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MODIFIED BROWNIAN MOTION SIMULATION AND CALCULATING ITÔ AND STRATONOVICH INTEGRALS

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Abstract

In this paper, we review the idea of Brownian motion and compute discretized modified Brownian paths. Finally, we experiment with idea for integration with respect to Brownian motion and illustrate the difference between Itô and Stratonovich integrals.

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1. INTRODUCTION

Itô and Stratonovich integral both have their uses in mathematical modeling. We define Stochastic Differential Equations (SDE) using the Itô and Stratonovich versions. The Itô integral can be defined in a manner similar to Riemann integral that is a limit in probability of Riemann sums, such a limit does not necessary exist path wise. The Itô integral of X with respect to W (Brownian motion) is a random variable[3]

$$\int_a^b X(t)dW : \Omega \to \Re$$

where Ω is a probability space, defined to be the L^2 limit of

$$\sum_{i=0}^{k-1} X(t_i)(W(t_{i+1}) - W(t_i))$$

as the mesh of the partition $0 = t_0 < t_1 < ... < t_k = T$ of $[0,T]$ tends to zero. The Stratonovich integral is another way to define stochastic integral.
In the definition of this integral, the same limiting procedure is used except for choosing the value of the process X at the midpoint of each subinterval instead of the left-hand, end-point i.e. $X((t_{i+1} + t_i)/2)$ is place of $X(t_i)$.

2. BROWNIAN MOTION

A scalar standard Brownian Motion, or standard wiener process, over $[0.T]$ is a random variable W(t) that depends continuously on $t \in [0,T]$ and satisfies the following three conditions:

1.W(0)=0 (prob . 1)
2. For $0 \le s \le t \le T$ the random variable given by the increment W(t) – W(s) is normally distributed with mean zero and variance t-s, equivalently

$$W(t) - W(s) \sim \sqrt{t-s}N(0,1)$$

where N(0,1) denotes a normally distributed random variable with zero mean and unit variance [1].
3. For $0 \le s \prec t \prec u \prec v \le T$ the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent.

For computational purposes it is useful to consider discretized Brownian Motion, where W(t) is specified at discrete t values.

We thus set $\delta_t = \dfrac{T}{N}$ for some positive integer N and let W_j denote $W(t_j)$ with $t_j = j\delta_t$.

Condition (2) says [1]

$$W_j = W_{j-1} + dW_j \qquad j = 1,\dots, N$$

where each dW_j is an independent random variable of the form $\sqrt{\delta t}\ N(0,1)$.
The following MATLAB code performs a simulation of discretized Brownian Motion over [0,1] with N= 1000.
Figure 1 shows the result note that for the purpose of visualization, the discrete data has been joined by straight lines . we will refer to an array W created by the algorithm in code 1 as a discretized Brownian path.

Code1. **Brownian path simulation**

```
randn('state',100)
t=1; n=500; dt=t/n;
dw=zeros(1,n);
w=zeros(1,n);
dw(1)=sqrt(dt)*randn;
w(1)=dw(1);
for j=2:n
dw(j)=sqrt(dt)*randn;
w(j)=w(j-1)+dw(j);
end
```

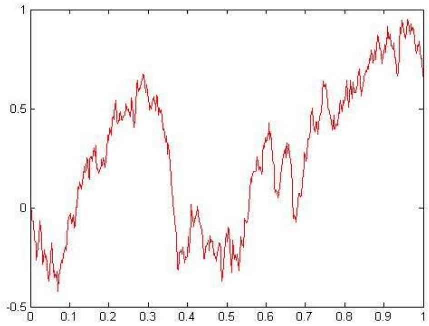

Figure 1

Figure 2 shows the same figure 1, but the key to writing efficient code 2 is avoiding loops and thereby computing directly with arrays rather than individual components.

Code2. Modified Brownian path simulation

```
randn('state',100)
t=1; n=500;dt=t/n;
dw=sqrt(dt)*randn(1,n);
w=cumsum(dw);
```

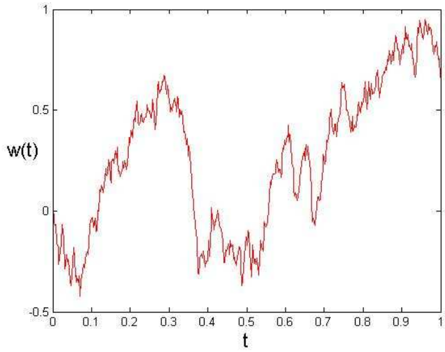

Figure 2

In code 3, we evaluate the function $u(w(t)) = \exp(t + \frac{1}{2} w(t))$ along 1000 discretized Brownian paths. Ten individual paths and the average of u(w(t)) over these paths is plotted. We see in Figure 2 that although u(w(t)) is non smooth along individual paths, its sample average appears to be smooth. In code3 avgerr records the maximum discrepancy between the sample average and the exact expected value over all points t_j.

