A SYMMETRIC INFORMATION DIVERGENCE MEASURE OF CSISZAR'S $F$ DIVERGENCE CLASS

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Abstract

Information divergence measures are often useful for comparing two probability distributions. A new divergence measure is introduced which is symmetric and belongs to the well known class of Csiszar's $f$ divergence. Its properties are studied and bounds in terms of some well known divergence measures are obtained. A numerical illustration to compare this measure with some known divergence measures is carried out.

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Additional Key Words and Phrases: Csiszar's $f$ divergence, symmetric divergence, information inequalities.

1. INTRODUCTION

A general class of divergence measure that includes several divergences used in measuring the distance or affinity between two probability distributions is the Csiszar's $f$ divergence class.

This class is introduced by using a convex function $f$ defined on $(0, \infty)$. An important property of this divergence is that many known divergences can be obtained from this measure by appropriately defining the convex function $f$. These measures have been applied in a variety of fields such as anthropology, genetics, finance, economics, analysis of contingency tables approximation of probability distribution, signal processing and pattern recognition. In this paper we have introduced a new symmetric information divergence measure which belongs to the Csiszar's $f$ divergence [2,3]. In section 2 we discuss Csiszar's $f$ Divergence and inequalities. A New symmetric measure is obtained in section 3. In section 4 we have derived information divergence inequalities providing bounds for the new measure in terms of some well known divergence measures. In section 5 the bounds obtained in previous section are checked numerically and the suggested measure is compared with other measures. Section 6 concludes the paper.

2. CSISZAR'S $f$ DIVERGENCE

Let $\Omega = \{x_1, x_2, \ldots\}$ be a set with at least two elements and $\mathbb{P}$ the set of all probability distribution $\mathbb{P} = (p(x) : x \in \Omega)$ For a convex function $f : (0, \infty) \rightarrow \mathbb{R}$, the $f$-divergence of the probability distributions $P$ and $Q$ by Csiszar, [2,3] and Ali & Silvey, [1] is defined as

$$C_f(P, Q) = \sum_{x \in \Omega} q(x) f\left(\frac{p(x)}{q(x)}\right)$$
Henceforth, for brevity we will denote $C_\varphi(P,Q), p, q$ and $\sum$ respectively.

Ostrowski [7] has discussed basic general properties of $f$ - divergences including their axiomatic properties and some important classes. During the recent past, there has been a considerable amount of work providing different kind of bounds on the distance, information and divergence measures [9, 4, 5 and 6]. Taneja and Kumar [8] unified and generalized three theorems studied by Dragomir [4, 5, 6] which provide bounds on $C(P,Q)$. The main result in [8] is the following theorem:

**Theorem 2.1** Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a mapping which is normalized, i.e., $f(1) = 0$ and suppose that

(i) $f$ is twice differentiable on $(r,R)$, $0 \leq r < 1 \leq R < \infty$, $(f’’)$ and $(f’’’)$ denote the first and second derivatives of $f$,

(ii) there exist real constants $m, M$ such that $m < M$ and $m \leq x^{s-1}f''(x) \leq M, \forall x \in (r,R), s \in \mathbb{R}$.

If $P, Q \in \mathcal{P}_2$ are discrete probability distributions with $0 < r \leq s < 1 \leq R < \infty$, then

$$m\Phi_i(P,Q) \leq C(P,Q) \leq M\Phi_i(P,Q),$$

and

$$m(\eta(P,Q) - \Phi_i(P,Q)) \leq C_\varphi(P,Q) - C(P,Q) \leq M(\eta(P,Q) - \Phi_i(P,Q)),$$

where

$$\Phi_i(P,Q) = \left\{ \begin{array}{ll}
2K_s(P,Q), & s \neq 0,1 \\
K(Q,P) & s = 0 \\
K(P,Q) & s = 1
\end{array} \right.$$ 

$$2K_s(P,Q) = [s(s-1)]^1[(\sum p^s q^{1-s})-1], \quad s \neq 0,1,$$

$$K(P,Q) = \sum p\ln\left(\frac{p}{q}\right), \quad (2.1)$$

$$C_\varphi(P,Q) = C_f\left(\frac{p^s}{q^s}, p\right) - C_f(P,Q) = \sum (p-q)^{s-1}\left(\frac{p}{q}\right), \quad (2.2)$$

and

$$\eta(P,Q) = C_{\varphi_s}\left(\frac{p^s}{q^s}, p\right) - C_{\varphi_s}(P,Q) \quad$$

$$\quad = \left\{ \begin{array}{ll}
((s-1)s^{-1}\Sigma(p-q)\left(\frac{p}{q}\right)^{s-1}, & s \neq 1 \\
\Sigma(p-q)\ln\left(\frac{p}{q}\right), & s = 1
\end{array} \right.$$ 

