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A STUDY OF FRACTIONAL INTEGRAL OPERATORS INVOLVING I-FUNCTION,GENRALIZED HURWITZ-LERCH-ZETA FUNCTION AND THE GENERAL CLASS OF POLYNOMIALS

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Abstract

The aim of the present study is to introduce two unified fractional integral operators involving products of generalized Hurwitz-Lerch Zeta function, the I-function and the general class of polynomials. First of all we give the complete existence conditions of our operators of study. Next, we develop the Mellin transforms and their inversions, the Mellin convolution, the associated Parseval-Goldstein theorem for these operators. Further, we obtain the images of the generalized hypergeometric function ${}_PF_Q$ together with application for these operators. In all, seven theorems and two corollaries have been established in this paper. On account of the general nature of kernels of our study, a number of similar results for simpler functions obtained earlier follow as special cases of our findings.

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1.Introduction:

Fractional integral operators play an important role in the theory of integral equations and in problems of mathematical physics. In the present paper, we shall study two fractional integral operators defined by the means of following equations:

$$R_x^{\eta,\rho}[f] = R_{x:z,v,a:z_1,g_j,G_j,h_j,H_j,g_{ji},G_{ji},h_{ji},H_{ji}:z_2}^{\eta,\rho:\delta_1,\delta_2:m,n,p_i,q_i,r:U,V,\mu_1,\mu_2}[f]$$
$$= x^{-\eta-\rho-1}\int_0^x t^\eta (x-t)^\rho \phi\left[z\left(\frac{t}{x}\right)^{\delta_1}\left(1-\frac{t}{x}\right)^{\delta_2},\nu,a\right]$$
$$I_{p_i,q}^{m,n}\left[z_1\left(\frac{t}{x}\right)^{\sigma_1}\left(1-\frac{t}{x}\right)^{\sigma_2}\right]$$
$$S_V^U\left[z_2\left(\frac{t}{x}\right)^{\mu_1}\left(1-\frac{t}{x}\right)^{\mu_2}\right] f(t)dt \tag{1}$$

$$W_x^{\eta,\rho}[f] = W^{\eta,\rho:\delta_1,\delta_2:m,n,p_i,q_i,r:U,V,\mu_1,\mu_2}_{x:z,v,a:z_1,g_j,G_j,h_j,H_j,g_{ji},G_{ji},h_{ji},H_{ji}:z_2}[f]$$
$$= x^{\eta}\int_x^{\infty} t^{-\eta-\rho-1}(t-x)^{\rho}\,\phi\left[z\left(\frac{x}{t}\right)^{\delta_1}\left(1-\frac{x}{t}\right)^{\delta_2},\nu,a\right]$$
$$I^{m,n}_{p_i,q_i;r}\left[z_1\left(\frac{x}{t}\right)^{\sigma_1}\left(1-\frac{x}{t}\right)^{\sigma_2}\right]$$
$$S_V^U\left[z_2\left(\frac{x}{t}\right)^{\mu_1}\left(1-\frac{x}{t}\right)^{\mu_2}\right]f(t)dt \tag{2}$$

Here $\phi(z,\nu,a)$ denotes the Generalized Hurwitz -Lerch Zeta function defined by [12,Eq.(3)]

$$\phi(z,\nu,a) = \sum_{k=0}^{\infty}(a+k)^{-\nu}z^k \tag{3}$$

$$(a \neq 0,-1,-2,.....;|z|<1, Re(\nu)>1 when\,|z|=1)$$

The above function is the generalization of the well known generalized (Hurwitz) zeta function $\zeta(\nu,a)$ and Riemann Zeta function $\zeta(\nu)$ [3, p.24; 1.10, Eq.(1); p.32, §1.12, Eq.(1)].

The I-function occurring in (1) and (2) is a generalization of the well known H-function and will be defined and represented as follows [15]

$$I^{m,n}_{p_i,q_i;r}\left[z\,\middle|\,\begin{matrix}(g_j,G_j)_{1,n},(g_{ji},G_{ji})_{n+1,p_i}\\(h_j,H_j)_{1,m},(h_{ji},H_{ji})_{m+1,q_i}\end{matrix}\right] = \frac{1}{2\pi\omega}\int_L \chi(\xi)z^{\xi}d\xi \tag{4}$$

where

$$\chi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(h_j - H_{j\xi})\prod_{j=1}^{n}\Gamma(1-g_j+G_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-h_{ji}+H_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(g_{ji}-G_{ji}\xi)\right]} \tag{5}$$

