Journal of Applied Mathematics, Statistics and Informatics (JAMSI), **7** (2011), No. 1

ANALYTIC AND PERIODIC SOLUTIONS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract

The analytic, entire and periodic complex or real-valued solutions of several classes of linear and non-linear systems of differential equations are studied using a functional-analytic approach. The obtained results include and generalize previously obtained results.

Mathematics Subject Classification 2000: 34A30, 34A34, 34C25, 34M99
Additional Key Words and Phrases: differential systems, analytic solutions, periodic solutions, linear, non-linear, singular

1. INTRODUCTION.

The existence of analytic, entire or periodic solutions of linear and non-linear systems of differential equations has been a topic which has attracted the interest of several researchers over the years. For the more standard results one may consult several classical books such as [3], [6] or [7]. In the present paper, a class of linear and non-linear systems of differential equations will be studied in the Hilbert space

$$[H_2(\Delta)]^k = \underbrace{H_2(\Delta) \times ... \times H_2(\Delta)}_{k-times}$$

or the Banach space

$$[H_1(\Delta)]^k = \underbrace{H_1(\Delta) \times ... \times H_1(\Delta)}_{k-times}$$

where $H_2(\Delta)$ is the Hilbert space of analytic functions in $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ defined by

$$H_2(\Delta) = \left\{ f : \Delta \to \mathbb{C} / f(z) = \sum_{n=1}^{\infty} c_n z^{n-1} \text{ analytic in } \Delta \right.$$

$$\left. \text{with } \sum_{n=1}^{\infty} |c_n|^2 < +\infty \right\},$$

Dedicated to my Professor Evangelos Ifantis for his 75th birthday
+ Article is published in memoriam.

with norm $\|f(z)\|^2_{H_1(\Delta)} = \sum_{n=1}^{\infty} |c_n|^2$ and, $H_1(\Delta)$ is the Banach space of analytic functions in Δ defined by

$$H_1(\Delta) = \Big\{ f : \Delta \to \mathbb{C} / f(z) = \sum_{n=1}^{\infty} c_n z^{n-1} \text{ analytic in } \Delta$$

$$\text{with } \sum_{n=1}^{\infty} |c_n| < +\infty \Big\},$$

with norm $\|f(z)\|_{H_1(\Delta)} = \sum_{n=1}^{\infty} |c_n|$.

The reason for studying the solutions of differential equations in the spaces $H_2(\Delta)$ or $H_1(\Delta)$ is justified by the fact that these two spaces: i) contain the important classes of polynomial solutions or convergent power-series solutions, ii) each one of their elements is only one function and not a class of equivalent functions as it happens, e.g. with the elements of the L^2 space of square integrable functions, iii) they appear naturally in various physical problems (see e.g. [11]) and, iv) they are suitable for the formulation and the solution of many problems in function theory. Moreover, the study of systems of differential equations in the above mentioned spaces will yield interesting information regarding their analytic, entire or periodic solutions.

The technique that will be used is a functional-analytic one, which reduces the study of the differential system to the study of an equivalent operator equation in an abstract separable Hilbert space or a Banach space imbedded in a separable Hilbert space. This functional analytic technique was presented systematically in [4], where the existence of analytic solutions of linear functional differential equations and functional differential systems was studied. More precisely, the basic result regarding the linear system

$$z^D \tilde{f}'(z) = A(z)\tilde{f}(z), \tag{1.1}$$

was the existence of at least $k-d$ linearly independent solutions in $[H_2(\Delta)]^k$, where $D = \operatorname{diag}(d_1, ..., d_k)$, d_i, $i = 1, ..., k$ non-negative integers, $d = \operatorname{trace}(D)$ and $A(z)$ a $k \times k$ matrix of bounded operators in $H_2(\Delta)$.

The same technique was used in [10], where it was proved that under certain conditions on the constant matrices $A_0 = \{a_{ij}(0)\}$ and $A_1 = \{a'_{ij}(0)\}$, $i, j = 1, ..., k$, the conjugate system of

$$z^D \tilde{f}'(z) = A(z)\tilde{f}(z), \tag{1.2}$$

has exactly $p(k-1)$ linearly independent solutions in $[H_2(\Delta)]^k$, where $D = \operatorname{diag}(p, ..., p)$, $p \geq 2$, $p \in \mathbb{N}$ and $A(z) = \{a_{ij}(z)\}$ a $k \times k$ matrix of functions, analytic in a neighborhood of the closed unit disc $\overline{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$.

Later in [5], the technique of [4] was extended in order to study the analytic solutions of non-linear differential equations. One of the equations studied was the scalar Briot-Bouquet equation [1]

$$zf'(z) = G(f(z)), \tag{1.3}$$

where $G(f(z))$ is analytic in $f(z)$ satisfying the conditions $G(0) = 0$, $G'(0) \neq 0$. The study of (1.3) led to the existence of families of complex-valued periodic solutions with period $\frac{2\pi}{|\tau|}$ of the non-linear equation

$$g''(t) = G(g(t)), \tag{1.4}$$

where $G(w)$ is analytic with $G(0) = -\tau^2 < 0$. If $G(0) = \tau^2 > 0$, then (1.4) has families of solutions of the form

$$g(t) = \sum_{n=1}^{\infty} b_n w^n e^{-n\tau t}, \quad \tau > 0, \quad b_1 = \lambda,$$

which converge absolutely for every $|w| \leq 1$ and $0 \leq t < +\infty$, provided that $|\lambda|$ is sufficiently small. These results were then applied to the second order non-linear differential system of the form:

$$\left.\begin{aligned} x''(t) = c_2 x + u(x,y) \\ y''(t) = c_2 y + v(x,y) \end{aligned}\right\}, \tag{1.5}$$

where u, v are sufficient to form an analytic function $h(g) = h(x + iy) = u(x,y) + iv(x,y)$ in a neighborhood of zero with $h(0) = h'(0) = 0$, to obtain families of real-valued solutions (periodic in the case where $c_2 = -\tau^2 < 0$ and of exponential type in the case where $c_2 = \tau^2 > 0$).

Recently, the formalism of [4] was used in [8] under the framework of a "discretization"technique, for the study of a single system, namely the Lorenz system. In the present paper, a systematic study of the analytic, entire or periodic solutions of linear and non-linear differential systems will be given, under appropriate conditions on the corresponding coefficient matrices. More precisely the analytic and entire solutions of the systems

$$z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{g}(z), \tag{1.6}$$

and

$$z^2\frac{d^2\tilde{f}(z)}{dz^2} + z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{g}(z) \tag{1.7}$$

will be studied. Also, starting from the study of the vector Briot-Bouquet equation

$$z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{G}(\tilde{f}(z)), \tag{1.8}$$

a generalization of the results of [5] will be given. As a consequence of the method used, simple proofs can be given regarding the study of higher order systems such as

$$z^2\frac{d^2\tilde{f}(z)}{dz^2} + z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{G}(\tilde{f}(z)). \tag{1.9}$$

Finally, conditions will be given such that the system

$$\frac{d^\nu \tilde{g}(t)}{dt^\nu} = A(re^{\pm i\mu t})\tilde{g}(t) + \tilde{G}(\tilde{g}(t)), \tag{1.10}$$

has complex-valued periodic solutions. The results concerning (1.10) yield information regarding the real periodic solutions of the system

$$\left.\begin{aligned} \frac{d^\nu \tilde{x}}{dt^\nu} &= A\tilde{x} - B\tilde{y} + \tilde{u}(\tilde{x},\tilde{y}) \\ \frac{d^\nu \tilde{y}}{dt^\nu} &= B\tilde{x} + A\tilde{y} + \tilde{v}(\tilde{x},\tilde{y}) \end{aligned}\right\}. \tag{1.11}$$

The rest of the paper is organized as follows: In Section 2, the main results are presented, whereas the method used is described in Section 3. The proofs of the main results are given in Section 5 and are based on general theorems formulated in terms of abstract operators which are given in Section 4.