The following information inequalities which are interesting from the information-theoretic point of view, are obtained from Theorem 2.1 and discussed in [5]

(i) The case $s=2$ provides the information bounds in terms of the Chi-square divergence $\chi^2(P,Q)$:

$$\frac{\eta}{2} \chi^2(P,Q) \leq C(P,Q) \leq \frac{2}{\eta} \chi^2(P,Q) \quad (2.3)$$

and
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\[ \frac{1}{2} \chi^2(P, Q) \leq [C_p(P, Q) - C(P, Q)] = \frac{1}{2} \chi^2(P, Q) \]  
(2.4)

where

\[ \chi^2(P, Q) = \sum_{e}^{W_{e} - W_{e}^2} \]  
(2.5)

(ii) For \( s = 1 \), the information bounds in terms of the Kullback-Leibler divergence, \( K(P, Q) \):

\[ mK(P, Q) \leq C(P, Q) \leq MK(P, Q), \]  
(2.6)

and

\[ mK(Q, P) \leq [C_p(P, Q) - C(P, Q)] \leq MK(P, Q). \]  
(2.7)

(iii) The case \( s = \frac{1}{2} \) provides the information bounds in terms of the Hellinger's discrimination, \( h(P, Q) \):

\[ 4mh(P, Q) \leq C(P, Q) \leq 4Mh(P, Q). \]  
(2.8)

and

\[ 4m \left( \frac{1}{2} \eta_{1/2}(P, Q) - h(P, Q) \right) \leq [C_p(P, Q) - C(P, Q)] \]  
(2.9)

where

\[ h(P, Q) = \sum_{e}^{1/2 - E_{e} - 1/2} \]  
(2.10)

(iv) For \( s = 0 \), the information bounds in terms of the Kullback-Leibler and \( \chi^2 \) – divergences:

\[ mK(Q, P) \leq C(P, Q) \leq MK(Q, P), \]  
(2.11)

and

\[ m [\chi^2(Q, P) - K(Q, P)] \leq [C_p(P, Q) - C(P, Q)] \leq M [\chi^2(Q, P) - K(Q, P)] \]  
(2.12)

3. NEW INFORMATION DIVERGENCE MEASURE

In this section we have found a new divergence measure which belongs to the Csiszar’s family and is symmetric with respect to the two probability distributions \( P \) and \( Q \). We have derived the new measure from the convex function \( f(u) \). We have also shown that second derivative of this function is positive for all \( u > 0 \), hence \( f(u) \) is convex. Further \( f(1) = 0 \). Thus we can say that the new measure is nonnegative and convex in the pair of probability distributions \( (P, Q) \in \Omega \).

We consider the function \( f: (0, \infty) \to \mathbb{R} \):

\[ f(u) = \frac{u + \alpha u^2 - \alpha u^2 - \alpha - 1}{u} \]  
(3.1)

and thus we have the new divergence measure.
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\[ R^* (P, Q) = \frac{\left( u^2 + v^2 + uv \right)^2}{u^2 v^2} \]  

(3.2)

\[ f'(u) = \frac{(u+1)^2 (u-1)^2}{u^3} \]  

(3.3)

\[ f''(u) = \frac{2(u^2 + 2u + 1)}{u^3} \left[ 20u^7 + 16u^6 + 6u^5 + 6u^4 + 6u^3 + 6u^2 + 16u + 20 \right] \]

\[ = 2 \frac{(u^2 + 2u + 1)}{u^3} \left[ 10u^7 + 5u^6 + 3u^5 + 3u^4 + 3u^3 + 3u^2 + 5u + 10 \right] \]

\[ = 2 \frac{(u^2 + 2u + 1)}{u^3} T(u) \]

(3.4)

Where \( T(u) = 10u^7 + 5u^6 + 3u^5 + 3u^4 + 3u^3 + 3u^2 + 5u + 10 \)

It follows that \( f''(u) > 0 \) for all \( u > 0 \)

Hence \( f(u) \) is convex for all \( u > 0 \).