and m,n,p_i,q_i are integers satisfying $0 \le n \le p_i, 1 \le m \le q_i$ $(i=1,...,r)$ is finite, G_j,H_j,G_{ji},H_{ji} are positive numbers and g_j,h_j,g_{ji},h_{ji} are complex numbers. The validity of existence of I-function are:

$$\left.\begin{aligned}&A_i > 0,\ |\arg z| < \tfrac{1}{2}A_i\pi;\\ &A_i \ge 0,\ |\arg z| \le \tfrac{1}{2}A_i\pi \text{ and } Re(B+1) \le 0\end{aligned}\right\} \tag{6}$$

where

$A_i = \sum_{j=1}^{n} G_j - \sum_{j=n+1}^{p_i} G_{ji} + \sum_{j=1}^{m} H_j - \sum_{j=m+1}^{q_i} H_{ji}\ \ \forall\, i = (1,2,...r)$
and

$B = \sum_{j=1}^{q_i} h_j - \sum_{j=1}^{p_i} g_j + \frac{1}{2}(p_i - q_i)\ \ \forall\, i = (1,2,...r)$

If r = 1, the I-function reduces to the H-function.

$S_V^U[x]$ also occurring in (1) and (2) denotes the general class of polynomials introduced by Srivastava [16, p.l, Eq.(1)]

$$S_V^U[x] = \sum_{R=0}^{V/U} (-V)_{UR}\, A_{V,R}\, \frac{z^R}{R!},\ V = 0,1,2 \tag{7}$$

where U is an arbitrary positive integer and the coefficients $A_{V,R}$ are arbitrary constants, real or complex.
and $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & if\ \ n=0 \\ \lambda(\lambda+1)...(\lambda+n-1) & \forall\ \ n \in \{1,2,3....\} \end{cases}$$

The above polynomial is quite general in nature and reduces to several classical polynomials [1,p.156-163].
On account of the importance of the fractional integral operators in the theory of integral equations and other allied topics, these operators have beenstudied from time to time by number of authors notably Riemann-Liouville [4], Weyl [4], Kiryakova [10], Kober [11], Kalla [8, 9], Saxena [13], Saxena and Kumbhat [14],Srivastava et al. [17], Goyal and Jain [5], Gupta and Soni [6], Jain and Sharma [7].A systematic survey of the fractional integral operators carried out by various authors has been presented in the famous paper by Srivastava and Saxena [19].
To be specific, we shall assume that the class of function f(t) occurring in (1) and (2) are such that:

$$f(t) = \begin{cases} 0\left\{|t|^{\varsigma}\right\}, & |t| \to 0 \\ 0\left\{|t|^{w_1}\, e^{-w_2|t|}\right\}, & |t| \to \infty \end{cases}$$

It is easy to verify that the operator defined by (1) exists if-
(i) The quantities $\delta_1, \delta_2, \sigma_1, \sigma_2, \mu_1, \mu_2$ are all positive (some of them may decrease to zero provided that the resulting operator has a meaning).

$$(ii)\ \ \mathrm{Re}\,(\eta+\varsigma) + \sigma_1 \min_{1\le j\le m}\left(\frac{\mathrm{Re}\,(h_j)}{H_j}\right) + 1 > 0$$

$$(iii)\ \ \mathrm{Re}\,(\rho) + \sigma_2 \min_{1\le j\le m}\left(\frac{\mathrm{Re}\,(h_j)}{H_j}\right) + 1 > 0$$

and the operator defined by (2) exists if $Re(w_2) > 0$ or $Re(w_2) = 0$ and

$$Re(\eta - w_1) + \sigma_1 \min_{1\le j\le m}\left(\frac{\mathrm{Re}\,(h_j)}{H_j}\right) + 1 > 0$$

and the set of conditions (i) and (iii) specified for the existence of the operator (1) are satisfied.

2. The Mellin transforms and the inversion formulae

Theorem 1 If M $\{f(t); s\}$ stands for the well known Mellin transform of the function f(t) defined by

$$M\left\{f\left(t\right);s\right\} = F\left(s\right) = \int_0^\infty t^{s-1} f\left(t\right) dt \tag{8}$$

and it exists, then

$$M\left\{R_x^{\eta,\rho}\left[f\right];s\right\} = G_1\left(s\right) M\left\{f\left(t\right);s\right\} \tag{9}$$