2. MAIN RESULTS.

THEOREM 2.1. *Consider the system*

$$z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{g}(z), \tag{2.12}$$

where $\tilde{g}(z) \in [H_2(\Delta)]^k$ *and* $A(z)$ *is a matrix of functions* $a_{ij}(z)$, $i,j = 1,2,...,k$ *analytic in* $\overline{\Delta} = \{z \in \mathbb{C} : |z| \le 1\}$. *Assume that the eigenvalues of the constant matrix* $A(0)$ *are different from the natural numbers* $0,1,2,....$ *Then the system (2.12) has a unique solution* $\tilde{f}(z) = (f_1(z),...,f_k(z))$ *in* $[H_2(\Delta)]^k$ *of the form:*

$$f_i(z) = \lambda_i + \sum_{n=1}^{\infty} b_{in} z^{n-1}, \quad i = 1,...,k, \tag{2.13}$$

where the vector $\tilde{\lambda} = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_k \end{pmatrix}^T$ *is the unique solution of the system*

$$-A(0)\tilde{\lambda} = \begin{pmatrix} g_1(0) & g_2(0) & ... & g_k(0) \end{pmatrix}^T. \tag{2.14}$$

The series in (2.13) converges absolutely for $z \in \overline{\Delta}$ *and if* $\tilde{g}(z) \in [H_1(\Delta)]^k$, *then the first derivatives of* $f_i(z)$, $i = 1,...,k$ *converge also absolutely for* $z \in \overline{\Delta}$.

THEOREM 2.2. *Consider the system*

$$z^2\frac{d^2\tilde{f}(z)}{dz^2} + z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{g}(z), \tag{2.15}$$

where $\tilde{g}(z) \in [H_2(\Delta)]^k$ *and* $A(z)$ *is a matrix of functions* $a_{ij}(z)$, $i,j = 1,2,...,k$ *analytic in* $\overline{\Delta}$. *Assume that the eigenvalues of the constant matrix* $A(0)$ *are different from* $0,1,2^2,....$ *Then the system (2.15) has a unique solution* $\tilde{f}(z) = (f_1(z),...,f_k(z))$ *in* $[H_2(\Delta)]^k$ *of the form:*

$$f_i(z) = \lambda_i + \sum_{n=1}^{\infty} b_{in} z^{n-1}, \quad i = 1,...,k, \tag{2.16}$$

where the vector $\tilde{\lambda} = \begin{pmatrix} \lambda_1 & \lambda_2 & ... & \lambda_k \end{pmatrix}^T$ *is the unique solution of the system*

$$-A(0)\tilde{\lambda} = \begin{pmatrix} g_1(0) & g_2(0) & ... & g_k(0) \end{pmatrix}^T. \tag{2.17}$$

The series in (2.16) converges absolutely for $z \in \overline{\Delta}$ *and if* $\tilde{g}(z) \in [H_1(\Delta)]^k$*, then the derivatives of* $f_i(z)$*,* $i = 1, ..., k$ *up to order two converge also absolutely for* $z \in \overline{\Delta}$*.*

THEOREM 2.3. *Assume that the components of* $\tilde{g}(z)$ *and the elements* $a_{ij}(z)$*,* $i, j = 1, ..., k$ *of the matrix* $A(z)$ *are entire functions. Then under the assumptions of theorem 2.1 (resp. 2.2), the system (2.12) (resp. (2.15)) has a unique entire solution with components given by (2.13) (resp. (2.16)).*

THEOREM 2.4. *Consider the system*

$$z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{G}(\tilde{f}(z)), \tag{2.18}$$

where $A(z)$ *is a matrix of functions* $a_{ij}(z)$*,* $i, j = 1, 2, ..., k$ *analytic in some neighborhood of* 0 *and* $\tilde{G}(\tilde{w})$*,* $\tilde{w} = (w_1, ..., w_k)$ *is a function analytic in a neighborhood of* $\tilde{0}$ *satisfying* $\tilde{G}(\tilde{0}) = \tilde{G}'(\tilde{0}) = \tilde{0}$*. Assume that the constant matrix* $A(0)$ *has at least one positive integer eigenvalue. Then there exists a one-parameter family of solutions of the system (2.18) of the form*

$$f_i(z) = \lambda z^m + b_{i1}(\lambda)z^{m+1} + b_{i2}(\lambda)z^{m+2} + ..., \quad i = 1, ..., k, \tag{2.19}$$

where m *is the greatest positive integer eigenvalue of* $A(0)$*. The series in (2.19) as well as its first derivative converge absolutely for every* $z \in \overline{\Delta}$ *and sufficiently small* $|\lambda|$*.*

REMARK 2.5. *Essentially the same result was proved in [12].*

THEOREM 2.6. *Consider the system*

$$z^2\frac{d^2\tilde{f}(z)}{dz^2} + z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{G}(\tilde{f}(z)), \tag{2.20}$$

where $A(z)$ *is a matrix of functions* $a_{ij}(z)$*,* $i, j = 1, 2, ..., k$ *analytic in some neighborhood of* 0 *and* $\tilde{G}(\tilde{w})$*,* $\tilde{w} = (w_1, ..., w_k)$ *is a function analytic in a neighborhood of* $\tilde{0}$ *satisfying* $\tilde{G}(\tilde{0}) = \tilde{G}'(\tilde{0}) = \tilde{0}$*. Assume that the constant matrix* $A(0)$ *has at least one eigenvalue with an integer square root and that* m^2 *is the greatest of these. Then there exists a one-parameter family of solutions of the system (2.20) of the form*

$$f_i(z) = \lambda z^m + b_{i1}(\lambda)z^{m+1} + b_{i2}(\lambda)z^{m+2} + ..., \quad i = 1, ..., k. \tag{2.21}$$

The series in (2.21) as well as its first two derivative converge absolutely for every $z \in \overline{\Delta}$ *and sufficiently small* $|\lambda|$*.*

REMARK 2.7. *Similar results can be proved for the systems:*

$$z^3\frac{d^3\tilde{f}(z)}{dz^3} + 3z^2\frac{d^2\tilde{f}(z)}{dz^2} + z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{G}(\tilde{f}(z)),$$

$$z^4\frac{d^4\tilde{f}(z)}{dz^4} + 6z^3\frac{d^3\tilde{f}(z)}{dz^3} + 7z^2\frac{d^2\tilde{f}(z)}{dz^2} + z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{G}(\tilde{f}(z)),$$

etc.

THEOREM 2.8. *Let $\nu > 0$ be an odd integer and assume that the constant matrix $A(0)$ has a purely imaginary nonzero eigenvalue. Let μ be the largest positive number for which $\pm i\mu^\nu$ is an eigenvalue of $A(0)$. Then for every $0 < r \leq 1$ the system:*

$$\frac{d^\nu \tilde{g}(t)}{dt^\nu} = A(re^{j\mu t})\tilde{g}(t) + \tilde{G}(\tilde{g}(t)), \tag{2.22}$$

where $j = \mp i$ for $\nu = 4\sigma - 3$ and $j = \pm i$ for $\nu = 4\sigma - 1$, has a family of periodic solutions $\tilde{g}(t) = (g_1(t), ..., g_k(t))$ of the form

$$g_i(t) = \lambda r e^{j\mu t} + b_{i1}(\lambda) r^2 e^{2jt} + ... \tag{2.23}$$

Moreover, the series in (2.23) together with their first ν derivatives converge absolutely for sufficiently small $|\lambda|$.

