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SEPARATE INVOLUTORY GROUPS

KUZENNYJ, M.F., TOMÁNEK, Ľ.

Abstract

The description of the finite p-groups of the type (p, p, p) was solved by many authors. We describe the groups generated by all involutions of the group. Our main result is formulated in the Theorem and the Corollary.

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A cyclic subgroup of order *2* of a group is called the involution. M. Hall characterized all finite p-groups as having only one subgroup of order *p* [see [2], Theorem 12.5.2]. The finite p-groups contain exactly three subgroup of order *p* and we denote this as a group of the type (*p, p, p*).

N. Blackburn [1] generalized the results of finite p-groups in 1961. In his work he describes the finite *p*-groups which do not have the group of the type (*p, p, p*), if $p > 2$. The description of finite *2*-groups of the type (*2,2,2*) is absent. The finite p- groups as having more than one subgroup of order *2* which do not have the group of the type (*2,2,2*) have been described by many authors [3],…,[7]. Many authors have described special case of the finite 2-groups of the type (*2,2,2*). Generally speaking, this problem has not been resolved up to the present time. We tried to resolve this problem in this paper. We describe the groups generated by all their involutions which we further call a separate involutory groups.

We use the standard notation and terminology of the theory of groups : $< a >$ the cyclic group generated by one element a; < a, b, c> the group generated by the elements a, b, c; $k = [a, b] = a^{-1}b^{-1}ab \quad \in G$ the commutator of the elements a,b; G' = [G, G] the group generated by all commutators is called the commutator (or the derivative of *G*) of the group *G;* *C(G)* the center of the group *G*, $C_G(H)$ the centralizer of the subgroup *H* in *G;* $H \leftthreetimes K$ the semidirect product of the groups *H*, *K*, *exp G* the exponent of the group G;

$|d|$ the order of the element $d \in G$; Ha the right coset of the subgroup H of the group G for $a \in G$; $H \lhd G$ H is a normal subgroup of G or H is a normal in G. If $H \lhd G$, then $aH = Ha$ for each element $a \in G$. S_n is the symmetric group of degree n. The subgroup A of the group G is a quasicentral subgroup if each subgroup of A is a normal in G.

Definition. A group G is said to be a separate involutory group if all of its elements which are not elements of an arbitrary subgroup S are the involutions. The subgroup S is called the involutory separator of group G.

Proposition 1.[[2]T.1.4.2]. A finite group G of order n is isomorphic to any subgroup of symmetric group S_n .

Proposition 2.[[2]T.6.5.3]. A group $G = H \lambda K$ is the semidirect product of the groups H, K if and only if $H \lhd G$, K is a subgroup of G, $H \cap K = E$ and $H \cup K = G$.

Theorem. Let G be a separate involutory group. Then G is one of the following types:

1. $E \neq G$ is the elementary Abelian 2 – group,
2. $G = C \lambda \langle d \rangle$, $|d| = 2$, C is the quasicentral abelian subgroup of G, $\exp C > 2$, $d^{-1} a d = a^{-1}$, for each element $a \in C$.

Proof. Let G be a separate involutory group, $|G| > 1$ and let $E \subseteq S \subset G$. According to the Definition there exists the element $x \in (G - S)$ of order 2.

Let G be an elementary abelian 2-group, it is clear that G is a group of the first type of the Theorem.

Suppose that $\exp G > 2$, $G = S \,.\, D$, $S \cap D = E$, $D \subseteq G$ and each element

$d \in D$ is the only representative of the right coset Sd . We get $|D| > 1$, $|d| = 2$. Hence $exp\ D = 2$. If $d_1, d_2 \in D$, $d_1 \neq d_2$, then $d_1^2 = d_2^2 = 1$. By properties D we have $d_1 \notin$ Sd_2 , $d_1.d_2 \notin S$, thus $|d_1.d_2| = 2$. Hence $(d_1d_2)^2 = d_1.d_2\ d_1.d_2 = 1$ also $d_1.d_2 = d_2^{-1}.d_1^{-1} = d_2.d_1$. Admittedly applies $x = a_1d_1$, $y = a_2d_2$ for all elements $x, y \in G$, $a_1, a_2 \in$ S, $d_1, d_2 \in D$.

Assume S $\subseteq C(G)$. If $x.y = a_1d_1 . a_2d_2 = a_1\, a_2\, d_1d_2 = a_2 d_2 . a_1d_1 = y.x$ for all elements $x, y \in G$, then $G' = [G,G] = E$. If $a.d \notin$ S for all elements $a \in$ S , $d \in D$ $1 \neq d$, then $1 = (ad)^2 = a^2d^2 = a^2$. Hence $exp\ S = 2$. Then G is an elementary abelian 2- group, i. e. G is a group of the first type of this Theorem.