where

$$G_1(s) = \sum_{k=0}^{\infty} \sum_{R=0}^{\frac{V}{U}} (a+k)^{-\nu} (-V)_{UR} A_{V,R} z^k \frac{z_2^R}{R!} I_{p_i+2,q_i+1;r}^{m,n+2} \left[z \left| \begin{matrix} A^* \\ B^* \end{matrix} \right. \right] \tag{10}$$

and

$$A^* = (-\eta + s - \delta_1 k - \mu_1 R, \sigma_1), (-\rho - \delta_2 k - \mu_2 R, \sigma_2), (g_j, G_j)_{l,n}, (g_{ji}, G_{ji})_{n+1,p_i}$$

$$B^* = (h_j, H_j)_{l,m}, (h_{ji}, H_{ji})_{m+1,q_i},$$
$$(-1 - \eta - \rho + s - (\delta_1 + \delta_2) k - (\mu_1 + \mu_2) R, (\sigma_1 + \sigma_2)) \tag{11}$$

provided that the conditions of the existence of the operator $R_x^{\eta,\rho}[f]$ are satisfied.
Proof. From (8) and (1), $M\{R_x^{\eta,\rho}[f]; s\} = \Delta$ (say) takes the following form

$$\Delta = \int_{x=0}^{\infty} x^{s-1} \left\{ x^{-\eta-\rho-1} \int_{t=0}^{x} t^{\eta} (x-t)^{\rho} \phi \left[z \left(\frac{t}{x}\right)^{\delta_1} \left(1 - \frac{t}{x}\right)^{\delta_2}, \nu, a \right] \right.$$
$$\left. I_{p_i,q_i;r}^{m,n} \left[z_1 \left(\frac{t}{x}\right)^{\sigma_1} \left(1 - \frac{t}{x}\right)^{\sigma_2} \right] S_V^U \left[z_2 \left(\frac{t}{x}\right)^{\mu_1} \left(1 - \frac{t}{x}\right)^{\mu_2} \right] f(t)\, dt \right\} dx \tag{12}$$

On changing the order of integration in the above equation (which is permissible under the conditions stated), we get

$$\Delta = \int_{t=0}^{\infty} t^{\eta} f(t) \left\{ \int_{x=t}^{\infty} x^{s-\eta-\rho-2} (x-t)^{\rho} \phi \left[z \left(\frac{t}{x}\right)^{\delta_1} \left(1 - \frac{t}{x}\right)^{\delta_2}, \nu, a \right] \right.$$
$$\left. I_{p_i,q_i;r}^{m,n} \left[z_1 \left(\frac{t}{x}\right)^{\sigma_1} \left(1 - \frac{t}{x}\right)^{\sigma_2} \right] S_V^U \left[z_2 \left(\frac{t}{x}\right)^{\mu_1} \left(1 - \frac{t}{x}\right)^{\mu_2} \right] dx \right\} dt \tag{13}$$

Now, we express the Generalized Hurwitz -Lerch Zeta function and the general class of polynomials S_V^U involved in the above equation in the series and generalized H-function i.e. I-function in terms of its well-known Mellin-Barnes contour integral and interchange the order of summation and integration in the result thus obtained. the equation given by (13) now yields the following result after a little simplification.

$$\Delta = \sum_{k=0}^{\infty} \sum_{R=0}^{[U/V]} (a+k)^{-\nu} (-V)_{UR} A_{V,R} z^k \frac{z_2^R}{R!}$$
$$\int_0^{\infty} t^{\eta+\delta_1 k+\sigma_1 \xi+\mu_1 R} f(t) \left(\frac{1}{2\pi\omega}\right) \int_L \chi(\xi)\, z_1^{\xi} d\xi$$
$$\left\{ \int_0^{\infty} x^{s-\eta-\rho-2-(\delta_1+\delta_2)k-(\sigma_1+\sigma_2)\xi-(\mu_1+\mu_2)R} (x-t)^{\rho+\delta_2 k+\sigma_2 \xi+\mu_2 R} dx \right\} dt \tag{14}$$

Evaluating the x-integral occurring in (14) with the help of a known result [4, p.201, Eq.(6)] and reinterpreting the resulting Mellin-Barnes contour integral so obtained in terms of I-function, we easily arrive at the desired theorem.
Theorem-2:-if $M\{f(t); s\}, M\{W_x^{\eta,\rho}[f]; s\}$ exists.

