g(z) \quad (z \in \Delta), \tag{1.2}$$

if there exists a function $\varphi(z)$, analytic in Δ such that

$$|\varphi(z)| \leq 1 \text{ and } f(z) = \varphi(z)g(z) \quad (z \in \Delta). \tag{1.3}$$

It may be noted here that (1.2) is closely related to the concept of quasi-subordination between analytic functions, which is a unification of the above concept of majorization and the following concept of subordination (see [9]).

If $f(z)$ and $F(z)$ are analytic in Δ, we say that $f(z)$ is subordinate to a function $F(z)$ written symbolically as $f \prec F$ or $f(z) \prec F(z)(z \in \Delta)$, if there exists a Schwarz function $\omega(z)$ which (by definition) is analytic in Δ with

$$\omega(0) = 0, \quad |\omega(z)| < 1 \quad (z \in \Delta),$$

such that

$$f(z) = F(\omega(z)) \quad (z \in \Delta).$$

If F is univalent in Δ, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\Delta) \subset F(\Delta)$.

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad (p \in \mathcal{N} = \{1, 2, 3, ...\}), \tag{1.4}$$

which are analytic and multivalent in the open unit disk Δ. In particular if $p = 1$, then $\mathcal{A}_1 = \mathcal{A}$.

For functions $f_j \in \mathcal{A}_p$ given by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, \quad (j = 1, 2; p \in \mathcal{N}), \tag{1.5}$$

we define the Hadamard product (or convolution) of f_1 and f_2 by

$$(f_1 * f_2)(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z).$$

Definition 1. Saitoh [10] introduced a linear operator

$$L_p(\alpha, \beta) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$$

defined by

$$L_p(\alpha, \beta) f(z) := \phi_p(\alpha, \beta; z) * f(z) \ \ (z \in \Delta, f \in \mathcal{A}_p)$$

where

$$\phi_p(\alpha, \beta; z) := \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} z^{n+p}$$

$$\left(\alpha \in \mathcal{R}, \beta \in \mathcal{R} \setminus \mathcal{Z}_0^-, \mathcal{Z}_0^- = \{0, -1, -2, ...\}; z \in \Delta\right).$$

and $(a)_n$ is Pochhammer symbol defned by

$$(a)_n = \begin{cases} 1 & for \ \ n = 0 \\ a(a+1)\ldots(a+n-1) & for\ n \in \mathcal{N} \end{cases}$$

The operator $L_p(\alpha, \beta)$ is an extension of the familiar Carlson- Shaffer operator [2]:

$$L(\alpha, \beta) := L_1(\alpha, \beta)$$

Definition 2. A function $g(z) \in \mathcal{A}_p$ is said to be in the class $S^{p,q}_{\alpha,\beta}[A, B; \gamma]$ of p-valent functions of complex order $\gamma \neq 0$ in Δ if and only if

$$\left[1 + \frac{1}{\gamma}\left\{\frac{z(L_p(\alpha, \beta)g(z))^{(q+1)}}{(L_p(\alpha, \beta)g(z))^{(q)}} - p + q\right\}\right] \prec \frac{1 + Az}{1 + Bz}, \tag{1.6}$$

$(z \in \Delta, -1 \leq B < A \leq 1, \alpha, \beta > 0, p \in \mathcal{N}, q \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}, \gamma \in \mathcal{C} - \{0\})$

In particular for $\alpha = \beta = 1$, our class will reduce into the class $S^*(A, B, \gamma, p, q)$ introduced earlier by Polatoğlu et al. [7]. Further if we put $p = 1, q = 0$, we shall obtain the class introduced by Polatoğlu and Özkan [8] which easily reduces to the class $S(\gamma)$ of starlike functions of complex order γ, introduced by Nasr and Aouf [5]. Also for $\gamma = 1 - \alpha$, $0 \leq \alpha < 1$, we get $S(1 - \alpha) = S^*(\alpha)$, the class of starlike functions of order α in Δ.

A majorization problem for the class $S(\gamma)$ has recently been investigated by Altinas et al. [1]. Also, majorization problems for the class $S^* = S^*(0)$ have been investigated by MacGregor [4]. In the very recent paper [3], Goyal and Goswami generalized these results for the class of multivalent functions using fractional derivatives operators. In the present paper, we investigate a majorization problem for the class $S_{\alpha,\beta}^{p,q}[A,B;\gamma]$.