REMARK 2.9. *Theorem 2.8 was proved in [5] in the scalar case for $\nu = 1$.*

THEOREM 2.10. *Let $\nu = 4\sigma - 2$ ($\nu = 4\sigma$) be an even integer and assume that the constant matrix $A(0)$ has a negative (positive) eigenvalue. Let μ be the largest positive number for which $-\mu^\nu$ (μ^ν) is an eigenvalue of $A(0)$. Then for every $0 < r \leq 1$ the system:*

$$\frac{d^\nu \tilde{g}(t)}{dt^\nu} = A(re^{\pm i\mu t})\tilde{g}(t) + \tilde{G}(\tilde{g}(t)), \tag{2.24}$$

has a family of periodic solutions $\tilde{g}(t) = (g_1(t), ..., g_k(t))$ of the form (2.23) with $j = \pm i$.

Consider now the $2k$-dimensional system of ordinary differential equations:

$$\left.\begin{aligned} \frac{d^\nu \tilde{x}}{dt^\nu} &= A\tilde{x} - B\tilde{y} + \tilde{u}(\tilde{x}, \tilde{y}) \\ \frac{d^\nu \tilde{y}}{dt^\nu} &= B\tilde{x} + A\tilde{y} + \tilde{v}(\tilde{x}, \tilde{y}) \end{aligned}\right\}, \tag{2.25}$$

where $\tilde{x(t)} = (x_1(t), ..., x_k(t))$, $\tilde{y} = (y_1(t), ..., y_k(t))$, A, B are $k \times k$ constant matrices and $\tilde{u}(\tilde{x}, \tilde{y}) = (u_1(\tilde{x}, \tilde{y}), ..., u_2(\tilde{x}, \tilde{y}))$, $\tilde{v}(\tilde{x}, \tilde{y}) = (v_1(\tilde{x}, \tilde{y}), ..., v_2(\tilde{x}, \tilde{y}))$ are continuously differentiable vector-valued functions of $2k$ variables the components of which satisfy the Cauchy-Riemann equations, i.e.

$$\frac{\partial u_i(\tilde{x}, \tilde{y})}{\partial x_j} = \frac{\partial v_i(\tilde{x}, \tilde{y})}{\partial y_j}, \quad \frac{\partial u_i(\tilde{x}, \tilde{y})}{\partial y_j} = -\frac{\partial v_i(\tilde{x}, \tilde{y})}{\partial x_j}, \quad i, j = 1, ..., k$$

and their derivatives satisfy the conditions

$$\frac{\partial u_i(\tilde{0}, \tilde{0})}{\partial x_j} = \frac{\partial v_i(\tilde{0}, \tilde{0})}{\partial y_j} = \tilde{0}$$

so that the function $\tilde{G}(\tilde{g}(t)) = \tilde{u}(\tilde{x}, \tilde{y}) + i\tilde{v}(\tilde{x}, \tilde{y})$ is analytic in $\tilde{g} = \tilde{x} + i\tilde{y}$ and $\tilde{G}(\tilde{0}) = \tilde{G}'(\tilde{0}) = \tilde{0}$. Then the system (2.25) can be rewritten as:

$$\frac{d^\nu \tilde{g}(t)}{dt^\nu} = (A + iB)\tilde{g}(t) + \tilde{G}(\tilde{g}(t)).$$

Theorem 2.8 implies the existence of real-valued periodic solutions of the system (2.25), provided that the matrix $A+iB$ has at least one purely imaginary nonzero eigenvalue. Theorem 2.10 implies also the existence of real-valued periodic solutions of the system (2.25), provided that the matrix $A+iB$ has at least one negative eigenvalue in the case where $\nu = 4\sigma - 2$ and a positive eigenvalue in the case where $\nu = 4\sigma$, $\sigma = 1, 2, \ldots$. Thus the following theorem is true:

THEOREM 2.11. *The system* (2.25) *has a family of real-valued periodic solutions with period* $\dfrac{2\pi}{\mu}$ *under one of the following conditions:*
(i) ν *is an odd integer and the matrix* $A+iB$ *has at least one purely imaginary nonzero eigenvalue* $\pm i\mu^{\nu}$, $\mu > 0$.
(ii) ν *is an even integer of the form* $\nu = 4\sigma - 2$, $\sigma = 1, 2, \ldots$ *and the matrix* $A+iB$ *has at least one negative eigenvalue* $-\mu < 0$.
(iii) ν *is an even integer of the form* $\nu = 4\sigma$, $\sigma = 1, 2, \ldots$ *and the matrix* $A+iB$ *has at least one positive eigenvalue* $\mu > 0$.

REMARK 2.12. *Theorem 2.11 was proved in [9] for* $\nu = 1$ *and in [5] in the scalar case for* $\nu = 2$.

For the system

$$\frac{d^{\nu}\tilde{g}(t)}{dt^{\nu}} = A(re^{-at})\tilde{g}(t) + \tilde{G}(\tilde{g}(t)), \tag{2.26}$$

the following theorem can be proved:

THEOREM 2.13. *If* ν *is odd (even) and the constant matrix* $A(0)$ *has at least one negative (positive) eigenvalue* $-\mu^{\nu}$ (μ^{ν}), *then for each positive integer* m, *the system (2.26) has a family* $\tilde{g}(t) = (g_1(t), \ldots, g_k(t))$ *of real-valued solutions of the form*

$$g_i(t) = \lambda r e^{-\mu t} + b_{i1}(\lambda) r^2 e^{-2jt} + \ldots. \tag{2.27}$$

which converge absolutely for sufficiently small λ, *every positive* t *and* $0 \le r \le 1$.

3. THE METHOD.

Let H be an abstract separable Hilbert space over the complex field, with the orthonormal base $\{e_n\}$, $n = 1, 2, 3, \ldots$. Use the symbols $(\cdot,\cdot)$ and $\|\cdot\|$ to denote scalar product and norm in H respectively. Let H_1 be the Banach space consisting of those elements f in H which satisfy the condition $\sum_{n=1}^{\infty} |(f, e_n)| < +\infty$, i.e. H_1 is imbedded in H in the sense that $f \in H_1$ implies $f \in H$ and $\|f\| \le \|f\|_1$. The norm in H_1 is denoted by $\|f\|_1 = \sum_{n=1}^{\infty} |(f, e_n)| < +\infty$. Finally let V be the shift operator on H:

$$Ve_n = e_{n+1}, \quad n = 1, 2, 3, \ldots$$

and V^* its adjoint:

$$V^*e_n = e_{n-1}, \quad n = 2, 3, \ldots \;, V^*e_1 = 0.$$

It can be seen that the following statements hold [4], [5]:
(i) Every point z in the interior of the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, belongs to the point spectrum of V^* and the set of proper elements $f_z = \sum_{n=1}^{\infty} z^{n-1} e_n$, $f_0 = e_1$, forms a complete system in H in the sense that, if f is orthogonal to f_z for every $z \in \Delta$, then $f = 0$.
(ii) The mapping

$$f(z) = (f_z, f) = \sum_{n=1}^{\infty} \overline{(f, e_n)} z^{n-1}, \quad z \in \Delta \tag{3.1}$$

is a one-to-one mapping from H onto $H_2(\Delta)$ which preserves the norm.
The element f in (3.1) is called the *abstract form* of $f(z)$. In general the abstract form of a function $G(f(z)) : H_2(\Delta)(H_1(\Delta)) \to H_2(\Delta)(H_1(\Delta))$ is a mapping $N(f) : H(H_1) \to H(H_1)$ for which the following relation holds:

$$G(f(z)) = (f_z, N(f)), \quad z \in \Delta.$$

(iii) H_1 is invariant under the operators V^m, $(V^*)^m$ and $\|V^m\|_1 = \|(V^*)^m\|_1 = 1$, $m = 1, 2, ...$