$$\text{then } M\{W_x^{\eta,\rho}[f]; s\} = G_2(s) M\{f(t); s\} \tag{15}$$

where $G_2(s)$ is obtained from $G_1(s)$ defined in (10), by replacing s by (1- s), provided that the conditions of the existence of the operator $W_x^{\eta,\rho}[f]$ are satisfied.
Proof: If we follow the lines of proof as given in Theorem 1 and make use of another well known result [4, p.185, eq.(7)], we easily obtain the Theorem 2. If we reduce the Generalized Hurwitz -Lerch Zeta function and general class of polynomials to unity and I-function to Fox H-function, then Theorem 1 and 2 give rise to theorem which are in essence same as those obtained by Saxena and Kumbhat [14, p.3-4, Eqs.(3.3), (3.6)].
On using the well known Mellin inversion theorem in (9) and (15) in succession, we arrive at the following interesting theorems.
Theorem 3

$$\frac{1}{2}\left[f(t+0)+f(t-0)\right]=\frac{1}{2\pi\omega}\lim_{\tau\to\infty}\int_{c-\omega\tau}^{c+\omega\tau}\frac{t^{-s}}{G_1(s)}M\left\{R_x^{\eta,\rho}\left[f\right];s\right\}\,ds \tag{16}$$

where f(t) is of bounded variation at the point $t=x(x>0)$ the conditions stated with the Theorem 1 are satisfied and $G_1(s)$ is defined by (10).
Theorem-4

$$\frac{1}{2}\left[f(t+0)+f(t-0)\right]=\frac{1}{2\pi\omega}\lim_{\tau\to\infty}\int_{c-\omega\tau}^{c+\omega\tau}\frac{t^{-s}}{G_2(s)}M\left\{W_x^{\eta,\rho}\left[f\right];s\right\}\,ds \tag{17}$$

where $f(t)$ is of bounded variation at the point $t=x(x>0)$, the conditions stated with the Theorem 2 are satisfied and $G_2(s)$ is obtained from $G_1(s)$ defined in (10) by replacing s by $1-s$.
Also when f(t) is continuous at $t=x(x>0)$ then the left hand side of (16) and (17) are equal to $f(t)$.
If we reduce the Generalized Hurwitz -Lerch Zeta function and general class of polynomials to unity and the I-function to the H-function by taking $r=1$ and subsequently to generalized hypergeometric function ${}_pF_q$ [18,p.18,Eq.(2.6.3)] by suitably adjusting the parameters, the theorem 3 and 4 yield the inversion formulae which are essentially the same as those given by Goyal and Jain [5, p.257, Eq.(3.10), (3.11)].

3. The Mellin Convolutions.

From a well known theorem by Titchmarsh [20, p.60, Th.44], we know that if $f\in L(0,\infty), g\in L(0,\infty)$ then $(f*g)\in L(0,\infty)$ where

$$(f*g)(x)=\int_0^\infty t^{-1}f\left(\frac{x}{t}\right)g(t)\,dt \tag{18}$$

Following the lines adopted by Buschman [2], we shall define a function $R^{\eta,\rho}(x)$ as follows

$$R^{\eta,\rho}(x)=x^{-\eta-\rho-1}(x-1)^{\rho}\,U(x-1)\,\phi\left[z\left(\frac{1}{x}\right)^{\delta_1}\left(\frac{x-1}{x}^{\delta_2}\right),\nu,a\right]$$
$$I_{p_i,q_i;r}^{m,n}\left[z_1\left(\frac{1}{x}\right)^{\sigma_1}\left(\frac{x-1}{x}\right)^{\sigma_2}\right]S_V^U\left[z_2\left(\frac{1}{x}\right)^{\mu_1}\left(\frac{x-1}{x}\right)^{\mu_2}\right] \tag{19}$$

where U denotes the well known unit step function. It can be easily verified that

$R^{\eta,\rho}(x) \in L(o,\infty)$ if

$$Re(\rho) + \sigma_2 \min_{1\le j\le m}\left(\frac{Re(h_j)}{H_j}\right) + 1 > 0$$

$$Re(\eta) + \sigma_1 \min_{1\le j\le m}\left(\frac{Re(h_j)}{H_j}\right) > 0$$

We can represent the operator (1) as a convolution of the form (18). Indeed, we have

$$R_x^{\eta,\rho}[f] = \int_0^\infty t^{-1}\left\{\left(\frac{x}{t}\right)^{-\eta-\rho-1}\left(\frac{x}{t}-1\right)^{\rho} U\left(\frac{x}{t}-1\right)\phi\left[z\frac{1}{\left(\frac{x}{t}\right)^{\delta_1}}\left(\frac{\frac{x}{t}-1}{\frac{x}{t}}\right)^{\delta_2},\nu,a\right]\right.$$