**Figure 3.1: Graph of the Convex Function \( f(u) \)**

Further \( f(1) = 0 \) thus we can say that the measure is non-negative and convex in the pair of probability distribution \( (P, Q) \in \Omega \)

Its corresponding non-symmetric function is given by

\[ f(u) = \frac{(u+1)^2 (u-1)^2}{u} \]

\[ f'(u) = \frac{(u-1)^2}{u^2} \left[ 20u^4 + 3u^3 + 2u^2 + 2u + 2 \right] \]

which is convex for \( u > 0 \).
4. Bounds for $R^*(P, Q)$

In this section we will derive information divergence inequalities providing bounds for $R^*(P, Q)$ in terms of some well known divergence measures, using the inequalities discussed in section 2.

**Proposition (4.1)**

Let $\chi^2(P, Q)$ and $R^*(P, Q)$ be defined as in (2.5) and (3.2) respectively. For $P, Q \in \mathbb{P}^2$ and $0 < r \leq \frac{p}{q} \leq R < \infty$ we have

\[
\frac{(R-p)^2}{R^2} T(R) \chi^2(P, Q) \leq R^*(P, Q) \leq \frac{(R-q)^2}{R^2} T(Q) \chi^2(P, Q)
\]

(4.1)

and

\[
\frac{(R-q)^2}{R^2} T(R) \chi^2(P, Q) \leq [R_q^*(P, Q) - R^*(P, Q)] \leq \frac{(R-p)^2}{R^2} T(Q) \chi^2(P, Q)
\]

(4.2)

where $R_q^*(P, Q) = (p - q) f'(\frac{p}{q})$

\[
= \frac{10 - p^2 [x^8 + 3x^6 + 3x^4 + 3x^2 + 8x + 10]}{p^2 q^2}
\]

(4.3)

and

$T(R) = \left[ 10R^7 + 8R^5 + 3R^4 + 3R^2 + 3R^2 + 8R + 10 \right]$

$T(r) = \left[ 10r^7 + 3r^6 + 3r^4 + 3r^2 + 8r + 10 \right]$

**Proof.** From the function $f(u)$ in (3.1) we have

\[
f'(u) = \frac{(u + 1)[(u - 1)^2][2u^2 + 2u^2 + u^2 + 2u + 4]}{u^2}
\]

And thus the divergence measure:

\[
R_q^*(P, Q) = \sum (p - q) f'(\frac{p}{q})
\]

\[
= \frac{10 - p^2 [x^8 + 3x^6 + 3x^4 + 3x^2 + 8x + 10]}{p^2 q^2}
\]

Further $f'(u) = \frac{10 - u^2}{u^2} T(u)$

Now if $u \in [a, b] \subset (0, \infty)$ then
Proposition (4.2) \[ 2 \frac{(R^2 - 1)^2}{R^4} \leq \frac{f'}{(u)} \leq \frac{2(R^2 - 1)^2}{u^2} \]

and accordingly \[ 2 \frac{(R^2 - 1)^2}{R^4} I(R) \leq \frac{f'}{(u)} \leq \frac{2(R^2 - 1)^2}{u^2} I(u) \]

(4.4)

where \( r \) and \( R \) are defined above.

Thus in view of (2.3), (2.4) and (4.4) we get inequalities (4.1) and (4.2) respectively.

The information bounds in terms of Kullback - Leibler divergence \( K(P, Q) \) defined in (2.1) are as follows

**Proposition (4.2)**

Let \( K(P, Q) \), \( R^2(P, Q) \) and \( R^2(P, Q) \) be defined as (2.1), (3.2) and (4.3) respectively. For \( P, Q, r \in P^2 \) and \( 0 < r < \frac{R}{q} < \infty \) we have

\[ 2 \frac{(R^2 - 1)^2}{R^4} K(P, Q) \leq R^2(P, Q) \leq \frac{2(R^2 - 1)^2}{q^2} K(P, Q) \]

and

\[ 2 \frac{(R^2 - 1)^2}{R^4} K(P, Q) \leq [R^2(P, Q)] - [R^2(P, Q)] \leq \frac{2(R^2 - 1)^2}{q^2} K(P, Q) \]

(4.6)

**Proof.** We have \( f' \) \( u \) \[ 2 \frac{(R^2 - 1)^2}{R^4} I(R) \leq \frac{f'}{(u)} \leq \frac{2(R^2 - 1)^2}{u^2} I(u) \]