Linear case

Taking into account (3.1) it follows that:
(a) The abstract form of $f^{(p)}(z)$ is the element $(C_0 V^*)^p f$, since

$$f^{(p)}(z) = (f_z, (C_0 V^*)^p f),$$

where $z \in \Delta$, $k = 1, 2, ...$, C_0 is the diagonal operator (of H)

$$C_0 e_n = n e_n, \quad n = 1, 2, ...$$

defined on the dense linear manifold spanned by $\{e_n\}$, which has a self-adjoint extension with discrete spectrum, i.e. the definition domain of C_0 can be extended to the range of the bounded (and compact) operator

$$B_0 e_n = \frac{1}{n} e_n, \quad n = 1, 2, ...$$

Moreover, the range of B_0^p in H, $p = 1, 2, ...$, i.e. the definition domain of C_0^p is isomorphic to the linear manifold in $H_2(\Delta)$ which consists of functions with derivatives up to order p in $H_2(\Delta)$.
(b) The abstract form of $z^m f(z)$ is the element $V^m f$, since

$$z^m f(z) = (f_z, V^m f),$$

where $z \in \Delta$, $m = 1, 2, ...$
(c) The abstract form of $a(z)f(z)$ is the element $a(V)f$, since

$$a(z)f(z) = (f_z, a(V)f),$$

where $a(z) = \sum_{n=1}^{\infty} \overline{a_n} z^{n-1}$ is analytic in a neighborhood of the closed unit disc $\overline{\Delta} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $a(V)$ is the operator $a(V) = \sum_{n=1}^{\infty} a_n V^{n-1}$, which is bounded on H with norm

$$\|a(V)\|_1 = \|a(z)\|_{H_1(\Delta)} = \sum_{n=1}^{\infty} |a_n|. \tag{3.2}$$

Also the following useful relations hold:

$$(C_0V^*)^p f = C_0(C_0+I)...(C_0+(p-1)I)(V^*)^p f, \tag{3.3}$$

$$V^p(C_0V^*)^p f = (C_0-I)(C_0-2I)...(C_0-pI)f, \tag{3.4}$$

for $p = 1,2,3,...$, where I is the identity operator on H.

Now, consider the Hilbert space $H^k = \underbrace{H\times ...\times H}_{k-times}$ consisting of the $k-$vector elements $\tilde{f} = (f_1,...,f_k) = \begin{pmatrix} f_1 & ... & f_k \end{pmatrix}^T$, where $f_i \in H$, $i = 1,...,k$.. The inner product in H^k is defined as $\left\langle \tilde{f}, \tilde{g} \right\rangle = \sum_{i=1}^{k} (f_i, g_i)$, where $\tilde{g} = (g_1,...,g_k) = \begin{pmatrix} g_1 & ... & g_k \end{pmatrix}^T$, $g_i \in H$, $i = 1,...,k$. Denote by $H_1^k = \underbrace{H_1\times ...\times H_1}_{k-times}$, the Banach space imbedded in H^k in the sense that $\tilde{f} \in H_1^k$ implies also that $\tilde{f} \in H_1$.

Due to (3.1), the Hilbert space $H_2(\Delta)$ can be considered as a realization of the abstract Hilbert space H. In the same way, the product space $[H_2(\Delta)]^k = \underbrace{H_2(\Delta)\times ...\times H_2(\Delta)}_{k-times}$ can be considered as a realization of the space H^k and the product space $[H_1(\Delta)]^k = \underbrace{H_1(\Delta)\times ...\times H_1(\Delta)}_{k-times}$ as a realization of the space H_1^k.

Definition 3.1. *The following useful operators can be defined in H^k:*

(1) The identity operator $\tilde{I}$: $\tilde{I}\tilde{f} = \tilde{f}$.
(2) The operator $\tilde{C}_0$: $\tilde{C}_0\tilde{f} = (C_0f_1,...,C_0f_k)$.
(3) The operator $\tilde{B}_0$: $\tilde{B}_0\tilde{f} = (B_0f_1,...,B_0f_k)$.
(4) The operator $\tilde{V}$: $\tilde{V}\tilde{f} = (Vf_1,...,Vf_k)$.

Remark 3.2. *1) The eigenvalues of $\tilde{C}_0$ in H^k are the values $n = 1,2,...$ each with multiplicity k. The eigenelements of $\tilde{C}_0$ which correspond to the eigenvalue n are the elements:*

$$\tilde{e}_{n1} = (e_n,0,0,...,0),\ \ \tilde{e}_{n2} = (0,e_n,0,...,0),\ \ ...,\ \ \tilde{e}_{nk} = (0,0,0,...,e_n).$$

Moreover, the set $\{\tilde{e}_{ni}\}$, $n = 1,2,...$, $i = 1,2,...,k$ of all the eigenelements of $\tilde{C}_0$, is an orthonormal base in H^k.
2) The operator $\tilde{B}_0$ is compact and self-adjoint.

At this point, taking into consideration what mentioned already, one can find the abstract form in H of the linear system

$$z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{g}(z), \tag{3.5}$$

which can also be written as:

$$zf_i'(z) = g_i(z) + \sum_{j=1}^{k} a_{ij}(z) f_j(z), \quad i = 1, ..., k, \tag{3.6}$$

where $A(z)$ is a matrix of functions $a_{ij}(z)$, $i, j = 1, 2, ..., k$ analytic in $\overline{\Delta}$.

More precisely, the abstract form, in H, of (3.6) is:

$$VC_0V^* f_i = g_i + \sum_{j=1}^{k} a_{ij}(V) f_j, \quad i = 1, ..., k,$$

which can also be written, due to (3.4), as:

$$(C_0 - I) f_i = g_i + \sum_{j=1}^{k} a_{ij}(V) f_j, \quad i = 1, ..., k.$$

Thus the abstract form, in H, of the system (2.12) is:

$$[\tilde{C}_0 - \tilde{I} - A(V)]\tilde{f} = \tilde{g}, \tag{3.7}$$

where $A(V) = (a_{ij}(V))_{i,j=1}^{k}$ a $k \times k$ matrix of bounded operators in H, which can be rewritten as:

$$[\tilde{I} - \tilde{B}_0 - \tilde{B}_0 A(V)]\tilde{f} = \tilde{B}_0 \tilde{g}. \tag{3.8}$$

In the same way it can be found that the abstract form in H of the linear system

$$z^2 \frac{d^2 \tilde{f}(z)}{dz^2} + z \frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{g}(z), \tag{3.9}$$

where $A(z)$ is a matrix of functions $a_{ij}(z)$, $i, j = 1, 2, ..., k$ analytic in $\overline{\Delta}$, is

$$[(\tilde{C}_0 - \tilde{I})^2 - A(V)]\tilde{f} = \tilde{g}, \tag{3.10}$$

where $A(V) = (a_{ij}(V))_{i,j=1}^{k}$ a $k \times k$ matrix of bounded operators in H.

Non-linear case

Consider now a function

$$\tilde{G}(\tilde{w}) = (G_1(\tilde{w}), ..., G_k(\tilde{w})) = \begin{pmatrix} G_1(\tilde{w}) & ... & G_k(\tilde{w}) \end{pmatrix}^T,$$

where $\tilde{w} = (w_1, ..., w_k) = \begin{pmatrix} w_1 & ... & w_k \end{pmatrix}^T$, such that its components have a power series expansion in the w_i, $i = 1, ..., k$ variables which start with terms of degree at least two. Suppose also that the series $G_i(\tilde{w})$, $i = 1, ..., k$ converge absolutely for $|w_i| < r$, $i = 1, ..., k$. Then $G_i(\tilde{f}(z))$ are defined for $\|f_i(z)\|_{H_1(\Delta)} < r$, i.e. they are defined from an open sphere $S(0, r/k)$ in $[H_1(\Delta)]^k$ into $H_1(\Delta)$. As in [5], it can be shown that $G_i(\tilde{f}(z))$ have continuous partial Fréchet derivatives for every $f_i(z) \in S(0, r/k)$.