$$\left. I_{p_i,q_i;r}^{m,n}\left[z_1\frac{1}{\left(\frac{x}{t}\right)^{\sigma_1}}\left(\frac{\frac{x}{t}-1}{\frac{x}{t}}\right)^{\sigma_2}\right] S_V^U\left[z_2\frac{1}{\left(\frac{x}{t}\right)^{\mu_1}}\left(\frac{\frac{x}{t}-1}{\frac{x}{t}}\right)^{\mu_2}\right]\right\} f(t)\,dt$$

$$= \left(R^{\eta,\rho} * f\right)(x) \text{ with the help of 18} \tag{20}$$

Again,if we define
$W^{\eta,\rho}(x) = x^{\eta}(1-x)^{\rho}\, U(1-x)\ \phi\left[z(x)^{\delta_1}(1-x)^{\delta_2}\ \nu, a\right]$

$$I_{p_i,q_i,;r}^{m,n}\left[z_1(x)^{\sigma_1}(1-x)^{\sigma_2}\right]\ S_V^U\left[z_2(x)^{\mu_1}(1-x)^{\mu_2}\right] \tag{21}$$

On proceeding in a manner as indicated above, we have

$$W_x^{\eta,\rho}[f] = \left(W^{\eta,\rho} * f\right)(x) \tag{22}$$

Also $W^{\eta,\rho}(x) \in L(o,\infty)$ for

$$Re(\eta) + \sigma_2 \min_{1\le j\le m}\left(\frac{Re(h_j)}{H_j}\right) + 1 > 0$$

$$Re(\rho) + \sigma_1 \min_{1\le j\le m}\left(\frac{Re(h_j)}{H_j}\right) + 1 > 0$$

4. An Analogue of the Parseval-Goldstein Theorem for Operators defined by (1) and (2)

Theorem 5 If

$$\varphi_1(x) = R_x^{\eta,\rho}[f_1] \tag{23}$$

$$\varphi_2(x) = W_x^{\eta,\rho}[f_2] \tag{24}$$

then

$$\int_0^\infty \varphi_1(x)f_2(x)dx = \int_0^\infty \varphi_2(x)f_1(x)dx \tag{25}$$

provided that the various integrals involved converge absolutely
Proof: We have from (1)

$$\int_0^\infty \varphi_1(x)f_2(x)dx = \int_0^\infty \left\{x^{-\eta-\rho-1}\int_0^x t^{\eta}(x-t)^{\rho}\ \phi\left[z(\frac{t}{x})^{\delta_1}(1-\frac{t}{x})^{\delta_2}\ \nu, a\right]\right.$$

$$I_{p_i,q_i;r}^{m,n}\left[z_1\left(\frac{t}{x}\right)^{\sigma_1}\left(1-\frac{t}{x}\right)^{\sigma_2}\right]S_V^U\left[z_2\left(\frac{t}{x}\right)^{\mu_1}\left(1-\frac{t}{x}\right)^{\mu_2}\right]f_1\left(t\right)dt\right\}f_2\left(x\right)dx \tag{26}$$

Changing the order of integrations in the right-hand side of the above equation (say ∇), which is permissible under the conditions stated, it takes the following form after a little simplification

$$\nabla=\int_0^\infty t^n f_1\left(t\right)\left\{\int_t^\infty x^{-\eta-\rho-1}\left(x-t\right)^\rho\phi\left[z\left(\frac{t}{x}\right)^{\delta_1}\left(1-\frac{t}{x}\right)^{\delta_2},\nu,a\right]\right.$$

$$I_{p_i,q_i;r}^{m,n}\left[z_1\left(\frac{t}{x}\right)^{\sigma_1}\left(1-\frac{t}{x}\right)^{\sigma_2}\right]S_V^U\left[z_2\left(\frac{t}{x}\right)^{\mu_1}\left(1-\frac{t}{x}\right)^{\mu_2}\right]f_2\left(x\right)dx\right\}dt \tag{27}$$

On reinterpreting the x-integral given in the above expression with the help of (2), we arrive at the required theorem.