(4.7)

Then \[ \inf g (u) = 2 \frac{(R^2 - 1)^2}{R^4} I(R) \]

(4.7)

and

\[ \sup g (u) = 2 \frac{(R^2 - 1)^2}{u^2} I(u) \]

(4.8)

The inequalities (4.5) and (4.6) follow from (2.6) and (2.7) using (4.7) and (4.8). The following proposition provides the information bounds in terms of the Hellinger's discrimination \( h(P, Q) \) and \( \tau_{\nu_1}(P, Q) \).
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**Proposition 4.3**

Let $\eta_0 (P, Q)$, $h(P, Q)$, $R^* (P, Q)$ and $R_{\phi}^* (P, Q)$ be defined as in (2.2), (2.9) respectively. For $P, Q \in \mathbb{P}^2$ and and $0 < r < \frac{1}{\phi} \leq \frac{1}{q} \leq \frac{1}{q} \leq \frac{1}{q} \leq \frac{1}{q}$

$$\frac{\text{E}[P - 1]}{\text{E}[P]} \cdot T(r) \leq h(P, Q) \leq \frac{\text{E}[R - 1]}{\text{E}[R]} \cdot T(R) \leq h(P, Q) \leq \frac{\text{E}[R - 1]}{\text{E}[R]} \cdot T(R) \leq h(P, Q)$$

(4.9)

And

$$\frac{8 (r^2 - 1)^2 T(r)}{r^2} \cdot \frac{1}{r^2} \leq h(P, Q) \leq [R_{\phi}^* (P, Q) - R^* (P, Q)]$$

(4.10)

Let $g = [r, R] \rightarrow \mathbb{R}$ be such that

$$g(u) = u \frac{\text{E}[R - 1]}{\text{E}[R]} = \frac{2 (u - 1)^2}{u^2} \cdot T(u)$$

$$\inf_{u \in (r, R)} g(u) = \frac{8 (r - 1)^2}{r} \cdot T(r)$$

(4.11)

$$\sup_{u \in (r, R)} g(u) = \frac{8 (R - 1)^2}{R} \cdot T(R)$$

(4.12)

Thus the inequalities (4.9) and (4.10) are established using (2.8), (2.9), (4.11) and (4.12)

Next follows the information bounds in terms of Kullback – Leibler and $\chi^2$ divergences.

**Proposition 4.4**

Let $K (P, Q)$, $\chi^2 (P, Q)$, $R^* (P, Q)$ and $R_{\phi}^* (P, Q)$ be defined as in (2.1), (2.4), (3.4) and (4.3) respectively.

For $P, Q \in \mathbb{P}^2$ and $0 < r < \frac{1}{\phi} \leq \frac{1}{q} \leq \frac{1}{q} \leq \frac{1}{q}$ we have

$$\frac{8 (r^2 - 1)^2}{r^2} \cdot T(r) \cdot K(q, P) \leq h(P, Q) \leq \frac{8 (R - 1)^2}{R^2} \cdot T(R) \cdot K(q, P)$$

(4.13)

And
Proof. We have
\[
\frac{1}{2} \int \frac{(u-1)^2}{u^2} T(u) \leq \frac{2 (p-1)^2}{2} T(R)\left(\frac{u}{p}\right)
\]
\[
\frac{1}{R^2} \left[ L^2 (Q,P) - K (Q,P) \right] \leq \frac{2 (R-1)^2}{R^4} T(R)\left(\frac{u}{p}\right)
\]
(4.14)

Thus (4.13) and (4.14) follows from (2.11) and (2.12) using (4.15) and (4.16)

5. NUMERICAL ILLUSTRATION

In this section we will numerically verify the bounds obtained in the previous section and also compare the new measure with some known divergence measures. Here
\[
J(P,Q) = K(P,Q) + K(Q,P) = \sum(p-q)\ln\frac{p}{q}
\]
is the Kullback-Leibler divergence
and
\[
\Psi(P,Q) = \chi^2(P,Q) + \chi^2(Q,P) = \sum\frac{(p-q)^2}{p+q}
\]
is the Symmetric Chi-square Divergence.