Denote by $N_i(\tilde{f})$ the abstract forms of $G_i(\tilde{f}(z))$, $i = 1, ..., k$ and let $\tilde{N}(\tilde{f}) = (N_1(\tilde{f}), ..., N_k(\tilde{f}))$ be the abstract form of $\tilde{G}(\tilde{f}(z)) = (G_1(\tilde{f}(z)), ..., G_k(\tilde{f}(z)))$. (The operator $\tilde{N}(\tilde{f})$ can be explicitly found in various cases arising in applications, as it has been found in [5] and [8], but this is unnecessary for the qualitative results we are interested in.)

The following is true:

LEMMA 3.3. *The operator $\tilde{N}$ is a C^1 mapping from an open sphere in H_1^k into H_1^k. Moreover*

$$\tilde{N}(\tilde{0}) = \tilde{N}'(\tilde{0}) = \tilde{0}, \tag{3.11}$$

where $\tilde{N}'(0)$ is the Fréchet derivative of $\tilde{N}$ at $\tilde{f} = \tilde{0}$.

The proof of this lemma is based on the relation:

$$\|f_1(z)f_2(z)\|_{H_1(\Delta)} \leq \|f_1(z)\|_{H_1(\Delta)} \cdot \|f_2(z)\|_{H_1(\Delta)}, \tag{3.12}$$

which follows from (3.2) and the relation $f_1(z)f_2(z) = (f_z, f_1(V)f_2)$. Indeed it is

$$\|f_1(z)f_2(z)\|_{H_1(\Delta)} = \|f_1(V)f_2\|_1 \leq \|f_1(V)\|_1 \cdot \|f_2\|_1 =$$

$$= \|f_1\|_1 \cdot \|f_2\|_1 = \|f_1(z)\|_{H_1(\Delta)} \cdot \|f_2(z)\|_{H_1(\Delta)}.$$

Consider now the subspace of H^k which is spanned by the $k(m+1)$ elements $\tilde{e}_{ni}$, $n = 1, ..., m+1$, $i = 1, ..., k$ and denote by $M(m)$ its orthogonal complement. Let $M_1(m)$ be the Banach space which is a subset of $M(m)$ as H_1^k is a subset of H^k, i.e.

$$M_1(m) = M(m) \cap H_1^k.$$

The space $M_1(m)$ is obviously invariant under the operators $\tilde{B}_0$ and $A(V)$. Moreover, the following lemma holds:

LEMMA 3.4. *Suppose that the element $\tilde{f} = \lambda\tilde{f}_0 + \tilde{y}$, where $\lambda \in \mathbb{C}$, $\tilde{y} \in M_1(m)$ and $\tilde{f}_0 = (\overline{\xi}_1 e_{m+1}, \overline{\xi}_2 e_{m+1}, ..., \overline{\xi}_k e_{m+1})$, $x_i \in \mathbb{C}$, belongs to the definition domain of the operator $\tilde{N}$. Then the element $\tilde{N}(\lambda\tilde{f}_0 + \tilde{y})$ belongs also to the space $M_1(m)$. As a consequence*

$$(N_i(\lambda\tilde{f}_0 + \tilde{y}), e_{n+1}) = 0, \quad i = 1, 2, ..., k \quad n = 1, 2, ..., m. \tag{3.13}$$

As before the abstract forms in H_1 of the non-linear systems

$$z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{G}(\tilde{f}(z)) \tag{3.14}$$

and

$$z^2\frac{d^2\tilde{f}(z)}{dz^2} + z\frac{d\tilde{f}(z)}{dz} = A(z)\tilde{f}(z) + \tilde{G}(\tilde{f}(z)), \tag{3.15}$$

where $A(z)$ is a matrix of functions $a_{ij}(z)$, $i, j = 1, 2, ..., k$ analytic in some neighborhood of 0 and $\tilde{G}(\tilde{w})$, $\tilde{w} = (w_1, ..., w_k)$ is a function analytic in a neighborhood of $\tilde{0}$ satisfying $\tilde{G}(\tilde{0}) = \tilde{G}'(\tilde{0}) = \tilde{0}$ are

$$[\tilde{C}_0 - \tilde{I} - A(V)]\tilde{f} = \tilde{g} + \tilde{N}(\tilde{f}) \tag{3.16}$$

and

$$[(\tilde{C}_0 - \tilde{I})^2 - A(V)]\tilde{f} = \tilde{g} + \tilde{N}(\tilde{f}), \tag{3.17}$$

respectively, where $\tilde{N}(\tilde{f})$ is the abstract form in H_1 of $\tilde{G}(\tilde{f}(z))$.

4. OPERATOR TYPE RESULTS.

In this section several useful results regarding operators in H^k or H_1^k will be given. These operators have already appeared in the abstract forms of the linear and non-linear systems of differential equations under consideration.

In the following, $A(z)$ will represent a $k \times k$ matrix of functions $a_{ij}(z)$, $i, j = 1, 2, ..., k$ analytic in $\overline{\Delta}$ and as a consequence the corresponding matrix $A(V)$ consisting of the bounded (in H) operators $a_{ij}(V)$, $i, j = 1, 2, ..., k$ (defined in section 3) will also be bounded in H^k.

Proposition 4.5. *The null space in H^k (H_1^k) of the operator $(\tilde{C}_0 - \tilde{I})^\nu - A(V)$, where $\nu \geq 1$ an integer, is trivial if the eigenvalues of the constant matrix $A(0)$ are different from the values $0, 1, 2^\nu, 3^\nu, ...$*

Corollary 4.6. *The inverse of the operator $(\tilde{C}_0 - \tilde{I})^\nu - A(V)$ is bounded in H^k (H_1^k).*

Proof Proof of Proposition 4.5. Suppose that $\tilde{f}$ is an element of the null space of $(\tilde{C}_0 - \tilde{I})^\nu - A(V)$ and that the eigenvalues of the constant matrix $A(0)$ are different from the values $0, 1, 2^\nu, 3^\nu, ...$ Then

$$(\tilde{C}_0 - \tilde{I})^\nu \tilde{f} = A(V)\tilde{f} \Rightarrow$$

$$\Rightarrow (C_0 - I)^\nu f_i = \sum_{j=1}^{k} a_{ij}(V) f_j, \quad i = 1, ..., k. \tag{4.18}$$

By taking the inner product of both parts of (4.18) with the element e_1, one obtains for every $i = 1, ..., k$:

$$((C_0 - I)^\nu f_i, e_1) = \left(\sum_{j=1}^{k} a_{ij}(V) f_j, e_1 \right) \Rightarrow$$

$$\Rightarrow \sum_{m=0}^{\nu} \begin{pmatrix} \nu \\ m \end{pmatrix} (-1)^m (C_0^{\nu-m} f_i, e_1) = \sum_{j=1}^{k} a_{ij}(0)(f_j, e_1) \Rightarrow$$

$$\Rightarrow \sum_{m=0}^{\nu} \begin{pmatrix} \nu \\ m \end{pmatrix} (-1)^m 1^{\nu-m} (f_i, e_1) = \sum_{j=1}^{k} a_{ij}(0)(f_j, e_1) \Rightarrow$$

$$\Rightarrow (1-1)^\nu (f_i, e_1) = \sum_{j=1}^{k} a_{ij}(0)(f_j, e_1), \quad i = 1, ..., k \Rightarrow$$

$$\Rightarrow A(0) \begin{pmatrix} (f_1, e_1) \\ .... \\ (f_k, e_k) \end{pmatrix} = \tilde{0} \Rightarrow \begin{pmatrix} (f_1, e_1) \\ .... \\ (f_k, e_1) \end{pmatrix} = \tilde{0} \Rightarrow$$

$$\Rightarrow (f_i, e_1) = 0, \quad \forall\, i = 1, ..., k, \tag{4.19}$$

since 0 is not an eigenvalue of $A(0)$.