5. Images

In this section, we obtain the images of the generalized hypergeometric function ${}_PF_Q$ in our operators of study.

$$(i)\;\; R_x^{\eta,\rho}\left[{}_PF_Q\left\{a_P:b_Q;z_3t^{\lambda_1}\left(x-t\right)^{\lambda_2}\right\}\right]$$

$$=x^{(\lambda_1+\lambda 2)l}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{R=0}^{[U/V]}\left(a+k\right)^{-\nu}\left(-V\right)_{UR}A_{V,R}\,z^k\frac{\left(a_P\right)_l}{\left(b_Q\right)_l}\frac{z_2^R}{R!}\frac{z_3^l}{l!}$$

$$I_{p_i+2,q_i+1;r}^{m,n+2}$$

$$\left[z_1\left|\begin{array}{c}(-\eta-\lambda_1 l-\delta_1 k-\mu_1 R,\sigma_1),(-\rho-\lambda_2 l-\delta_2 k-\mu_2 R,\sigma_2),C^*\\ D^*,(-1-\eta-\rho-\left(\lambda_1+\lambda_2\right)l-\left(\delta_1+\delta_2\right)k-\left(\mu_1+\mu_2\right)R,\sigma_1+\sigma_2)\end{array}\right.\right] \tag{28}$$

where

$$\left.\begin{array}{l}C^*=(g_j,G_j)_{1,n},(g_{ji},G_{ji})_{n+1,p_i}\\ D^*=(h_j,H_j)_{1,m},(h_{ji},H_{ji})_{m+1,q_i}\end{array}\right\} \tag{29}$$

provided that the following conditions are satisfied

(i) The quantities $\delta_1,\delta_2,\sigma_1,\sigma_2,\mu_1,\mu_2,\lambda_1,\lambda_2$ are all positive (some of them may however decrease to zero provided that the resulting image has meaning).

$$(i)\quad Re\left(\eta\right)+\sigma_1\min_{1\le j\le m}\left(\frac{Re\left(h_j\right)}{H_j}\right)+1>0$$

$$(ii)\quad Re\left(\rho\right)+\sigma_2\min_{1\le j\le m}\left(\frac{Re\left(h_j\right)}{H_j}\right)+1>0$$

Proof: From (1), we have

$$R_x^{\eta,\rho}\left[{}_PF_Q\left\{a_P:b_Q\;;z_3t^{\lambda_1}(x-t)^{\lambda_2}\right\}\right]$$

$$=x^{-\eta-\rho-1}\int_0^x t^\eta(x-t)^\rho\phi\left[z(\frac{t}{x})^{\delta_1}(1-\frac{t}{x})^{\delta_2}\;\nu,a\right]I_{p_i,q_i,r}^{m,n}\left[z_1\left(\frac{t}{x}\right)^{\sigma_1}\left(1-\frac{t}{x}\right)^{\sigma_2}\right]$$

$$S_V^U\left[z_2\left(\frac{t}{x}\right)^{\mu_1}\left(1-\frac{t}{x}\right)^{\mu_2}\right]\ {}_PF_Q\left\{a_P:b_Q\ ;z_3t^{\lambda_1}(x-t)^{\lambda_2}\right\}dt \tag{30}$$

Now expressing the Generalized Hurwitz -Lerch Zeta function and general class of polynomials S_V^U involved in the right hand side of (30) in series form using (3) and (7) respectively, the I-function in terms of Mellin-Barnes contour integral using (4). Also expressing generalized hypergeometric function ${}_pF_q$ in terms of its well known series and interchanging the order of summation and integration in the result thus obtained (which is permissible under the conditions stated) then the right hand side of (30) (say $\wp$) takes the following form after little simplification

$$\wp=\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{R=0}^{V/U}(a+k)^{-\nu}(-V)_{UR}A_{V,R}\ z^k\frac{(a_P)_l}{b_Q)_l}\frac{z_2^R}{R!}\frac{z_3^l}{l!}\left(\frac{1}{2\pi\omega}\right)\int_L\chi(\xi)z_1^{\xi}d\xi$$

$$\left\{x^{(\lambda_1+\lambda_2)l-1}\int_0^x\left(\frac{t}{x}\right)^{\eta+\lambda_1l+\delta_1k+\mu_1R+\sigma_1\xi}\left(1-\frac{t}{x}\right)^{\rho+\lambda_2l+\delta_2k+\mu_2R+\sigma_2\xi}dt\right\} \tag{31}$$

Finally, on evaluating the t-integral involved in the above expression (31) and re-interpreting the result thus obtained in terms of I-function, we easily arrive at the required result after a little simplification.