Ex. 5.1 (Symmetrical):

Let P be the binomial probability distribution for the random variable X with parameter (n=8 p=0.5) and Q its approximated normal probability distribution.

| Table 5.1 Binomial Probability Distribution (n=8 p=0.5) |
|---|---|---|---|---|---|---|---|---|---|
| X  | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
| p  | .004 | .031 | .109 | .219 | .219 | .109 | .031 | .004 |
| q  | .005 | .030 | .104 | .220 | .282 | .220 | .104 | .030 | .005 |
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| \( p(x) \) | .774 | 1.042 | 1.0503 | .997 | .968 | .997 | 1.0503 | 1.042 | .774 |
| \( q(x) \) |

The measure \( R^* (P, Q) \), \( \Psi (P, Q) \), \( \chi^2 (P, Q) \), \( J(P, Q) \) are

\[
R^* (P, Q) = 0.00020185, \quad \chi^2 (P, Q) = 0.00145837, \quad \Psi (P, Q) = 0.00305063, \quad J (P, Q) = 0.0015184
\]

It is noted that

\[
\frac{\text{Lower bound}}{\text{Upper bound}} = \frac{0.00015764}{0.00853097}
\]

thus \( 0.00015764 < R^* (P, Q) = 0.00020185 < 0.00853097 \)

width of the interval is 0.00837335.

Example 5.2 (Asymmetrical)

Let \( P \) be the binomial probability distribution for the random variable \( X \) with parameter \( n = 8 \) \( p = 0.4 \) and \( Q \) its approximated normal probability distribution. Then

Table 2 Binomial Probability Distribution (n=8, p=0.4)

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>.017</td>
<td>.090</td>
<td>.209</td>
<td>.279</td>
<td>.232</td>
<td>.124</td>
<td>.041</td>
<td>.008</td>
<td>.001</td>
</tr>
<tr>
<td>( q(x) )</td>
<td>.020</td>
<td>.082</td>
<td>.198</td>
<td>.285</td>
<td>.244</td>
<td>.124</td>
<td>.037</td>
<td>.007</td>
<td>.0007</td>
</tr>
<tr>
<td>( \frac{p(x)}{q(x)} )</td>
<td>.850</td>
<td>1.102</td>
<td>1.056</td>
<td>.979</td>
<td>.952</td>
<td>1.001</td>
<td>1.097</td>
<td>1.194</td>
<td>1.401</td>
</tr>
</tbody>
</table>

From the above data measures \( R^* (P, Q), \chi^2 (P, Q), \Psi (P, Q) \) and \( J (P, Q) \) are calculated
R* (P, Q) = 0.04883318, \( \Psi (P, Q) = 0.00657063 \)
\( \chi^2 (P, Q) = 0.00333883, \quad J (P, Q) = 0.0032778 \)

Lower bound = 
\[
\frac{(R - 1)^2}{R^0} \left[ 10R^7 + 3R^6 + 3R^5 + 3R^4 + 3R^3 + 3R^2 + 3R^1 + 10 \right] \chi^2 (P, Q) = 0.01590959
\]
Upper bound = 
\[
\frac{(R - 1)^2}{r^4} \left[ 10r^7 + 3r^6 + 3r^5 + 3r^4 + 3r^3 + 3r^2 + 3r^1 + 10 \right] \chi^2 (P, Q) = 0.09986605
\]

Thus 0.01590959 < R* (P, Q) = 0.04883318 < 0.09986605

The width of the interval is 0.08395646.

The following figure shows the behavior of R* (P, Q) [New measure], \( \Psi (P, Q) \) [Symmetric Chi-square divergence] and \( J (P, Q) \) [Symmetric Kullback-Leibler divergence] we have considered \( p = (a, 1-a) \) and \( q = (1-a, a) \) \( a \in (0, 1) \). It is clear from the figure 5.1 that measure R* (P, Q) and \( \Psi (P, Q) \) have a steeper slope than \( J (P, Q) \).
6. CONCLUDING REMARKS

The Csiszar's $f$-divergence is a general class of divergence measures, which includes several divergences used in measuring the distance or affinity between two probability distributions. This class is introduced by using a convex function $f$, defined on $(0,\infty)$. An important property of this divergence is that many known divergences can be obtained by appropriately defining the convex function $f$. Non-parametric measures for the Csiszar's $f$-divergences are also available. We have introduced a new symmetric divergence measure by considering a convex function and have investigated its properties. Further we have established its bounds in terms of known divergence measures. We have also compared the new measure with some well known divergence measures.
7. REFERENCES


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