Code 3 . A sample function along a Brownian path

```
randn('state',100)
t=1;
n=500;
dt=t/n;
t=[dt:dt:1];
m=1000;
dw=sqrt(dt)*randn(m,n);
w=cumsum(dw,2);
u=exp(repmat(t,[m,1])+0.5*w);
 umean=mean(u);
 avgerr=norm((umean-exp(9*t/8)),'inf')
```

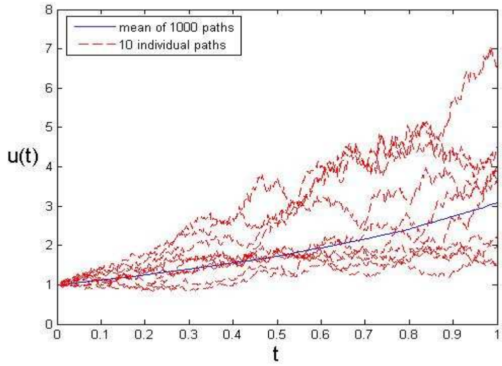

Figure 3

3. STOCHASTIC INTEGRALS

Consider a suitable function h . The integral $\int_0^T h(t)dt$ may be approximated by the Rieman sum

$$\sum_{j=0}^{N-1} h(t_j)(t_{j+1} - t_j) \qquad (1)$$

Where the discrete points $t_j = j\delta_t$. In a similar way we may consider a sum of the form

$$\sum_{j=0}^{N-1} h(t_j)(W(t_{j+1}) - W(t_j)) \qquad (2)$$

which by analogy with (1) , may be regarded as an approximation to a stochastic integral

$$\int_0^T h(t)dW(t)$$

here we are integrating h with respect to Brownian Motion. An alternative (1) is given by

$$\sum_{j=0}^{N-1} h(\frac{t_j + t_{j+1}}{2})(t_{j+1} - t_j) \quad (3)$$

and alternative (2) is

$$\sum_{j=0}^{N-1} h(\frac{t_j + t_{j+1}}{2})(W(t_{j+1}) - W(t_j)) \quad (4)$$

If h(t) ≡ w(t) , then sum (4) requires w(t) to evaluated at $t = \frac{(t_j + t_{j+1})}{2}$.

It can be shown that forming $\frac{(W(t_j) + W(t_{j+1}))}{2}$ and adding an independent $N(0, \frac{\Delta t}{4})$ increment gives a value for $W(\frac{t_j + t_{j+1}}{2})$ that maintains the three condition listed at the section 2. Note that the two “Stochastic Riemann sums” (3) and (4) give markedly different answers. Further experiments with smaller δ_t reveal that this mismatch does not go away as $\delta_t \to 0$.

This highlights a significant difference between deterministic and stochastic integration, in defining a stochastic integral as the limiting case of a Riemann sum, rise to what known as the Itô integral, whereas the midpoint sum (4) produces the Stratonovich integral. The Itô version is the limiting case of

$$\sum_{j=0}^{N-1} W(t_j)(W(t_{j+1}) - W(t_j)) = \frac{1}{2}\sum_{j=0}^{N-1}(W(t_{j+1})^2 - W(t_j)^2 - (W(t_{j+1}) - W(t_j))^2)$$

$$= \frac{1}{2}(W(T)^2 - W(0)^2 - \sum_{j=0}^{N-1}(W(t_{j+1}) - W(t_j))^2). \quad (5)$$

It is clearly that term $\sum_{j=0}^{N-1}(W(t_{j+1}) - W(t_j))^2$ have expected value T and variable to close to the constant T. this argument can be made precise, leading to

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T \quad (6)$$

for Itô integral.

The stratonovich version is the limiting case of

$$\sum_{j=0}^{N-1}(\frac{W(t_j) + W(t_{j+1})}{2} + \Delta Z_j)(W(t_{j+1}) - W(t_j)) \quad (7)$$

where each ΔZ_j is independent $N(0, \frac{\Delta t}{4})$.

This sum is written to

$$\frac{1}{2}(W(T)^2 - W(0)^2) + \sum_{j=0}^{N-1} \Delta Z_j (W(t_{j+1}) - W(t_j))$$

In which the term $\sum_{j=0}^{N-1} \Delta Z_j (W(t_{j+1}) - W(t_j))$ has expected value 0 and variance $O(\delta_t)$.

Thus, in place of (6), we have

$$\int_0^T W(t)dW(t) = \frac{1}{2}W(T)^2$$

Code4. Approximate stochastic integrals

```
randn('state',100)
t=1;n=500;
dt=t/n;
dw=sqrt(dt)*randn(1,n);
w=cumsum(dw);
ito=sum([0,w(1:end-1)].*dw)
strat=sum((0.5*([0,w(1:end-1)]+w)+0.5*sqrt(dt)*randn(1,n)).*dw)
itoerr=abs(ito-0.5*(w(end)^2-t))
straterr=abs(strat-0.5*(w(end)^2))
```

4. CONCLUSION

The quantities Itôerr and straterr in code 4 record the amount by which the Riemann Sums Itô and strat differ from their respective $\delta_t \to 0$ limits (6) and (7). Finally, we find using the modified Brownian motion Itô error and stratonovich error 0.0158 and 0.0186 respectively.

5. ACKNOWLEDGMENTS

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6. REFERENCES

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