(ii) $\qquad W_x^{\eta,\rho}\left[{}_PF_Q\left\{a_P:b_Q\ ;z_3t^{\lambda_1}(t-x)^{\lambda_2}\right\}\right]$

$$=x^{(\lambda_1+\lambda_2)l}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{R=0}^{V/U}(a+k)^{-\nu}(-V)_{UR}\ A_{V,R}\ \ z^k\frac{(a_P)_l}{(b_Q)_l}\frac{z_2^R}{R!}\frac{z_3^l}{l!}$$

$$I^{m,n+2}_{p_i+2,q_i+1;r}$$

$$\left[z_1\left|\begin{matrix}(1-\eta-\delta_1k-\mu_R+(\lambda_1+\lambda_2)l,\sigma_1),(-\rho-\lambda_2l-\delta_2k-\mu_2R,\sigma_2),C^*\\ D^*,(-\rho-\eta-(\delta_1+\delta_2)k-(\mu_1+\mu_2)\,R+\lambda_1l,\sigma_1+\sigma_2)\end{matrix}\right.\right] \tag{32}$$

where C^* and D^* are same as defined in (29) provided that the following conditions are satisfied.
(i) The quantities $\delta_1,\delta_2,\sigma_1,\sigma_2,\mu_1,\mu_2,\lambda_1,\lambda_2$ are all positive (some of them may however decrease to zero provided that the resulting image has meaning).

$$(ii)\quad Re\,(\rho)+\sigma_2\min_{1\le j\le m}\left(\frac{Re\,(h_j)}{H_j}\right)+1>0$$

$$(iii)\quad Re\,(\eta)+\sigma_1\min_{1\le j\le m}\left(\frac{Re\,(h_j)}{H_j}\right)>0$$

To prove the result (32), we follow the same method as given earlier for the proof of result (28).

6. Applications

Now we shall make use of analogue of Parseval Goldstein Theorem 5 earlier in establishing two interesting theorems which are believed to be new.

Theorem 6 If

$$\varphi(x) = R_x^{\eta,\rho}\left[f\right] \tag{33}$$

$$\text{then} \int_0^\infty {}_PF_Q\left\{a_P : b_Q\ ; z_3 t^{\lambda_1}(t-x)^{\lambda_2}\right\}\varphi(x)dx$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{R=0}^{V/U}(a+k)^{-\nu}(-V)_{UR}\ A_{V,R}\ z^k \frac{(a_p)_l}{(b_Q)_l}\ \frac{z_2^R}{R!}\ \frac{z_3^l}{l!}$$

$$I^{m,n+2}_{p_i+2,q_i+1;r}$$

$$\left[z_1 \left| \begin{matrix} (1-\eta-\delta_1 k-\mu_1 R+(\lambda_1+\lambda_2)l,\sigma_1),(-\rho-\lambda_2 l-\delta_2 k-\mu_2 R,\sigma_2),C^* \\ D^*,(-\rho-\eta-(\delta_1+\delta_2)k-(\mu_1+\mu_2 R+\lambda_1 l,\sigma_1+\sigma_2) \end{matrix}\right.\right]$$

$$M\left\{f(x);1+(\lambda_1+\lambda_2)l\right\} \tag{34}$$

where C* and D* are the same as given in (29) and the conditions of existence of the operator $R_x^{\eta,\rho}\left[f\right]$ mentioned earlier are satisfied and integrals occurring in (34) are absolutely convergent.

Proof: On substituting the result given by (32) and (33) in the analogue of the Parseval Goldstein theorem given by (25), we easily arrive at the required theorem after a little simplification.

If we reduce the general class of polynomial $S_V^U$to Laguerre Polynomial L_V^α [1, p.158, A.8] and take $\delta_2 = \sigma_2 = \mu_2 = \lambda_2 = 0$ in Theorem 6, then we get the following interesting corollary

Corollary 6.1 Let us there hold the same conditions as in Theorem 6, then

$$\int_0^\infty {}_PF_Q\left\{a_P : b_Q : z_3 t^\lambda\right\}\Psi\left(x\right)dx$$

$$= \Gamma\left(1+\rho\right)\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{R=0}^{[V]}\left(a+k\right)^{-\nu}\begin{pmatrix} V+a \\ V\end{pmatrix}\frac{(-V)_R}{(1+\alpha)_R}\frac{(a_P)_l}{(b_Q)_l}z^k\frac{z_2^R}{R!}\frac{z_3^l}{l!}$$

$$I^{m,n+1}_{p_i+1,q_i+1;r}\left[z_1\left|\begin{matrix}(1-\eta-\delta k-\mu R+\lambda l,\sigma),C^* \\ D^*,(-\rho-\eta-\delta k-\mu R+\lambda l,\sigma)\end{matrix}\right.\right]M\left\{f\left(x\right);1+\lambda l\right\} \tag{35}$$

where

$$\Psi\left(x\right) = x^{-\eta-\alpha-1}\int_0^x t^\eta\left(x-t\right)^\rho$$

$$\phi\left[z\left(\frac{t}{x}\right)^\delta,\nu,a\right]I^{m,n}_{p_i,q_i;r}\left[z_1\left(\frac{t}{x}\right)^\sigma\right]L_V^{(\alpha)}\left[z_2\left(\frac{t}{x}\right)^\mu\right]f\left(t\right)dt \tag{36}$$

and C^* and D^* occurring in (35) are the same as given in (29).