In the same way, by taking consecutively the inner product of both parts of (4.18) with the elements e_s, $s = 2, 3...$ and using the fact that $(f_i, e_{s-1}) = 0$, $i = 1, ..., k$, $s = 2, 3, ...$ one obtains for every $i = 1, ..., k$:

$$((C_0 - I)^\nu f_i, e_s) = \left(\sum_{j=1}^{k} a_{ij}(V) f_j, e_s \right) \Rightarrow$$

$$\Rightarrow \sum_{m=0}^{\nu} \binom{\nu}{m} (-1)^m (C_0^{\nu-m} f_i, e_s) = \sum_{j=1}^{k} a_{ij}(0)(f_j, e_s) \Rightarrow$$

$$\Rightarrow \sum_{m=0}^{\nu} \binom{\nu}{m} (-1)^m 2^{\nu-m} (f_i, e_s) = \sum_{j=1}^{k} a_{ij}(0)(f_j, e_s) \Rightarrow$$

$$\Rightarrow (s-1)^\nu (f_i, e_s) = \sum_{j=1}^{k} a_{ij}(0)(f_j, e_s), \quad i = 1, ..., k \Rightarrow$$

$$\Rightarrow A(0) \begin{pmatrix} (f_1, e_s) \\ \cdots\cdots\cdots \\ (f_k, e_s) \end{pmatrix} = (s-1)^\nu \begin{pmatrix} (f_1, e_s) \\ \cdots\cdots\cdots \\ (f_k, e_s) \end{pmatrix} \Rightarrow$$

$$\Rightarrow (f_i, e_s) = 0, \quad \forall\, i = 1, ..., k,$$

since $(s-1)^\nu$, $s = 2, 3, ...$ is not an eigenvalue of $A(0)$. Thus $(f_i, e_s) = 0$, $s = 1, 2, 3, ...$ and $f_i = 0$, $i = 1, ..., k$. $\square$

PROPOSITION 4.7. *Let m^ν be the greatest positive integer eigenvalue of the constant matrix $A(0)$. Then the bounded, on H_1^k, operator $(\tilde{I} - \tilde{B}_0)^\nu - A(V)\tilde{B}_0^\nu$ restricted on $M_1(m)$ has a bounded inverse.*

PROOF. As in [5], it can be proved that the operator $\tilde{B}_0^\nu$ is a compact operator on $M_1(m)$. Since $A(V)$ is a bounded operator, the operator $(\tilde{I} - \tilde{B}_0)^\nu - A(V)\tilde{B}_0^\nu$ has the form $\tilde{I} - \tilde{K}$, where $\tilde{K} = \tilde{I} - (\tilde{I} - \tilde{B}_0)^\nu + A(V)\tilde{B}_0^\nu$ is a compact operator on $M_1(m)$. Thus the Fredholm alternative (see e.g. [2]) implies the invertibility of $(\tilde{I} - \tilde{B}_0)^\nu - A(V)\tilde{B}_0^\nu$, if the null space of $\tilde{I} - \tilde{K}$ is trivial.

Let $\tilde{g}$ be an element of the null space of $\tilde{I} - \tilde{K}$ in $M_1(m)$ and set $\tilde{f} = \tilde{B}_0^\nu \tilde{g}$. Then

$$(\tilde{I} - \tilde{K})\tilde{f} = \tilde{0} \Rightarrow (\tilde{I} - \tilde{B}_0)^\nu \tilde{f} - A(V)\tilde{B}_0^\nu \tilde{f} = \tilde{0} \Rightarrow$$

$$\Rightarrow (\tilde{C}_0 - \tilde{I})\tilde{f} = A(V)\tilde{f}. \tag{4.20}$$

Since $\tilde{g} \in M_1(m)$, it follows that $\tilde{f} \in M_1(m)$ because $M_1(m)$ is invariant under the operator $\tilde{B}_0$. Thus $\tilde{f}$ is orthogonal to all $\tilde{e}_{ni}$, $n = 1, ..., m+1$, $i = 1, ..., k$, which means that

$$(f_i, e_n) = 0, \quad i = 1, ..., k, \quad n = 1, ..., m+1. \tag{4.21}$$

By taking the inner product of both parts of (4.20) with the element e_{m+2}, one obtains for every $i = 1, ..., k$:

$$((C_0 - I)^\nu f_i, e_{m+2}) = \sum_{j=1}^{k} (a_{ij}(V) f_j, e_{m+2}) \Rightarrow$$

$$\Rightarrow (m+1)^\nu (f_i, e_{m+2}) = \sum_{j=1}^{k} a_{ij}(0)(f_j, e_{m+2}), \quad i = 1, ..., k \Rightarrow$$

$$\Rightarrow A(0) \begin{pmatrix} (f_1, e_{m+2}) \\ .... \\ (f_k, e_{m+2}) \end{pmatrix} = (m+1)^\nu \begin{pmatrix} (f_1, e_{m+2}) \\ .... \\ (f_k, e_{m+2}) \end{pmatrix} \Rightarrow$$

$$\Rightarrow (f_i, e_{m+2}) = 0, \quad i = 1, ...k,$$

since m^ν is the greatest positive integer eigenvalue of the constant matrix $A(0)$. Continuing in the same way it can be proved that

$$(f_i, e_{m+s}) = 0, \quad i = 1, ...k, \quad s = 3, 4, ...$$

This, combined with (4.21) gives $\tilde{f} = \tilde{0}$, which means that the null space of $\tilde{I} - \tilde{K}$ is trivial in $M_1(m)$. □

REMARK 4.8. *The range of $\tilde{B}_0^\nu$ is invariant under the operator $(\tilde{I} - \tilde{K})^{-1}$ in H^k (H_1^k).*

PROPOSITION 4.9. *Assume that the eigenvalues of the constant matrix $A(0)$ are different from the values $0, 1, 2^\nu, 3^\nu, ...$ Then for every $\tilde{g} \in H^k$ (H_1^k), there exists an element $\tilde{x} \in H^k$ (H_1^k) such that the unique solution of*

$$(\tilde{C}_0 - \tilde{I})^\nu \tilde{f} - A(V)\tilde{f} = \tilde{g} \tag{4.22}$$

to be given by

$$\tilde{f} = \tilde{\eta}_0 + \tilde{B}_0^\nu \tilde{x}, \tag{4.23}$$

where $\tilde{\eta}_0 = (\overline{\lambda}_1 e_1, ..., \overline{\lambda}_k e_1)$ and the vector $\tilde{\lambda} = (\lambda_1, ..., \lambda_k)$ satisfies

$$-A(0)\tilde{\lambda} = \tilde{g'}, \quad \text{where} \quad \tilde{g'} = \begin{pmatrix} (g_1, e_1) \\ .... \\ (g_k, e_1) \end{pmatrix}.$$