2. MAJORIZATION PROBLEM FOR THE CLASS $S_{\alpha,\beta}^{p,q}[A,B;\gamma]$

We begin by proving

Theorem 2.1 *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in S_{\alpha,\beta}^{p,q}[A,B;\gamma]$ and $\alpha \geq |\gamma(A-B)+\alpha B|$. If $(L_p(\alpha,\beta)f(z))^{(q)}$ majorized by $(L_p(\alpha,\beta)g(z))^{(q)}$ in Δ, then*

$$| (L_p(\alpha+1,\beta)f(z))^{(q)} | \leq | (L_p(\alpha+1,\beta)g(z))^{(q)} | \; for \; | z | \leq r_0, \tag{2.1}$$

where $r_0 = r_0(\gamma,\alpha,A,B)$ is the smallest positive real root of the equation

$$r^3 \mid \alpha B + \gamma(A-B) \mid -(\alpha + 2|B|)r^2 - [|\gamma(A-B)+\alpha B| + 2]\, r + \alpha = 0 \tag{2.2}$$

$$(-1 \leq B < A \leq 1, \alpha \; and \; \beta > 0, \gamma \in \mathcal{C} - \{0\})$$

Proof. Since $g \in S_{\alpha,\beta}^{p,q}[A,B;\gamma]$, we find from (1.6)

$$\left[1 + \frac{1}{\gamma}\left\{\frac{z(L_p(\alpha,\beta)g(z))^{(q+1)}}{(L_p(\alpha,\beta)g(z))^{(q)}} - p + q\right\}\right] = \frac{1+A\omega(z)}{1+B\omega(z)} \tag{2.3}$$

$$(-1 \leq B < A \leq 1, \alpha,\ \beta > 0, p,q \in \mathcal{N}, p > q \; and \; \gamma \in \mathcal{C} - \{0\})$$

where $\omega(z) = c_1 z + c_2 z^2 + ..., \omega \in \mathcal{P}, \mathcal{P}$ denotes the well known class of the bounded analytic functions in Δ and satisfies the conditions

$$\omega(0) = 0,\ and \ \mid \omega(z) \mid \leq \mid z \mid (z \in \Delta) \tag{2.4}$$

From (2.3), we get

$$\frac{z(L_p(\alpha,\beta)g(z))^{(q+1)}}{(L_p(\alpha,\beta)g(z))^{(q)}} = \frac{p-q+[\gamma(A-B)+(p-q)B]\omega(z)}{1+B\omega(z)}. \tag{2.5}$$

Now using the following, easily verified, identity

$$z(L_p(\alpha,\beta)g(z))^{(q+1)} = \alpha\,(L_p(\alpha+1,\beta)g(z))^{(q)} - (\alpha-p+q)\,(L_p(\alpha,\beta)g(z))^{(q)} \tag{2.6}$$

in (2.5) and making simple calculations, we obtain (recall that $\alpha \geq |\alpha B + \gamma(A-B)|$ and $|z| < 1$))

$$| (L_p(\alpha,\beta)g(z))^{(q)} | \leq \frac{\alpha[1+|B||z|]}{\alpha - |\alpha B + (A-B)\gamma||z|} | (L_p(\alpha+1,\beta)g(z))^{(q)} | \tag{2.7}$$

Next since $(L_p(\alpha,\beta)f(z))^{(q)}$ is majorized by $(L_p(\alpha,\beta)g(z))^{(q)}$ in the unit disk Δ, we have from (1.3) that

$$(L_p(\alpha,\beta)f(z))^{(q)} = \varphi(z)\,(L_p(\alpha,\beta)g(z))^{(q)}$$

Differentiating it with respect to 'z' and multiplying by 'z', we get

$$z(L_p(\alpha,\beta)f(z))^{(q+1)} = z\varphi'(z)\,(L_p(\alpha,\beta)g(z))^{(q)} + z\varphi(z)(L_p(\alpha,\beta)g(z))^{(q+1)}$$

Using (2.6) in the above equation, it yields

$$(L_p(\alpha+1,\beta)f(z))^{(q)} = \frac{z}{\alpha}\varphi'(z)\,(L_p(\alpha,\beta)g(z))^{(q)} + \varphi(z)\,(L_p(\alpha+1,\beta)g(z))^{(q)} \quad (2.8)$$

Thus, noting that $\varphi \in \mathcal{P}$ satisfies the inequality (see, e.g. Nehari [6])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \qquad (z \in \Delta) \quad (2.9)$$

and making use of (2.7) and (2.9) in (2.8), we get

$$|\,(L_p(\alpha+1,\beta)f(z))^{(q)}\,| \leq$$

$$\left[|\varphi(z)| + \left(\frac{1-|\varphi(z)|^2}{1-|z|^2}\right)\left(\frac{|z|(1+|B||z|)}{\alpha-|\alpha B+(A-B)\gamma||z|}\right)\right] |\,(L_p(\alpha+1,\beta)g(z))^{(q)}\,| \quad (2.10)$$

which upon setting $|z| = r$ and $|\varphi(z)| = \rho$ $(0 \leq \rho \leq 1)$ leads to the inequality

$$|\,(L_p(\alpha+1,\beta)f(z))^{(q)}\,| \leq \Psi(r,\rho)|\,(L_p(\alpha+1,\beta)g(z))^{(q)}\,| \quad (2.11)$$

where

$$\Psi(r,\rho) = \frac{-r(1+|B|r)\rho^2 + (1-r^2)\rho\,[\alpha-|\alpha B+(A-B)\gamma)|r] + r(1+|B|r)}{(1-r^2)[\alpha-|\alpha B+(A-B)\gamma|r]} \quad (2.12)$$