Theorem 7

$$\text{If} \qquad \Psi\left(x\right) = W_x^{\eta,\rho}\left[f\right] \tag{37}$$

then

$$\int_0^\infty {}_PF_Q\left\{a_P : b_Q; z_3 t^{\lambda_1}\left(x-t\right)^{\lambda_2}\right\}\Psi\left(x\right)dx$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{R=0}^{[V/U]}\left(a+k\right)^{-\nu}\left(-V\right)_{UR}A_{V,R}z^k\frac{\left(a_P\right)_l}{\left(b_Q\right)l}\frac{z_2^R}{R!}\frac{z_3^l}{l!}$$

$$I^{m,n+2}_{p_i+2,q_i+1;r}$$

$$\left[z_1\left|\begin{matrix}\left(-\eta-\lambda_1 l-\delta_1 k-\mu_1 R,\sigma_1\right)\left(-\rho-\lambda_2 l-\delta_2 k-\mu_2 R,\sigma_2\right),C^*\\ D^*,\left(-1-\eta-\rho-\left(\lambda_1+\lambda_2\right)l-\left(\delta_1+\delta_2\right)k-\left(\mu_1+\mu_2\right)R,\sigma_1+\sigma_2\right)\end{matrix}\right.\right]$$

$$M\left\{f\left(x\right);1+\left(\lambda_1+\lambda_2\right)l\right\} \tag{38}$$

where C^* and D^* are same as given in (29) and the conditions of the existence of the operator $W_x^{\eta,\rho}[f]$ are satisfied and the integrals occurring in (38) are absolutely convergent.

On substituting the results given by (28) and (37) in (25), we get the required theorem after a little simplification.

If we reduce the Generalized Hurwitz -Lerch Zeta function to well known Reimann Zeta function, I-function to Fox H-function and the polynomial S_V^U to Bateman polynomial Z_V [1, p.161, A.16] and further let $\delta_1=\delta_2=\sigma_1=\mu_1=\lambda_1=0, z=1$ in Theorem 7, then we arrive at the following result

Corollary 7.1 Let us there hold the same conditions as in Theorem 7, then

$$\int_0^\infty {}_PF_Q\left\{a_P : b_Q; z_3\left(x-t\right)^{\lambda}\right\}\Psi\left(x\right)dx$$

$$= \Gamma\left(1+\eta\right)\frac{\prod_{j=1}^{Q}\Gamma\left(b_j\right)}{\prod_{j=1}^{P}\Gamma\left(a_j\right)}\sum_{k=0}^{\infty}\sum_{R=0}^{V}\left(k\right)^{-\nu}\frac{\left(-V\right)_R\left(1+V\right)_R z_2^R}{R!R!R!}$$

$$\int_0^\infty H^{0,1:m,n;1,P}_{1,1:p,q;P,Q+1}$$

$$\left[\begin{matrix}z_1\\ -z_3 x^{\lambda}\end{matrix}\left|\begin{matrix}\left(-\rho-\mu R;\sigma,\lambda\right)\left(g_j,G_j\right)_{1,p};\left(1-a_j,1\right)_{1,P}\\ \left(-1-\eta-\rho-\mu R;\sigma,\lambda\right):\left(h_j,H_j\right)_{1,q};\left(0,1\right),\left(1-b_j,1\right)_{1,Q}\end{matrix}\right.\right]f\left(x\right)dx \tag{39}$$

where

$$\varphi(x)=x^{\eta}\int_x^\infty t^{-\eta-\rho-1}(t-x)^{\rho}\zeta(\nu)\,H^{m,n}_{p,q}\left[z_1\left(1-\tfrac{x}{t}\right)^{\sigma}\left|\begin{matrix}(g_j,G_j)_{1,p}\\ (h_j,H_j)_{1,q}\end{matrix}\right.\right]$$

$$Z_V\left[z_2\left(1-\frac{x}{t}\right)^{\mu}\right]f(t)dt \tag{40}$$

The H-function occurring in the right hand side of (40) stands for H-function of two variables [18, p.82, Eq.(6.1.1)].

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