PROOF. First of all $\tilde{\eta}_0 \in H^k$ (H_1^k) and as a consequence $\tilde{f} \in H^k$ (H_1^k). By substitution of (4.23) into (4.22) one obtains:

$$(\tilde{C}_0 - \tilde{I})^\nu \tilde{B}_0^\nu \tilde{x} - A(V)\tilde{B}_0^\nu \tilde{x} = \underbrace{\tilde{g} + A(0)\tilde{\eta}_0}_{\tilde{u}}. \tag{4.24}$$

Since the eigenvalues of $A(0)$ are different from the values $0, 1, 2^\nu, 3^\nu, ...$ it can be proved as in the proof of Proposition 4.5, that the element $\tilde{B}_0^\nu \tilde{x}$ is uniquely determined by (4.24). Moreover due to Remark 4.8, it follows that $\tilde{x} \in H^k$ (H_1^k). □

PROPOSITION 4.10. *Consider the operator equation*

$$(\tilde{C}_0 - \tilde{I})^\nu \tilde{f} = A(V)\tilde{f} + \tilde{N}(\tilde{f}), \tag{4.25}$$

where $\tilde{N}$ a non-linear mapping defined in an open sphere of H_1^k which satisfies (3.11) and (3.13). Assume that the constant matrix $A(0)$ has at least one eigenvalue *which belongs to the set $S = \{1, 2^\nu, 3^\nu, ...\}$. Then there exists a family of solutions of (4.25) in H_1^k of the form*

$$f_i = \overline{\lambda} e_{m+1} + \overline{b_{i1}(\lambda)} e_{m+2} + ..., \quad i = 1, 2, ..., k, \tag{4.26}$$

where m^ν is the greatest eigenvalue of $A(0)$ which belongs to S.

PROOF. Let's seek for solutions of (4.25) of the form

$$\tilde{f} = \overline{\lambda}\tilde{f}_0 + \tilde{B}_0^\nu \tilde{x}, \tag{4.27}$$

where $\tilde{x} \in M_1(m)$, $\tilde{f}_0 = (\overline{\xi}_1 e_{m+1}, ..., \overline{\xi}_k e_{m+1})$ and $\tilde{\xi} = (\xi_1, ..., \xi_k)$ is the eigenvector of $A(0)$ corresponding to its eigenvalue m^ν.

It is possible to find an open sphere centered at the origin of the space $\mathbb{C} \times M_1(m)$ such that the element defined by (4.27) to belong to the definition domain of $\tilde{N}$. Then by noticing that
a) $\tilde{f}_0$ is an eigenvector of $(\tilde{C}_0 - \tilde{I})^\nu$ with corresponding eigenvalue m^ν, which leads to the relation

$$(\tilde{C}_0 - \tilde{I})^\nu = A(0)\tilde{f}_0,$$

b) since $A(z)$ can be rewritten as $A(0) + zA_1(z)$, the matrix $A(V)$ can also be written as $A(V) = A(0) + A_1(V)\tilde{V}$
and substituting (4.27) into (4.26), one obtains:

$$(\tilde{I} - \tilde{B}_0)^\nu \tilde{x} = \overline{\lambda} A_1(V)\tilde{V}\tilde{f}_0 + A(V)\tilde{B}_0^\nu \tilde{x} + \tilde{N}(\overline{\lambda}\tilde{f}_0 + \tilde{B}_0^\nu \tilde{x}).$$

Observe that $\tilde{V}\tilde{f}_0 \in M_1(m)$ and the operators $\tilde{B}_0^\nu$, $A_1(V)$ and $A(V)$ leave the space $M_1(m)$ invariant. Also condition (3.13) implies that the operator $\tilde{N}(\overline{\lambda}\tilde{f}_0 + \tilde{B}_0^\nu \tilde{x})$ maps elements of $M_1(m)$ into $M_1(m)$. Thus the function

$$F(\lambda, \tilde{x}) = (\tilde{I} - \tilde{B}_0)^\nu \tilde{x} - \overline{\lambda} A_1(V)\tilde{V}\tilde{f}_0 - A(V)\tilde{B}_0^\nu \tilde{x} - \tilde{N}(\overline{\lambda}\tilde{f}_0 + \tilde{B}_0^\nu \tilde{x})$$

is a C^1 mapping from an open sphere of $\mathbb{C} \times M_1(m)$ into $M_1(m)$ and satisfies

$$F(0, \tilde{0}) = \tilde{0}, \quad \frac{\partial F}{\partial \tilde{x}}(0, \tilde{0}) = (\tilde{I} - \tilde{B}_0)^\nu - A(V)\tilde{B}_0^\nu.$$

Then Proposition 4.7 and the implicit function theorem imply the existence of an $\tilde{x} \in M_1(m)$ for sufficiently small $|\lambda|$. From the form of the element (4.27), it follows that the components of the solution $\tilde{f}$ of (4.25) have the form (4.26). $\square$

5. PROOF OF MAIN RESULTS.

PROOF PROOF OF THEOREM 2.1. According to what mentioned in Section 3, the abstract form of (2.12) is

$$[\tilde{C}_0 - \tilde{I} - A(V)]\tilde{f} = \tilde{g}, \tag{5.1}$$

where $A(V) = (a_{ij}(V))_{i,j=1}^k$ a $k \times k$ matrix of bounded operators in H.

The Fredholm alternative can be applied to (3.8) since $\tilde{B}_0$ is a compact operator and $a_{ij}(V)$ are bounded operators. Thus, either (3.8) has a unique solution in H or the corresponding homogeneous equation

$$[\tilde{I} - \tilde{B}_0 - \tilde{B}_0 A(V)]\tilde{f} = \tilde{0} \tag{5.2}$$

has a non-trivial solution. But (5.2) is equivalent to

$$[\tilde{C}_0 - \tilde{I} - A(V)]\tilde{f} = \tilde{0},$$

and according to Proposition 4.5 (for $\nu = 1$), it has only the trivial solution. As a consequence, (3.7) has a unique solution in H or equivalently, the system (2.12) has a unique solution in $[H_2(\Delta)]^k$, which proves the first part of the theorem. (Notice that a) the element $\tilde{B}_0\tilde{g}$ belongs in H^k and b) the range of the operator $\tilde{B}_0$ is invariant under the operator $[\tilde{I}_0 - \tilde{B}_0 - \tilde{B}_0 A(V)]^{-1}$. This last statement can be proved as in [5].)

For the second part of the theorem, according to Proposition 4.9 the element

$$\tilde{f} = \tilde{n}_0 + \tilde{B}_0\tilde{x}, \tag{5.3}$$

where $\tilde{n}_0 = (\overline{\lambda}_1 e_1, \overline{\lambda}_2 e_1, ..., \overline{\lambda}_k e_1)$ and $\tilde{x}$ an element of H^k, is the unique solution of equation (3.7). Thus

$$f_i = \overline{\lambda}_i e_1 + B_0 x_i, \quad i = 1, ..., k$$

or after using (3.1):

$$(f_z, f_i) = (f_z, \overline{\lambda}_i e_1) + (f_z, B_0 x_i) \Rightarrow$$

$$\Rightarrow f_i(z) = \lambda_i + \sum_{n=1}^{\infty} \frac{1}{n} \overline{(x_i, e_n)} z^{n-1}, \quad i = 1, ..., k,$$

which is of the form (2.13).