In order to determine r_0, we note that

$$r_0 = \max\{r \in [0,1] : \Psi(r,\rho) \leq 1, \forall \rho \in [0,1]\}$$

$$= \max\{r \in [0,1] : \chi(r,\rho) \geq 0, \forall \rho \in [0,1]\},$$

where

$$\chi(r,\rho) = (1-r^2)\Big[\alpha-|\alpha B+(A-B)\gamma)|r\Big] -$$

$$-\rho(1-r^2)\Big[\alpha-|\alpha B+(A-B)\gamma)|r\Big] - (1-\rho^2)(1+|B|r)r. \quad (2.13)$$

A simple calculation shows that the inequality $\chi(r,\rho) \geq 0$ is equivalent to

$$u(r,\rho) = (1-r^2)\Big[\alpha-|\alpha B+(A-B)\gamma)|r\Big] - (1+|B|r)\,r(1+\rho) \geq 0,$$

Obviously the function $u(r,\rho)$ takes its minimum value at $\rho = 1$, i.e.

$$\min\{u(r,\rho) : \rho \in [0,1]\} = u(r,1) = v(r),$$

where

$$v(r) = r^3\mid \alpha B + \gamma(A-B)\mid -(\alpha+2|B|)r^2 - [|\gamma(A-B)+\alpha B|+2]\,r + \alpha.$$

It follows that $v(r) \geq 0$ for all $r \in [0, r_0]$, where $r_0(\gamma,\alpha,A,B)$ is the smallest positive real root of the equation (2.2). In fact, as one can see easily, in any case, i.e. either $|\alpha B + (A - B)\gamma| \neq 0$, or if it is equal to zero, (2.2) has a unique root in the interval (0, 1) and this is the smallest positive root of equation (2.2). This completes the Theorem 2.1.

Setting $A = 1$ and $B = -1$, we get

Corollary 2.1. *Let the function $f \in \mathcal{A}_p$ and suppose that $g \in S^{p,q}_{\alpha,\beta}[1, -1; \gamma]$ and $\alpha \geq |2\gamma - \alpha|$. If $(L_p(\alpha, \beta)f(z))^{(q)}$ is mazorized by $(L_p(\alpha, \beta)g(z))^{(q)}$ in Δ, then*

$$| (L_p(\alpha + 1, \beta)f(z))^{(q)} | \leq | (L_p(\alpha + 1, \beta)g(z))^{(q)} | \; for \; | z | \leq r_1, \tag{2.14}$$

where

$$r_1 = r_1(\gamma, \alpha) = \begin{cases} \frac{k - \sqrt{k^2 - 4\alpha|2\gamma - \alpha|}}{2|2\gamma - \alpha|}, & if \; 2\gamma \neq \alpha \\ \frac{\alpha}{\alpha + 2}, \quad if \quad 2\gamma \neq \alpha \end{cases} \tag{2.15}$$

$$(k = 2 + \alpha + |2\gamma - \alpha|; and \; \gamma \in \mathcal{C} - \{0\}).$$

Setting $p = 1$ and $q = 0$ in Corollary 2.1 we will get the following result:

Corollary 2.2. *Let the function $f \in \mathcal{A}$ and suppose that $g \in S^{1,0}_{\alpha,\beta}[1, -1; \gamma]$. If $L(\alpha, \beta)f(z)$ is mazorized by $L(\alpha, \beta)g(z)$ in Δ, then*

$$| L(\alpha + 1, \beta)f(z) | \leq | L(\alpha + 1, \beta)g(z) | \; for \; | z | \leq r_1,$$

where r_1 is given by (2.15).

Further putting $\alpha = 1$ and $\beta = 1$ in Corollary 2.2, we get

Corollary 2.3. *Let the function $f \in \mathcal{A}$ and suppose that $g \in S^{1,0}_{1,1}[1, -1; \gamma] := S(\gamma)$. If $f(z)$ is mazorized by $g(z)$ in Δ, then*

$$| f'(z)) | \leq | g'(z) | \; (| z | \leq r_2),$$

$$r_2 = r_2(\gamma) = \begin{cases} \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}, & if \; 2\gamma \neq 1 \\ \frac{1}{3}, \quad if \quad 2\gamma = 1 \end{cases}$$

which is a known result obtained by Altinas et al. [1], which contains another known result by MacGregor [4] for $\gamma = 1$.

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S. P. GOYAL
Department of Mathematics
University of Rajasthan, Jaipur - 302055, India
E-Mail: somprg@gmail.com

SANJAY KUMAR BANSAL
Department of Mathematics
Bansal School of Engg. and Tech. Jaipur-303904, India
E-Mail: bansalindian@gmail.com

PRANAY GOSWAMI
Department of Mathematics
Amity University Rajasthan Jaipur- 302002 , India
E-Mail: pranaygoswami83@gmail.com

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