Assume $S \not\subset C(G)$. If $a \in$ S, $1 \neq d \in D$, $ad \notin$ S, then $(ad)^2 = 1$.Hence $ad = d^{-1}a^{-1}$. Then applies $d^{-1}ad = d^{-2}a^{-1} = a^{-1}$. If exp S = 2 , then $a^{-1} = a \in C(G)$. We proved

S $\subseteq C(G)$. This is a contradiction with the assumption $S \not\subset C(G)$. Hence $exp\ S > 2$. Assume that there exists an element a$\in$ S so that $\langle a\rangle \subset S \not\subset C(G)$. Then for an element $1 \neq d \in D$ we have $d^{-1}ad = a^{-1}$, $|a| > 2$. For all elements $x \in$ S, we get

$d^{-1}axd = (ax)^{-1} = x^{-1}. a^{-1} = d^{-1}xdd^{-1}ad = (xa)^{-1} = a^{-1}.x^{-1}$. Then $x^{-1}a^{-1} = a^{-1}x^{-1}$ implies $xa = ax$ for $x, a \in$ S. Hence S'=$[S, S] = E$ for all elements $a, x \in$ S. Furthermore S is the quasicentral abelian subgroup of the group G, and also $S \lhd G$.

Let denote $C = C_G(\langle a\rangle)$. Because $\langle a\rangle \lhd G$, then C is a normal in G , $S \subseteq C$. Let a be the element of the cyclic group $\langle a\rangle$ for which applies $\langle a\rangle \not\subset C(G)$, then $C \subset G$. By the Proposition 1 a group $G/C \cong \mathrm{Aut}(\langle a\rangle)$ and $G/C \neq E$ is a finite group. Let suppose $|G : C| > 2$, then there exist the elements $d_1, d_2 \in G$ for which applies C$d_1 \neq$ Cd_2. We have $|d_1. d_2| = 2$, and $d_2d_1\, a d_1^{-1} d_2^{-1} = d_2a^{-1}d_2^{-1} = a$. Hence $d_1.d_2 \in$ C, implies Cd_1 = Cd_2. This is a contradiction with C$d_1 \neq$ Cd_2. Hence $|G : C| = 2$, $d \in (D - C)$, $|d| = 2$ and by the Proposition 2 $G = C \lambda \langle d\rangle$.

Because $S \subseteq C$,then $x \in (G - C)$ and this implies $x \in (G - S)$. Then $x^2 = 1$, and C is the involutory separator of group G. Analogicly the same as for S we get $d^{-1}ad = a^{-1}$ for each element $a \in C$ and $exp\ C > 2$. Hence G is the group of the second type of this Theorem.

Conversely, we assume that G is a group of one of the types of this Theorem. Then $|G| > 1$. If G is a group of the first type of this Theorem, then in this case $S = E$. If $S \subset G$ then $|x| = 2$ applies for each element $x \in (G - S)$. By the Definition G is a separate involutory group.

Let G be a group of the second type of this Theorem, and $x \in (G - S)$. Because $x = a.d$ for each element a of C, we have $x^2 = adad = aa^{-1}d^2 = 1$. Hence G is a separate involutory group, too. The proof of the Theorem is complete.

By the Theorem the next Corollary follows.

Corollary. The separate involutory groups G are generated by all their involutions.

Proof. Let G be the group of one of the types of the Theorem. If G is a group of the first type of the Theorem, it is clear that the elementary abelian 2 – groups are generated by their involutions.

Suppose that G is a group of the second type of the Theorem. Let M be the set of the involutions of the group G. We denote $N = \langle M \rangle$. We want to prove $G = N$.

Let $ad \in G$ and $(ad)^2 = adad = a.a^{-1}d^2 = 1$ for each element a of $C = C_G(\langle a \rangle)$ and from the preceding follows that $ad \in N$. It implies $G \subset N$. Conversely, if $d \in M \subset N$, then $add = a \in N$. By the Proposition 2 we get $C \lambda \langle d \rangle \subseteq N \subseteq G = C \lambda N$. We proved that $G = N$. The proof of the Corollary is complete.

SEPARATE INVOLUTORY GROUPS

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Kuzennij, M.F.
Institut matematiki NAN
Ukraina

Tománek ,Ľudovít
University of Žilina, Faculty of Science
Univerzitná 8215/1, 01026 Žilina , Slovakia,
e-mail: ludovit.tomanek@fpv.uniza.sk

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