Since $\tilde{x} \in H^k$ it follows that $\tilde{B}_0\tilde{x} \in H_1^k$ and thus the series (2.13) converges absolutely. The first derivatives of $f_i(z)$ converge also absolutely for $z \in \overline{\Delta}$ because $\tilde{B}_0$ leaves invariant the space H_1^k and the element $\tilde{f} = \tilde{n}_0 + \tilde{B}_0\tilde{x}$ belongs in the definition domain of $\tilde{C}_0$ in H_1^k. □

Proof Proof of Theorem 2.2. The proof is the same with the proof of theorem 2.1. The only difference is that Propositions 4.5 and 4.9 are now applied for $\nu = 2$. □

Proof Proof of Theorem 2.3. It suffices to set $z = rw$ and obtain solutions of the corresponding system in $|w| < 1$ for every $r > 0$. □

Proof Proof of Theorems 2.4 and 2.6. The proofs of these theorems follow easily by:
a)using the abstract forms in H_1^k: (3.16) and (3.17) of (2.18) and (2.20), respectively, obtained in Section 3 and
b) applying Proposition 4.10 for $\nu = 1$ and $\nu = 2$, respectively.

The form of the series (2.19) or (2.21) is obtained as in the proof of theorem 2.13 by using the mapping (3.1). □

Lemma 5.1. *Assume that the operator equation*

$$(-ia)^{\nu}(\tilde{C}_0 - \tilde{I})^{\nu}\tilde{f} = A(V)\tilde{f} + \tilde{N}(\tilde{f}), \quad a \in \mathbb{R}, \tag{5.4}$$

where $\tilde{N}$ satisfies (3.11) and (3.13), has a solution $\tilde{f}$ in H_1^k of the form:

$$\tilde{f} = \overline{\lambda}\tilde{f}_0 + \tilde{B}_0^{\nu}\tilde{x}, \tag{5.5}$$

where $\tilde{f}_0 = (\xi e_{m+1}, ..., \xi e_{m+1})$, $\lambda \in \mathbb{C}$ *and* $\tilde{x} \in M_1(m)$, $f_i(z) = (f_z, f_i)$ *and* $G_i(\tilde{f}(z)) = (f_z, N_i(\tilde{f}))$, $i = 1, ..., k$ *Then the function*

$$\tilde{g}(t) = \tilde{f}(\underbrace{re^{iat}}_{z}), \quad g_i(t) = \lambda r^m e^{iamt} + b_{i1}(\lambda)\rho^{m+1} e^{ia(m+1)t} + ... \tag{5.6}$$

is a complex valued periodic solution of the system:

$$\frac{d^\nu \tilde{g}(t)}{dt^\nu} = A(re^{iat})\tilde{g}(t) + \tilde{G}(\tilde{g}(t)). \tag{5.7}$$

Moreover the series in (5.6) converge absolutely for every real t *and* $0 \leq r \leq 1$.

Proof. Since the solution $\tilde{f}$ of (5.4) has the form (5.5), the corresponding functions $f_i(z) = (f_z, f_i)$, $i = 1, ..., k$ together with their derivatives up to order ν belong to $H_1(\Delta)$. Thus $g_i(t)$, $i = 1, ..., k$ are $\nu-$times differentiable functions with respect to t for every $0 \leq r \leq 1$.

For $\nu = 1$ it is:

$$\frac{dg_i(t)}{dt} = iaz\frac{df_i(z)}{dz} = ia(f_z, VC_0V^* f_i) \overset{(3.4)}{=} ia(f_z, (C_0 - I)f_i) =$$

$$= (f_z, \overline{ia}(C_0 - I)f_i) = (f_z, (-ia)(C_0 - I)f_i).$$

Similarly, it can be found that for $\nu \geq 2$ the following holds:

$$\frac{d^\nu g_i(t)}{dt^\nu} = (ia)^\nu (f_z, (C_0 - I)^\nu f_i) = (f_z, (-ia)^\nu (C_0 - I)^\nu f_i).$$

Thus from (5.4) it follows:

$$(-ia)^\nu (C_0 - I)^\nu f_i = \sum_{j=1}^{k} A_{ij}(V) f_j + N_i(\tilde{f}), \quad i = 1, ..., k \Rightarrow$$

$$\Rightarrow (f_z, (-ia)^\nu (C_0 - I)^\nu f_i) = \sum_{j=1}^{k} (f_z, A_{ij}(V) f_j) + (f_z, N_i(\tilde{f})), \quad i = 1, ..., k \Rightarrow$$

$$\Rightarrow \frac{d^\nu g_i(t)}{dt^\nu} = \sum_{j=1}^{k} a_{ij}(re^{iat}) g_i(t) + G_i(\tilde{g(t)}), \quad i = 1, ..., k$$

which is system (5.7) and in this way the solutions of (5.4) and (5.7) are connected. The form of the series in (5.6) is obtained by using the mapping (3.1) and the form of (5.5). □

Remark 5.2. *In the following it would be more useful to write (5.4) as:*

$$(\tilde{C}_0 - \tilde{I})^\nu \tilde{f} - \frac{i^\nu}{a^\nu} A(V)\tilde{f} = \underbrace{\frac{1}{(-ia)^\nu}\tilde{N}(\tilde{f})}_{\tilde{h}(\tilde{f})}, \tag{5.8}$$

where $\tilde{h}(\tilde{f})$ *satisfies the same conditions as* $\tilde{N}(\tilde{f})$. *Observe also that* ν *is an integer and thus it can be written in one of the forms:*

$$\nu = 4\sigma - 3, \quad \nu = 4\sigma - 1, \quad \nu = 4\sigma, \quad \nu = 4\sigma - 2, \quad \sigma = 1, 2, ...$$

and the corresponding values of i^ν *are* i, $-i$, 1 *and* -1.

Proof Proof of Theorem 2.8. Let μ be the largest positive number for which $i\mu^\nu$ ($-i\mu^\nu$) is an eigenvalue of $A(0)$. Set in (5.8) $a=-\mu$ ($a=\mu$) if $\nu=4\sigma-3$ and $a=\mu$ ($a=-\mu$) if $\nu=4\sigma-1$. In both cases the greatest eigenvalue of the matrix $\frac{i^\nu}{a^\nu}A(0)$ is 1 which is of the form n^ν, $n=1,2,....$ Thus, according to Proposition 4.10, equation (5.4) has a family of solutions of the form (4.26) and as a consequence, due to Lemma 5.1, the system (2.22) has a family of solutions of the form (2.23). $\square$

Proof Proof of Theorem 2.10. The proof is similar to the proof of Theorem 2.8. It suffices to mentions that now for $a=\mu$ and $\nu=4\sigma-2$ ($a=-\mu$ and $\nu=4\sigma$) the greatest eigenvalue of the matrix $\frac{i^\nu}{a^\nu}A(0)$ is 1. $\square$

The proof of this theorem is analogous to the proof of Theorem 2.8 and is based on the lemma:

Lemma 5.3. *Assume that the operator equation*

$$(-a)^\nu(\tilde{C}_0-\tilde{I})^\nu\tilde{f}=A(V)\tilde{f}+\tilde{N}(\tilde{f}), \quad a>0, \tag{5.9}$$

where $\tilde{N}$ *satisfies (3.11) and (3.13), has a solution* $\tilde{f}$ *in* H_1^k *of the form:*

$$\tilde{f}=\overline{\lambda}\tilde{f}_0+\tilde{B}_0^\nu\tilde{x}, \tag{5.10}$$

where $\tilde{f}_0=(\xi e_{m+1},...,\xi e_{m+1})$, $\lambda\in\mathbb{C}$ *and* $\tilde{x}\in M_1(m)$, $f_i(z)=(f_z,f_i)$ *and* $G_i(\tilde{f}(z))=(f_z,N_i(\tilde{f}))$, $i=1,...,k$ *Then the function* $\tilde{g}(t)=\tilde{f}(re^{-at})$ *is a solution of (2.26) for every* $0\le r\le 1$ *and every positive* t.

Acknowledgment

I would like to express my warm thanks to my former student and now Assistant Professor Eugenia Petropoulou for her many helpful remarks and suggestions.

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Received March 2010